Emergent Gauge Symmetries for Non-Abelian Theories

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1 Introduction

We seek to better understand the distinction between gauge and physical symmetries in a field theory and the possible mechanisms by which gauge redundancies may emerge from physical symmetries. The gauge symmetries we observe in nature today can be thought of as low energy approximations emerging from more fundamental principles [1]. General relativity (GR) is an incomplete theory, evident by the singular structure it predicts at the center of black holes. This is because GR is not able to describe how gravity behaves in the quantum regime. In developing a quantum theory of gravity, one approach is to consider the diffeomorphism invariance of GR an effective feature. This approach is typically deemed impossible due to the Weinberg-Witten Theorem (WWT). However, the specific process by which gauge symmetries may emerge is not well understood, so whether or not WWT directly applies isn't immediately obvious. There is a proposed mechanism of gauge emergence in the context of an abelian theory [2]. In order for this mechanism to be useful it must also be consistent for non-abelian theories, since GR is a non-abelian gauge theory. Here, we generalize the abelian example of emergence shown in [2] to a non-abelian symmetry and show that the emergence mechanism remains consistent. We then consider how it may or may not apply to the WWT to see if future considerations of emergent gravity from field theories may be fruitfull. Overall, this paper will provide insight into the nature of emergent gauge theories and their possible applications towards gravity.

2 Background

2.1 Gauge vs Physical Symmetries

Before discussing how gauge redundancies may arise in a system from something initially physical, it is important to be able to distinguish gauge and physical symmetries.

Physical symmetries describe an invariance under a transformation that produces a physically distinguishable result. For instance, consider the rotational symmetry of a Rutherford scattering problem as illustrated in Fig.1. The particle's path is deflected by the electromagnetic interaction, and there is a certain probability associated with detecting it in region A. Rotating this scenario about the z-axis doesn't change this probability, so this set up has a symmetry under rotations. However, this transformation does change the path the deflected particle actually takes, making this a physical symmetry.

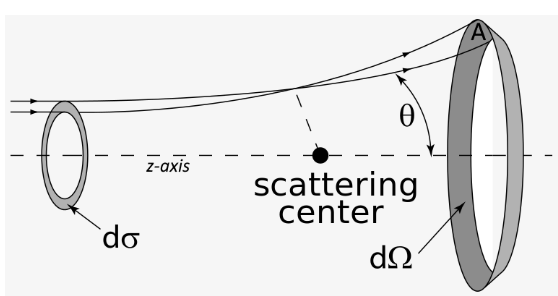

Figure 1: Rotational Symmetry in Rutherford Scattering

Gauge symmetries describe an invariance under a transformation that produces a physically indistinguishable result; they are a consequence of mathematical redundancy in our description of physics. The most commonly used example of a gauge symmetry is the one present in electromagnetism (EM). Transforming the electric and magnetic potentials as in Eq.1 leaves the electric and magnetic fields unchanged. The electric and magnetic fields are the physical, measurable quantities of EM. This nonphysical transformation comes from ambiguities in the definitions of the potentials allowed to us by the Bianchi identities of EM.

$$\begin{pmatrix} V \\ \vec{A} \end{pmatrix} = A^{\mu} \to (A^{\mu})' = \begin{pmatrix} V + \frac{\partial \chi}{\partial t} \\ \vec{A} + \vec{\nabla}\chi \end{pmatrix} = A^{\mu} + \partial^{\mu}\chi \tag{1}$$

Understanding how to distinguish physical and gauge symmetries is crucial in understanding the emergence of gauge symmetry since we want to be able to locate exactly when and how physical symmetries may become mathematical redundancies. One systematic way help make this separation clear is to analyze Noether currents [2].

2.2 Noether Currents

Noether's Theorem states that we can define a conserved quantity for a system whose Lagrangian density is invariant (up to a four gradient) with respect to some specified field transformation. Consider the action for a relativistic field theory:

$$S = \int \mathcal{L}(\varphi_i(x^{\mu}), \partial_{\mu}\varphi_i(x^{\mu}))d^4x \quad [i = 1, ..., n] \tag{2}$$

For simplicity, we will drop the i index and consider a system with one field. By varying the action with respect to arbitrary field deformations and requiring it to be stationary we get equations of motion via Hamilton's Principle. This yields an Euler-Lagrange equation describing the evolution of each field in the theory:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\right) = \frac{\delta S}{\delta \varphi} = 0 \tag{3}$$

Noether's Theorem follows a similar analysis, except instead of arbitrary field deformations we are considering a specific symmetry transformation of the field. Say

$$\varphi \to \varphi + \delta\varphi \tag{4}$$

is an infinitesimal transformation corresponding to a symmetry of the Lagrangian up to a four gradient. This means

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}, \quad \delta\mathcal{L} = \partial^{\mu}(B_{\mu}). \tag{5}$$

Using the multivariate chain rule, we can write the infinitesimal change in the Lagrangian density as

$$\delta\mathcal{L} = \partial^{\mu}(B_{\mu}) = \frac{\partial \mathcal{L}}{\partial \varphi}\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\delta\partial_{\mu}\varphi \tag{6}$$

Substituting in Eq.3 we get:

$$\partial^{\mu}(B_{\mu}) = \partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\delta\varphi\right) + \frac{\partial S}{\partial \varphi}\delta\varphi \tag{7}$$

By requiring the solutions to be on shell, meaning they satisfy equations of motion (Eq.3), the second term on the right hand side of Eq.7 goes to zero, and what's remaining is the statement of a conserved quantity defined as the Noether current [3]:

$$0 = \partial_\mu(J^\mu), \quad \text{with} \quad J^\mu \equiv \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi - \partial^\mu(B_\mu)\right) \tag{8}$$

Note that the Lagrangian density doesn't need to be totally invariant, as long as it is invariant at least up to a total four gradient then we are still able to define a conserved quantity. The zeroth component of the Noether current is defined as the charge density, while the rest of it specifies components of the current density vector. A Noether charge can then be written as [3]:

$$Q = \int J^0 d^3x \tag{9}$$

These quantities can be used to systematically distinguish gauge and physical symmetries. Noether charges are non-trivial when corresponding to a physical symmetry [2]. When corresponding to a gauge symmetry, the Noether charges are trivial, and the associated Noether current takes a special superpotential form:

$$J^\mu = W^\mu + \partial_\nu N^{\nu\mu} \tag{10}$$

In Eq.10, W^μ is zero on shell and $N^{\nu\mu}$ represents a second rank tensor with antisymmetric spacetime indices. From this, we can say that the Noether current for a gauge symmetry is trivially conserved by index structure when evaluated on shell [4].

3 Emergent Gauge Symmetry for Non-Abelian Theories

By analyzing Noether currents we now have the ability to distinguish gauge and physical symmetries, and from this we can start to identify the properties we desire from an emergent theory:

- a Lagrnagian density symmetric with respect to some combination of field transformations
- a Noether current associated with these transformations that is not of the superpotential form, meaning they represent a physical symmetry
- a natural constraint on the fields that allows the Noether current to simplify to a superpotential form, meaning with this constraint the symmetry goes from being physical to gauge
- a non-trivial theory leftover after the constraint is applied to the fields
- symmetry transformations that preserve the superpotential Noether current, meaning that the gauge symmetry stays emerged and doesn't become physical again after subsequent symmetry transformations

Barcelo et al. give an example of an emergent theory satisfying all of these properties. What they use is very similar to EM, but with an additional term:

$$\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_\xi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - A_\mu)\psi + \frac{\xi}{2}(\partial^\mu A_\mu)(\partial^\nu A_\nu), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{11}$$

The transformations they consider are also similar to those of EM:

$$\psi \to \psi' = e^{-i\chi}\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{i\chi} \tag{12}$$

$$A^\mu \to (A^\mu)' = A^\mu + \partial^\mu\chi, \quad \text{with} \quad \Box\chi = 0 \tag{13}$$

The only difference is that in this theory the parameter χ must obey a homogeneous wave equation in order to define a symmetry of the Lagrangian with the additional ξ term. This additional term also allows these symmetries to be physical. Applying the constraint $\partial^\mu(A_\mu) = 0$ allows EM with its familiar gauge redundancy to emerge. In this theory, the Lorentz gauge is a physical constraint allowing EM with its typical gauge redundancy to emerge. Similar abelian emergence properties have also been observed in some condensed matter systems [5].

For this mechanism to be relevant in constructing an emergent theory of gravity, we must ensure it is consistent for non-abelain symmetries. Our immediate goal is to try and generalize the abelian emergent theory to a non-abelian setting. There is a natural non-abelian generalization of the EM portion of this theory, the Yang-Mills theory [6]:

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \bar{\psi}\gamma^\mu(iI\partial_\mu - \lambda^a A^a_\mu)\psi, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \tag{14}$$

Defining $U \equiv e^{-i\lambda^a\chi^a}$, and $\bar{U} \equiv e^{i\lambda^a\chi^a}$, the full set of transformations for the Yang-Mills theory can be written as:

$$(\lambda^a A^a_\mu) \to (\lambda^a A^a_\mu)' = U\lambda^a A^a_\mu \bar{U} + i\partial_\mu(U)\bar{U} \tag{15}$$

$$\psi \to \psi' = U\psi \tag{16}$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\bar{U} \tag{17}$$

The extra term in the abelian emergence theory was not as obvious to generalize, but we might try something as simple as just adding generator indices to the gauge fields:

$$\mathcal{L}_\xi = \frac{\xi}{2}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) \tag{18}$$

With these non-abelian generalizations, we now want to go through and make sure it has the properties we would expect of an emergent theory.

3.1 Symmetry of the Lagrangian

We know the transformations given in Equations 15-17 are symmetries of the Yang-Mills theory as per the non-abelian gauging procedure [7]. But the additional ξ term may impact this, so we have to check it. The full theory we are considering is:

$$\begin{aligned}
\mathcal{L} &= \overbrace{-\frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu} + \bar{\psi}\gamma^\mu(i\partial_\mu - \lambda^a A^a_\mu)\psi}^{\mathcal{L}_{YM}} + \overbrace{\frac{\xi}{2}(\partial^\mu A^a_\mu)^2}^{\mathcal{L}_\xi} \\
&= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\,\nu} - \partial^\nu A^{a\,\mu}) - \frac{1}{2}f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\,\nu}A^{c\,\nu} - \frac{1}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{d\,\mu}A^{e\,\nu} \\
&\quad + \frac{\xi}{2}(\partial^\mu A^a_\mu)^2 + \bar{\psi}\gamma^\mu(i\partial_\mu - \lambda^a A^a_\mu)\psi \\
&\text{with} \quad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu
\end{aligned}$$

First, we can linearize the transformation of the full set of gauge fields in Eq.15 by restricting the transformation to first order in χ^a. From this we can factor out a common generator matrix and deduce the individual gauge field transformation:

$$\begin{aligned}
(\lambda^a A^a_\mu)' &= (1 - i\lambda^a\chi^a)\lambda^b A^b_\mu(1 + i\lambda^c\chi^c) + i(1 - i\lambda^a\chi^a)\partial_\mu(-i\lambda^b\chi^b)(1 + i\lambda^c\chi^c) \\
&= \lambda^a A^a_\mu - i\lambda^a\chi^a\lambda^b A^b_\mu + i\lambda^b A^b_\mu\lambda^c\chi^c + \lambda^b\partial_\mu\chi^b \\
&= \lambda^a A^a_\mu + \lambda^a\partial_\mu(\chi^a) + iA^b_\mu\chi^c \\
&= \lambda^a A^a_\mu + \lambda^a\partial_\mu(\chi^a) - f^{bca}\lambda^a A^b_\mu\chi^c \\
&= \lambda^a(A^a_\mu + \partial_\mu(\chi^a) - f^{bca}A^b_\mu\chi^c) \\
\implies (A^a_\mu)' &= A^a_\mu + \partial_\mu(\chi^a) - f^{bca}A^b_\mu\chi^c
\end{aligned}$$

Using this linearized transformation of the gauge fields and the spinor and adjoint spinor field transformations, we can check if they represent a symmetry of our theory.

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L}'_{YM} + \mathcal{L}'_{\xi} = \frac{\xi}{2}(\partial^{\mu}(A^a_{\mu}) + \Box\chi^a - f^{bca}\partial^{\mu}(A^b_{\mu}\chi^c))(\partial^{\nu}(A^a_{\nu}) + \Box\chi^a - f^{dea}\partial^{\mu}(A^d_{\mu}\chi^e))$$

$$\text{Keeping first order in } \chi^a:$$

$$\begin{aligned}\delta\mathcal{L}_{\xi} &= \xi\partial^{\mu}(A^a_{\mu})\Box\chi^a - \xi f^{bca}\partial^{\mu}(A^a_{\mu})\partial^{\nu}(A^b_{\nu}\chi^c) \\ &= \xi\partial^{\mu}(A^a_{\mu})(\Box\chi^a - f^{bca}\partial^{\nu}(A^b_{\nu}\chi^c))\end{aligned}$$

In order to define a symmetry of the Lagrangian (and therefore the action) we must restrict the values of χ^a to ones that satisfy this non-homogeneous wave equation:

$$\Box\chi^a = f^{bca}\partial^{\nu}(A^b_{\nu}\chi^c) \tag{19}$$

Setting structure constants to zero we obtain the same condition on χ we did in the abelian case.

3.2 Equation of Motion

We can use the Euler-Lagrange Equation to obtain the equation of motion for the gauge fields:

$$\frac{\partial\mathcal{L}}{\partial(A^a_{\mu})} = \partial_{\nu}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_{\nu}A^a_{\mu})}\Big) \tag{20}$$

Working on part of the RHS of Eq.20:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_{\alpha}A^h_{\beta})} &= \frac{\partial}{\partial(\partial_{\alpha}A^h_{\beta})}\Big(-\frac{1}{4}(\partial_{\mu}A^a_{\nu} - \partial_{\nu}A^a_{\mu})(\partial^{\mu}A^{a\,\nu} - \partial^{\nu}A^{a\,\mu}) \\ &\qquad - \frac{1}{2}f^{abc}(\partial_{\mu}A^a_{\nu} - \partial_{\nu}A^a_{\mu})A^{b\,\nu}A^{c\,\nu} + \frac{\xi}{2}(\partial^{\mu}A^a_{\mu})^2\Big) \\ &= (\partial^{\beta}A^{h\,\alpha} - \partial^{\alpha}A^{h\,\beta}) - \frac{1}{2}f^{abc}(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{ah} - \delta^{\alpha}_{\nu}\delta^{\alpha}_{\mu}\delta^{ah})A^{b\,\mu}A^{c\,\nu} \\ &\qquad + \frac{\xi}{2}\eta^{\gamma\mu}\eta^{\rho\nu}(\delta^{\alpha}_{\mu}\delta^{\beta}_{\gamma}\delta^{ah}\partial_{\nu}A^a_{\rho} + \delta^{\alpha}_{\nu}\delta^{\beta}_{\rho}\delta^{ah}\partial_{\mu}A^a_{\gamma}) \\ &= (\partial^{\beta}A^{h\,\alpha} - \partial^{\alpha}A^{h\,\beta}) - \frac{1}{2}f^{hbc}A^{b\,\alpha}A^{c\,\beta} + \frac{1}{2}f^{hbc}A^{b\,\beta}A^{c\,\alpha} \\ &\qquad + \frac{\xi}{2}\eta^{\beta\alpha}\eta^{\rho\nu}\partial_{\nu}A^h_{\rho} + \frac{\xi}{2}\eta^{\gamma\mu}\eta^{\alpha\beta}\partial_{\mu}A^h_{\gamma} \\ &= -\frac{1}{2}(\partial^{\alpha}A^{h\,\beta} - \partial^{\beta}A^{h\,\alpha}) - \frac{1}{2}f^{hbc}A^{b\,\alpha}A^{c\,\beta} \\ &\qquad + \frac{1}{2}(\partial^{\beta}A^{h\,\alpha} - \partial^{\alpha}A^{h\,\beta}) + \frac{1}{2}f^{hbc}A^{b\,\beta}A^{c\,\alpha} \\ &\qquad + \xi\eta^{\alpha\beta}\partial^{\mu}A^h_{\mu} \\ &= -\frac{1}{2}F^{h\,\alpha\beta} + \frac{1}{2}F^{h\,\beta\alpha} + \xi\eta^{\alpha\beta}\partial^{\mu}A^h_{\mu}\end{aligned}$$

With a relabeling of indices, we get the RHS side of Eq.20:

$$\partial_{\nu}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_{\nu}A^a_{\mu})}\Big) = -\frac{1}{2}\partial_{\nu}(F^{a\,\nu\mu}) + \frac{1}{2}\partial_{\nu}(F^{a\,\mu\nu}) + \xi\partial^{\mu}(\partial^{\nu}A^a_{\nu}) \tag{21}$$

Up to this point we have not restricted ourselves to any particular symmetry group/ from of the structure constants. However, if we restrict ourselves to semi-simple and compact groups, then we can write the following cyclic/anti-cyclic relationships of the structure constant indices:

$$f^{abc} = f^{bca} = f^{cab} = -f^{cba} = -f^{bac} = -f^{acb} \tag{22}$$

In restricting ourselves to this from of the structure constants, we get the following anti symmetry in space-time indices:

$$F^{a\,\mu\nu} = -F^{a\,\nu\mu}, \quad \text{with} \quad F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{23}$$

Using Eq.23 we can further simplify Eq.21:

$$\partial_\nu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^a_\mu)} \Big) = \partial_\nu F^{a\,\mu\nu} + \xi \partial^\mu (\partial^\nu A^a_\nu) \tag{24}$$

For the LHS of Eq.20:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(A^h_\gamma)} = \frac{\partial}{\partial(A^h_\gamma)} \Big(&- \frac{1}{2} f^{abc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \eta^{\mu\alpha} \eta^{\nu\beta} A^b_\alpha A^c_\beta \\
&- \frac{1}{4} f^{abc} f^{ade} \eta^{\mu\alpha} \eta^{\nu\beta} A^b_\mu A^c_\nu A^d_\alpha A^e_\beta - \bar{\psi} \gamma^\mu (i\partial_\mu - \lambda^a A^a_\mu) \psi \Big) \\
= &-\frac{1}{2} f^{abc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \eta^{\mu\alpha} \eta^{\nu\beta} \delta^\gamma_\alpha \delta^{hb} A^c_\beta - \frac{1}{2} f^{abc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \eta^{\mu\alpha} \eta^{\nu\beta} \delta^\gamma_\beta \delta^{hc} A^b_\alpha \\
&- \frac{1}{4} f^{abc} f^{ade} \eta^{\mu\alpha} \eta^{\nu\beta} \delta^\gamma_\nu \delta^{hb} A^c_\nu A^d_\alpha A^e_\beta - \frac{1}{4} f^{abc} f^{ade} \eta^{\mu\alpha} \eta^{\nu\beta} \delta^\gamma_\mu \delta^{hc} A^b_\mu A^d_\alpha A^e_\beta \\
&- \frac{1}{4} f^{abc} f^{ade} \eta^{\mu\alpha} \eta^{\nu\beta} \delta^\gamma_\alpha \delta^{hd} A^b_\mu A^c_\nu A^e_\beta - \frac{1}{4} f^{abc} f^{ade} \eta^{\mu\alpha} \eta^{\nu\beta} \delta^\gamma_\beta \delta^{he} A^b_\mu A^c_\nu A^d_\alpha - \bar{\psi} \gamma^\gamma \lambda^h \psi \\
= &-\frac{1}{2} f^{ahc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \eta^{\mu\gamma} \eta^{\nu\beta} A^c_\beta - \frac{1}{2} f^{abh} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \eta^{\mu\alpha} \eta^{\nu\gamma} A^b_\alpha \\
&- \frac{1}{4} f^{ahc} f^{ade} \eta^{\gamma\alpha} \eta^{\nu\beta} A^c_\nu A^d_\alpha A^e_\beta - \frac{1}{4} f^{abh} f^{ade} \eta^{\mu\alpha} \eta^{\gamma\beta} A^b_\mu A^d_\alpha A^e_\beta \\
&- \frac{1}{4} f^{abc} f^{ahe} \eta^{\mu\gamma} \eta^{\nu\beta} A^b_\mu A^c_\nu A^e_\beta - \frac{1}{4} f^{abc} f^{adh} \eta^{\mu\alpha} \eta^{\nu\gamma} A^b_\mu A^c_\nu A^d_\alpha - \bar{\psi} \gamma^\gamma \lambda^h \psi \\
= &-\frac{1}{2} f^{ahc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \eta^{\mu\gamma} \eta^{\nu\beta} A^c_\beta - \frac{1}{2} f^{abh} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \eta^{\mu\alpha} \eta^{\nu\gamma} A^b_\alpha \\
&- \frac{1}{2} f^{ahc} f^{ade} \eta^{\gamma\alpha} \eta^{\nu\beta} A^c_\nu A^d_\alpha A^e_\beta - \frac{1}{2} f^{abh} f^{ade} \eta^{\mu\alpha} \eta^{\gamma\beta} A^b_\mu A^d_\alpha A^e_\beta - \bar{\psi} \gamma^\gamma \lambda^h \psi \\
= &-\frac{1}{2} f^{ahc} A^c_\mu (\partial^\gamma A^{a\,\mu} - \partial^\mu A^{a\,\gamma} + f^{ade} A^{d\,\gamma} A^{e\,\mu}) - \bar{\psi} \gamma^\gamma \lambda^h \psi \\
&- \frac{1}{2} f^{abh} A^b_\mu (\partial^\mu A^{a\,\gamma} - \partial^\gamma A^{a\,\mu} + f^{ade} A^{d\,\mu} A^{e\,\gamma}) \\
= &-\frac{1}{2} f^{ahc} A^c_\mu F^{a\,\gamma\mu} - \frac{1}{2} f^{abh} A^b_\mu F^{a\,\mu\gamma} - \bar{\psi} \gamma^\gamma \lambda^h \psi
\end{aligned}$$

With a relabeling of indices, we get the LHS side of Eq.20:

$$\frac{\partial \mathcal{L}}{\partial(A^a_\mu)} = -\frac{1}{2} f^{cab} A^b_\nu F^{c\,\mu\nu} - \frac{1}{2} f^{cba} A^b_\nu F^{c\,\nu\mu} - \bar{\psi} \gamma^\mu \lambda^a \psi \tag{25}$$

This is the farthest we can simply without assuming any form of the structure constants. But if we again assume semi-simple compact groups as we did on the RHS calculation, we can use Equations 22 and 23 to combine the first two term to get:

$$\frac{\partial \mathcal{L}}{\partial(A^a_\mu)} = -f^{abc} A^b_\nu F^{c\,\mu\nu} - j^{a\,\mu}, \quad \text{with} \quad j^{a\,\mu} \equiv \bar{\psi} \gamma^\mu \lambda^a \psi \tag{26}$$

Using Eq.24 and Eq.26 we can write the equation of motion:

$$j^{a\,\mu} = -\partial_\nu F^{a\,\mu\nu} - f^{abc} A^b_\nu F^{c\,\mu\nu} - \xi \partial^\mu (\partial^\nu A^a_\nu) \tag{27}$$

3.3 Noether Current

Using the definition of the Noether current we previously derived and applying it to our theory which has with three fields, we get:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} \frac{\partial (A^a_\nu)'}{\partial \beta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \frac{\partial \psi'}{\partial \beta} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \frac{\partial \bar{\psi}'}{\partial \beta} \tag{28}$$

Here we let β be some constant that parameterizes the transformations of the fields given in Equations 15-17 up to first order in χ^a:

$$A^a_\mu \to (A^a_\mu)' = A^a_\mu + \beta \partial_\mu(\chi^a) - \beta f^{bca} A^b_\mu \chi^c \tag{29}$$

$$\psi \to \psi' = (1 - i\beta\lambda^a \chi^a)\psi \tag{30}$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}(1 + i\beta\lambda^a\chi^a) \tag{31}$$

The third term in Eq.28 vanishes since the Lagrangian density does not depend on the change in the adjoint spinor field. We can re purpose Equations 24 and 26 with a simple index relabeling for the Noether current definition. The rest of the needed quantities are as follows:

$$\begin{aligned}
\frac{\partial (A^a_\nu)'}{\partial \beta} &= \frac{\partial}{\partial \beta}(A^a_\nu + \beta\partial_\nu(\chi^a) - \beta f^{bca} A^b_\nu \chi^c) = \partial_\nu(\chi^a) - f^{bca} A^b_\nu \chi^c \\
\frac{\partial \psi'}{\partial \beta} &= \frac{\partial}{\partial \beta}((1 - i\beta\lambda^a\chi^a)\psi) = -i\lambda^a \chi^a \psi \\
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} &= \frac{\partial}{\partial(\partial_\mu \psi)}(\bar{\psi}\gamma^\mu(i\partial_\mu - \lambda^a A^a_\mu)\psi) = i\bar{\psi}\gamma^\mu
\end{aligned}$$

Now finding the Noether current:

$$\begin{aligned}
J^\mu &= (F^{a\,\nu\mu} + \xi\eta^{\nu\mu}(\partial^\gamma A^a_\gamma))(\partial_\nu(\chi^a) - f^{bca} A^b_\nu \chi^c) + (i\bar{\psi}\gamma^\mu)(-i\lambda^a\chi^a\psi) \\
&= -F^{a\,\mu\nu}\partial_\mu(\chi^a) + f^{bca} F^{a\,\mu\nu} A^b_\nu \chi^c + \xi(\partial^\mu A^a_\mu)\partial^\nu(\chi^a) - f^{bca}\xi(\partial^\nu A^a_\nu) A^{b\,\mu}\chi^c + j^{a\,\mu}\chi^a
\end{aligned}$$

Now we can evaluate the Noether Current on shell by replacing $j^{a\,\mu}$ with its value as determined by the equation of motion in Eq.27:

$$\begin{aligned}
J^\mu\big|_{\text{shell}} &= -F^{a\,\mu\nu}\partial_\nu(\chi^a) + f^{bca} F^{a\,\mu\nu} A^b_\nu \chi^c + \xi(\partial^\nu A^a_\nu)\partial^\mu(\chi^a) - f^{bca}\xi(\partial^\nu A^a_\nu) A^{b\,\mu}\chi^c \\
&\quad + (-\partial_\nu F^{a\,\mu\nu} - f^{abc} A^b_\nu F^{c\,\mu\nu} - \xi\partial^\mu(\partial^\nu A^a_\nu))\chi^a \\
&= -F^{a\,\mu\nu}\partial_\nu(\chi^a) - f^{bca} F^{a\,\mu\nu} A^b_\nu \chi^c + \xi(\partial^\nu A^a_\nu)\partial^\mu(\chi^a) - f^{bca}\xi(\partial^\nu A^a_\nu) A^{b\,\mu}\chi^c \\
&\quad - \partial_\nu F^{a\,\mu\nu}\chi^a - f^{abc} A^b_\nu F^{c\,\mu\nu}\chi^a - \xi\partial^\mu(\partial^\nu A^a_\nu)\chi^a \\
&= -F^{a\,\mu\nu}\partial_\nu(\chi^a) - f^{bca} F^{a\,\mu\nu} A^b_\nu \chi^c + \xi(\partial^\nu A^a_\nu)\partial^\mu(\chi^a) - f^{bca}\xi(\partial^\nu A^a_\nu) A^{b\,\mu}\chi^c \\
&\quad - \partial_\nu F^{a\,\mu\nu}\chi^a - f^{cba} A^b_\nu F^{a\,\mu\nu}\chi^c - \xi\partial^\mu(\partial^\nu A^a_\nu)\chi^a \\
&= -\partial_\nu(\chi^a F^{a\,\mu\nu}) + \xi(\partial^\nu A^a_\nu)\partial^\mu(\chi^a) - f^{bca}\xi(\partial^\nu A^a_\nu) A^{b\,\mu}\chi^c \\
&\quad + A^b_\nu F^{a\,\mu\nu}\chi^c(f^{bca} - f^{cba}) - \xi\partial^\mu(\partial^\nu A^a_\nu)\chi^a
\end{aligned}$$

In obtaining the equating of motion we assumed we were working with semi-simple compact groups, meaning $f^{bca} = -f^{cba}$, and one of the terms above cancels out. We are left with:

$$J^\mu\big|_{\text{shell}} = -\partial_\nu(\chi^a F^{a\,\mu\nu}) + \xi\Big(\varphi^a\partial^\mu(\chi^a) - \partial^\mu(\varphi^a)\chi^a - f^{abc}\varphi^a A^{b\,\mu}\chi^c\Big), \quad \text{with} \quad \varphi^a \equiv \partial^\gamma A^a_\gamma \tag{32}$$

In evaluating $\partial_\mu\big(J^\mu\big|_{\text{shell}}\big)$, we see the first term is trivially conserved by index structure. The four gradients are symmetric and the gauge field tensor is anti symmetric under the exchange of space time indices μ and ν. So in their contraction this term will vanish. The second term however, does not evaluate to zero on shell. This means the Noether current is not of the superpotential form meaning this theory's local symmetry is a physical one.

Now we want to find a constraint on the gauge fields that allow the Noether current to reduce to a superpotential. In Eq.32, every term other than the trivially conserved quantity contains a φ^a. So setting $\varphi^a = \partial^\mu(A^a_\mu) = 0$ will allow a gauge symmetry to emerge. This is similar to the result in the abelian case, but now we have the generator index a. One interesting consequence in the non-ableian case is that we have multiple gauge fields, one for each generator of the group. This is the Noether current assocaited with the transformation on A^a_μ, where a can take N different values for a group with N generators. From this, we see we can take any combination of gauge fields and have this projection allow an emergent gauge symmetry for those respective fields. Forcing this constraint simultaneously on all of the gauge fields will recover the full Yang-Mills theory. This can be seen by looking at the Helmholtz composition of the gauge fields [2]:

$$A^a_\mu = \bar{A}^a_\mu + \partial_\mu \zeta^a, \quad \partial^\mu \bar{A}^a_\mu = 0 \tag{33}$$

This essentially splits the vector field into its spin-1 and spin-0 contributions, where only the spin-0 contribution is dependent on φ^a. This can be seen by using the definition of φ^a and then writing ζ in terms of φ and the Green's Function for the d'Alembertian:

$$\varphi^a = \partial^\mu(A^a_\mu) = \partial^\mu(\bar{A}^a_\mu + \partial_\mu \zeta^a) = \Box \zeta^a \tag{34}$$

$$\implies \zeta = \int d^4x' G(x,x')\varphi(x') \tag{35}$$

This means that as $\varphi^a \to 0$, $A^a_\mu \to \bar{A}^a_\mu$ with $\partial^\mu(\bar{A}^a_\mu) = 0$. The projection leaves us with a non-trivial result, it leaves dynamic gauge fields in the Lorenz gauge with a newly generated mathematical redundancy. In this theory, we can consider the Lorenz gauge a physical condition that allows the Yang-Mills theory to emerge in the Lorenz gauge when this constraint is applied to all of the gauge fields.

The next important thing to consider is whether or not the symmetries preserve the emergent gauge symmetry. To do this, we consider how φ^a transforms:

$$\varphi^a \to (\varphi^a)' = \partial^\mu(\bar{A}^a_\mu + \partial_\mu(\chi^a) - f^{bca}A^b_\mu \chi^c) = \Box \chi^a - f^{bca}\partial^\mu(A^b_\mu \chi^c) = 0 \tag{36}$$

By imposing the condition we derived in Eq.19, we see that when φ^a is set to zero, it stays that way meaning the gauge symmetry stays emerged.

4 Weinberg-Witten Theorem

The Weinberg-Witten (WW) Theorem has been used to shoot down a broad range of emergent theories. The theorem states that massless particles charged under a conserved Lorentz covariant current cannot have a spin larger than 1/2, while massless particles charged under a conserved Lorentz covariant stress-energy tensor cannot have a spin greater than 1 [8]. Despite this, many well-known theories have been able to circumvent these restrictions by using formulations that do not satisfy all of the assumptions in the Weinberg-Witten Theorem. Table 1 gives examples of particles from other well known theories and how they have managed to circumvent the Weinberg-Witten Theorem.

If theories following the same analysis as used in this paper are to have a chance of describing quantum gravity they will need to be able to consistently define a massless spin-2 particle i.e. the graviton. This has not been possible with previous emergent theories because the Weinberg Witten Theorem has restricted the maximum spin that can be found on massless particles in such theories. If this theory is to hold it must be shown that it violates one or more of the assumptions made in WW so that the theorem cannot be applied.

Particle, Spin	Conserved Current?	Massless?	Lorentz Covariance?	Conclusion
Weinberg-Witten $\leq \frac{1}{2}$ for conserved currents ≤ 1 for conserved stress-energy	Must have conserved current or conserved stress-energy	Massless particles required	Lorentz covariant current or stress-energy required	All columns must be satisfied for WW to apply
Photon, 1	No conserved current	Massless	No current	WW does not apply
W± and Z bosons, 1	Conserved currents	Massive	Currents not Lorentz covariant	WW does not apply
Gluons, 1	Conserved currents	Massless	Currents not Lorentz covariant	WW does not apply
Graviton, 2	Conserved stress-energy tensor	Massless	Stress-energy tensor not Lorentz covariant	WW does not apply

Table 1: Weinberg-Witten and its Exceptions

4.1 Relationship between Lorentz Covariance and Gauge Invariance

The theory analyzed in this paper is a gauge theory, as a result it was necessary to find a relationship between gauge invariance and Lorentz covariance in order to decide whether or not WW applies to our theory. Lorentz covariance is a restriction on how we allow equations and mathematical objects to transform. In short, Lorentz invariance is equivalent to form invariance under a Lorentz transformation. As an example, the transformation rule for some Lorentz covariant quantity p^μ would look like

$$p^\mu \to \Lambda^\mu_\nu p^\nu, \tag{37}$$

whereas a non-Lorentz covariant quantity would have a transformation rule that looks something like

$$p^\mu \to \Lambda^\mu_\nu p^\nu + \partial^\mu \chi, \tag{38}$$

where χ could be some scalar field. The important observation here is that in equation 37 the right hand and left hand sides take the same functional form, whereas in equation 38 they do not.

Now, when we try to construct a massless spin-1 quantum vector field we find that its transformation rule looks like [8]:

$$A^\mu \to \Lambda^\mu_\nu A^\nu + \partial^\mu \Omega. \tag{39}$$

This does not at first appear to be Lorentz covariant; however, we can observe that this transformation rule takes the exact same form as a gauge transformation, like in equation 1. This is crucial since we are dealing with a gauge theory, so we can conclude that this apparently non-Lorentz covariant vector field is indeed Lorentz covariant because the extra derivative term that it picks up when undergoing a Lorentz transformation is associated with a gauge transformation and is non-physical as a result. This allows us to conclude that gauge invariance implies Lorentz covariance and that a quantity that can be shown to not be gauge invariant is also not Lorentz covariant.

4.2 Application of the Weinberg-Witten Theorem to the Emergent Theory

Our analysis of the relationship between gauge invariance and Lorentz covariance gives us a systematic approach to decide whether or not WW applies to our theory. First, we will reproduce equation 27, which is our equation of motion:

$$j^{a\,\mu} = -\partial_\nu F^{a\,\mu\nu} - f^{abc} A^b_\nu F^{c\,\mu\nu} - \xi \partial^\mu (\partial^\nu A^a_\nu). \tag{40}$$

This current, $j^{a\mu}$, comes from matter fields and is conserved under the covariant derivative, D_μ. We desire a massless current that is conserved under ∂_μ. To do this we will set $j^{a\mu}$ to zero and move $-\partial_\nu F^{a\,\mu\nu}$ to the left hand side. This gives us a current on the right hand side that is trivially conserved under ∂_μ as a result of the index structure on the left hand side. This is demonstrated in equation 41.

$$\partial_\nu F^{a\,\mu\nu} = -f^{abc} A^b_\nu F^{c\,\mu\nu} - \xi \partial^\mu (\partial^\nu A^a_\nu) \tag{41}$$

Where our current will now be defined as

$$J^{a\nu} = -f^{abc} A^b_\nu F^{c\,\mu\nu} - \xi \partial^\mu (\partial^\nu A^a_\nu). \tag{42}$$

The final step in this analysis is to figure out how $J^{a\mu}$ transforms by plugging in the gauge transformation for our vector fields, A^a_μ, which gives us

$$J^{a\nu'} = -f^{abc}(A^b_\mu + \partial_\mu(\chi^b) - f^{acb}A^a_\mu\chi^c)F^{c\,\mu\nu} - \xi\partial^\mu(\partial^\nu((A^a_\mu + \partial_\mu(\chi^a) - f^{bca}A^b_\mu\chi^c)). \tag{43}$$

This tells us that our current, $J^{a\mu}$, is not gauge invariant and not Lorentz covariant. This is a critically important result because it indicates that WW does not apply to this theory because this current, while conserved, is not Lorentz covariant, which was is one of the key requirements for WW to be valid.

5 Cost Analysis

This project is purely theoretical research and there are no associated laboratory or software costs, the only expenses will be for labor and overhead. The total cost is summarized in Table 2. The stated value of labor is for both fall and spring semesters (16 weeks) assuming 10 hours of work a week, and assuming the standard 50% of labor overhead for university work.

Table 2: Cost Analysis

Item	**Unit Cost**	**Quantity**	**Unit**	**Cost**
Labor - 2 Students	\$11.00	320	hour	\$7040.00
Overhead - 50% of Labor	-	-	-	\$3250.00
				Total: \$10290.00

6 Conclusion

The proposed emergence mechanism [2] is not just a degenerate feature special to abelian theories, it is consistent for non-abelian theories. This is important because to have any chance of applying this idea to gravity it must work for non-abelian symmetries. Treating gravity as an emergent phenomenon from a field theory perspective is often ruled out due to the Weinberg-Witten Theorem, but here we show how its assumptions aren't satisfied in the context of our emergent theory. One question that still needs to be addressed is the nature of Lorentz covariance with respect to this symmetry. When constructing a spin-1 vector field we end up requiring gauge invaraince in order to maintain Lorentz covariance. If this symmetry is initially physical, as shown here, this means we can no longer argue away the non-covariant components of the vector fields as being non-physical [9]. One possibility is also considering Lorentz covariance an effective feature. Some research has been done with this idea ([2]) but more consideration is needed for how it may work in the context of quantizing this non-abelian emergent theory. In general, we show that research in emergent gauge symmetries is not necessarily doomed by the WW and is worth more consideration on how the diffeomorhpism invariance of general relativity may emerge from more fundamental principles.

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