Thursday, March 1, 2018
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Recall that a p-form $F^{(p)}$ is a totally anti-symmetric $(0,p)$-tensor.
The exterior derivative is defined a $(dF)_{\mu_1 \cdots \mu_{p+1}} = (p+1)!\, \partial_{[\mu_1} F_{\mu_2 \cdots \mu_{p+1}]}$
and takes a p-form to a $(p+1)$-form.
Fun facts: $d^2 = 0$, in D-dimensions $p \in [0, D]$, p-forms are naturally associated to p-surfaces.

Let's begin w/ a somewhat unmotivated classification scheme.
For a differentiable manifold M of dimension D, let $\Omega^p(M)$ be the space of all p-forms.
What do I mean by this?
Well we first construct a set of basis one-forms $dx^{\mu_1}, dx^{\mu_2}, \cdots, dx^{\mu_D}$ which span the space. subspace. Then any p-form on this space can be written as a linear combination of independent p-dimensional collections of basis forms, i.e. $F^{(p)} = F_{\mu^1 \mu^2 \cdots \mu^p}\, dx^{\mu_1} \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_p}$ where we should remember to sum over repeated indices.
For $D=3$, $p=2$ and basis dx, dy, dz this is $F^{(2)} = F_{xy}\, dx \wedge dy + F_{yz}\, dy \wedge dz + F_{zx}\, dz \wedge dx$.
So all possible assignments of (F_{xy}, F_{yz}, F_{zx}) fill out the space of 2-forms $\Omega^2(M)$ on this 3D space. Notice that these linear combinations provide a natural vector-space structure.

Okay now consider only the p-forms on $\Omega^p(M)$ which satisfy $A^{(p)} \in \Omega^p(M)$ s.t. $dA^{(p)} = 0$.
We call the set of forms $\{A^{(p)}\} = A^p(M)$ "closed".
It should be clear that $A^p(M)$ has a vector-space structure. $d(A_1^{(p)} + A_2^{(p)}) = dA_1^{(p)} + dA_2^{(p)} = 0$, and $d(k A^{(p)}) = k\, dA^{(p)} = 0$.

Now also consider only the p-forms on $\Omega^p(M)$ which satisfy $B^{(p)} \in \Omega^p(M)$ s.t. $B^{(p)} = dC^{(p-1)}$.
We call the set of forms $\{B^{(p)}\} = B^p(M)$ "exact".
These too have a vector-space structure since $B_1^{(p)} + B_2^{(p)} = dC_1^{(p-1)} + dC_2^{(p-1)} = d(C_1^{(p-1)} + C_2^{(p-1)})$ and $k B^{(p)} = k\, dC^{(p-1)} = d\, k C^{(p-1)}$.

Is there any overlap between $A^p(M)$ and $B^p(M)$? Of course! In fact since $d^2 = 0$, any of the exact forms $B^{(p)}$ must be closed since $dB^{(p)} = d\,dC^{(p-1)} = 0$.

What is really useful though are the forms on $\{A^{(p)}\}$ which are not in set $\{B^{(p)}\}$, i.e. the set of closed but not exact forms. These end up giving us very useful topological information about a space.

So we introduce the quotient (vector-space) $H^p(M) \equiv \dfrac{A^p(M)}{B^p(M)}$ that is, two "vectors" in $A^p(M)$ are considered to be the same, if they differ by an element of $B^p(M)$, e.g. $A_1^{(p)} = A_2^{(p)} + dC^{(p-1)}$.

For an intuitive idea of vector space quotients consider $\mathbb{R}^3_{xyz}$ quotiented by $\mathbb{R}^2_{xy}$.
Then the vectors $\vec{V}_1 = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$ and $\vec{V}_2 = \vec{V}_1 + U_x \hat{i} + U_y \hat{j}$ are considered the same.
But this means that the only real freedom of choice in $\frac{\mathbb{R}^3_{xyz}}{\mathbb{R}^2_{xy}}$ is the choice of V_z, i.e. the quotient is a 1D vector space

Unfortunately, the dimensions of the vector spaces $A^p(M)$, $B^p(M)$ and $H^p(M)$ are not so obvious.
However, the dimension of $H^p(M)$, also called the pth Betti number of M is a useful topological characteristic of M.

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Some examples and calculations

$H^0(M)$: If $p=0$ then we are just talking about smooth functions $f \in C^\infty(M)$ on M.
Elements of $A^0(M)$ are closed, i.e. $dA^{(0)}=0$ or $df = \frac{\partial f}{\partial x^\mu} = 0$, so locally these functions are locally constant.
However if M is composed of disconnected pieces, e.g.:

then a locally constant need not have the same value on each disconnected piece, e.g. it may be $f(x_1^\mu)=0$ and $f(x_2^\mu)=1$.
Clearly $A^0(M)$ forms an N-dimensional vector space where N is the number of disconnected components of M (where the constant values of f on each piece serve as the vector components).
So that is $A^0(M)$. What about $B^0(M)$? Well $B^{(0)}$ would require $B^{(0)} = dC^{(-1)}$ which is impossible so $B^0(M)$ is empty.
Therefore $H^0(M) = \frac{A^0(M)}{B^0(M)} = A^0(M) = \mathbb{R}^k$ and the 0th Betti number k simply counts the number of disconnected components of M.

To evaluate $H^p(M)$ for higher p, we need to consider specific spaces M.

$H^1(\mathbb{R})$: A 1-form can be written as $F^{(1)} = F(x)dx$.
Since this is a top form, i.e. $p=D$, we must have $dF^{(1)}=0$.
Hence all 1-forms in this case are closed, i.e. $A^1(\mathbb{R}) = \Omega^1(\mathbb{R})$.
What about exact forms? When can $B^{(1)} = dC^{(0)}$ on $\mathbb{R}$?
More explicitly: $B^{(1)} = B(x)dx = dC^{(0)} = \frac{dC(x)}{dx}dx$

Given any smooth $B(x)$ can we find a $C(x)$ s.t. $B(x) = \frac{dC(x)}{dx}$?
Of course, the fundamental theorem of calculus guarantees that $C(x) = \int B(x)dx$ does the trick.

So we have found that $A^1(\mathbb{R}) = \Omega^1(\mathbb{R}) = B^1(\mathbb{R})$
Hence $H^1(\mathbb{R}) = \frac{A^1(\mathbb{R})}{B^1(\mathbb{R})} = 0$ and the 1st Betti number of $\mathbb{R}$ is 0.

In fact this can be generalized to arbitrary $\mathbb{R}^n$ leading to the Poincaré Lemma:

$$H^p(\mathbb{R}^n) = \begin{cases} \mathbb{R} & p=0 \\ 0 & p\neq 0 \end{cases}$$

$p=0$: This is just the statement that $\mathbb{R}^n$ has one connected component.
$p\neq 0$: This says that all closed p-forms on $\mathbb{R}^n$ are also exact, i.e. $A^p(\mathbb{R}^n) = B^p(\mathbb{R}^n)$.

The importance of this result is that it tells us that the difference between closed and exact forms is a global property of the manifold. Why? Because locally any manifold looks like $\mathbb{R}^n$, so locally any closed form is exact. The problem arises when extending it to the entire space. But that is what topology is all about! Topology doesn't care about local structure, only global.

$H^1(S^1)$: This time it is more natural to use a basis form $d\theta$ (instead of dx).
$F^{(1)} = F(\theta)\,d\theta$ and again we find that since $p = D$ then $dF^{(1)} = 0$
so once again $A^1(S^1) = \Omega^1(S^1)$, i.e. all 1-forms on S^1 are closed.
What about exact?
From our previous argument we want to find $B(\theta)\,d\theta = \frac{dC(\theta)}{d\theta}\,d\theta$
which we could try to get from $C(\theta) = \int B(\theta)\,d\theta$.
The new ingredient here is that all of the functions involved must be periodic in θ (that's what it means to live on S^1). So we could imagine expand $B(\theta)$ in a Fourier series $B(\theta) = \frac{1}{2}a_0 + \sum_{n=1} a_n \cos(n\theta) + \sum_{n=1} b_n \sin(n\theta)$.
What we discover is that $C(\theta)$ is periodic only if $a_0 = 0$.
This is because for any $n \neq 0$ we are just integrating sin or cos which gives back periodic functions.
So the set of exact 1-forms $B^1(S^1)$ is almost the same as $A^1(S^1)$, the difference being parameterized by a single number, i.e. a_0.
Hence $H^1(S^1) = \frac{A^1(S^1)}{B^1(S^1)} = \mathbb{R}$ and the 1st Betti number of S^1 is 1.

It would be nice, though tedious to work through more examples. Instead I will quote a few interesting and useful results, and then try to give all of this an interpretation.

Poincaré Duality Theorem: On a closed ($\sim$ not infinite) manifold of dimension D, M^D

$$b_p(M^D) = b_{D-p}(M^D) \quad \text{where } b_p(M^D) \text{ is the } p\text{th Betti \# for } M^D$$

Checking: $M^1 = S^1$ $\quad b_0 = 1 = b_1$ since $H^0(S^1) = \mathbb{R} = H^1(S^1)$

Cohomology of S^2: $H^0(S^2) = \mathbb{R}^1$ since it has one connected component
$H^1(S^2) = 0$ must be proven independently
$H^2(S^2) = \mathbb{R}^1$ by Poincaré Duality Theorem

$$S^n: \begin{cases} H^0(S^n) = \mathbb{R} \\ \quad \vdots \qquad\ 0 \\ H^n(S^n) = \mathbb{R} \end{cases}$$

Cohomology of T^2: $H^0(T^2) = \mathbb{R}^1$ since connected
$H^1(T^2) = \mathbb{R}^2$ must be proven independently
$H^2(T^2) = \mathbb{R}^1$ by Poincaré Duality Theorem

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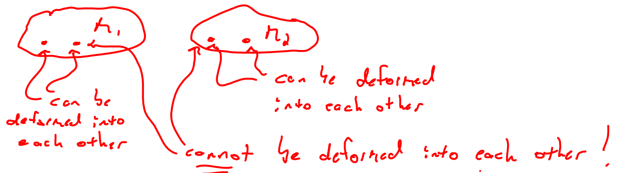
To give this all an interpretation, to fill in some of the unproven results, and to hint at why this is useful, recall:

A p-form $A^{(p)}$ is naturally associated w/ a p-dimensional surface Σ_p since $\int_{\Sigma_p} A^{(p)}$ is well defined.

So another way to think about cohomology is to think about topologically distinct p-dimensional surfaces in a space M. In fact this is the basic idea of homology, from which cohomology can be obtained by a dualization argument. It turns out we have to focus on closed surfaces (w/out boundary).

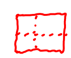
Consider $H^0(M)$, that is how many types of 0-dimensional surfaces (points) cannot be deformed into each other. This clearly counts the number of connected pieces of M.

since



S^2: $H^0(S^2) = \mathbb{R}$ since connected
$H^1(S^2) = 0$ since any 1-dimensional closed surface on S^2 can be contracted to a point
$H^2(S^2) = \mathbb{R}$ we can wrap S^2 w/ 1 non-contractible 2D surface

T^2: $H^0(T^2) = \mathbb{R}$ since connected
$H^1(T^2) = \mathbb{R}^2$ since we have 2 independent non-contractible cycles
$H^2(T^2) = \mathbb{R}^2$ we can wrap the torus w/ 1 non-contractible 2D surface.

 or 

Why is this useful? Well it allows us to determine whether 2 spaces are topologically the same. In these examples it is pretty obvious, but remember many spaces are defined abstractly, e.g. embedded surfaces defined by a constraint equation. Often we can't "draw" these and look for non-contractible cycles, but in terms of forms we are just working w/ "functions" defined on the space. And we can comfortably say that if X and Y are topologically equivalent then $H^p(X) = H^p(Y)$ for all p.

What about physics? Well one use that I can think of is that in String Theory we have extra dimensions which are typically considered to form small compact manifolds, and we also have Dp-branes. So it becomes useful to know how the Dp-branes can wrap the manifolds and be stable (non-contractible).