Review: Notation \( \langle \hat{a} \rangle \equiv \int dx \hat{a}^*(x) \hat{a}(x) \)

Observables correspond to Hermitian Operators \( \hat{a} \equiv \langle \hat{a} \rangle^* \)

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\( \langle \hat{a} \hat{b} \rangle = \langle \hat{a} \rangle \langle \hat{b} \rangle + \langle \hat{b} \rangle \langle \hat{a} \rangle \)

\( \langle \hat{a} \rangle = \langle \hat{b} \rangle = \langle 1 \rangle = 0 \)

Each Hermitian \( \hat{a} \) generates a set of eigenstates \( \{ \psi_i \} \) via \( \hat{a} \psi_i(x) = \lambda_i \psi_i(x) \) i.e., \( \hat{a}^2 \chi_i = \lambda_i^2 \chi_i \)

For: \( \hat{b} \equiv \hat{b}(x) = \frac{\delta(x-p)}{\sqrt{2\pi}} \)

\( \langle \hat{b} \rangle \equiv \langle \hat{b} \rangle^* \)

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Having orthonormal \( \{ \psi_i \} \) we can express any \( \phi(x) \) in Hilbert space as:

\[ \phi(x) = \int \phi(y) \psi_i(y) \psi_i^*(x) dy \]

\[ \langle \phi | \hat{b} | \psi_i \rangle = \int \phi(y) \psi_i^*(x) \delta(x-y) dy \]

\[ \text{Let's consider the Fourier transform of } \phi(x) \]

\[ \phi(x) = \int_\infty^{-\infty} \phi(p) \frac{e^{ipx}}{\sqrt{2\pi}} dp \]

\[ \langle \phi | \hat{b} | \psi_i \rangle = \int_{\infty}^{-\infty} \phi(p) \frac{e^{ipx}}{\sqrt{2\pi}} dp \cdot \frac{1}{\sqrt{2\pi}} e^{-ipx} \]

\[ \Rightarrow \int_{\infty}^{-\infty} \frac{1}{\sqrt{2\pi}} \phi(p) e^{-ipx} dp \]

\[ \hat{b} \phi(x) = \int_{\infty}^{-\infty} \phi(p) \frac{e^{ipx}}{\sqrt{2\pi}} dp \]

\[ \text{Thus, the action of } \hat{b} \text{ is a complete basis for the } H_k \text{ space.} \]
Having looked at the two “continuous” operators \( \hat{q}, \hat{p} \) what about other \( \hat{A} \)? We find continuous, discrete, both.

Things we can say about “discrete” \( \hat{A} \).

1. Eigenvalues are real (actually obvious since \( \langle \hat{A} \rangle = \langle \hat{A}^* \rangle \))
   \[ \langle \hat{A}_1 \hat{A}_2 \rangle = \langle \hat{A}_2 \hat{A}_1 \rangle \text{ take } \hat{A}_1 = \hat{q}_1 \hat{h} \]
   \[ \langle \hat{A}_1 \hat{q}_1 \rangle = \langle \hat{q}_1 \hat{A}_1 \rangle \]
   \[ q_1 \langle \hat{q}_1 \rangle = q_1^* \langle \hat{q}_1 \rangle \]
   \[ q_1 = q_1^* \Rightarrow \text{ } q_1 \text{ is real} \]

2. Discrete states (eigenstates) of \( \hat{A} \) are orthogonal.
   \[ \langle \hat{A}_1 \hat{A}_2 \rangle = \langle \hat{A}_2 \hat{A}_1 \rangle \text{ take } \hat{A}_1 = \hat{q}_1, \hat{A}_2 = \hat{q}_2 \]
   \[ \langle \hat{A}_1 \hat{q}_1 \rangle = \langle \hat{q}_1 \hat{A}_1 \rangle \text{ assume } \hat{q}_1 \neq \hat{q}_2 \]
   \[ q_1 \langle \hat{q}_1 \rangle = q_1^* \langle \hat{q}_1 \rangle \]
   \[ q_1^* = q_1 \neq q_2 \Rightarrow \langle \hat{q}_1 \hat{q}_2 \rangle = 0 \]

3. The discrete states form a complete set.
Our goal has been to extract from \( \Psi(\mathbf{r},t) \) information about the quantum state of the system in Hilbert space. Each Hermitian operator generates an orthogonal basis of Hilbert space:

\[
| \Psi(\mathbf{r},t) \rangle = \sum_n C_n(t) | \phi_n(\mathbf{r}) \rangle
\]

where \( C_n(t) = \int \psi_n(\mathbf{r}) \Psi(\mathbf{r},t) d\mathbf{r} \equiv \langle \psi_n | \Psi(t) \rangle \)

To program:

1. Get \( \Psi \) from TISE
2. Pick what you want to measure (e.g., position, \( \hat{p} \))
3. Form the Hamiltonian operator \( \hat{A} = \hat{A}(\mathbf{r},\hat{p}) \)
4. Expand \( \Psi \) in the appropriate basis of \( \hat{A} \).
5. Expansion coefficients \( C_n \) give the probabilities

\[
E = \sum_n \langle \psi_n | \hat{A} | \psi_n \rangle q_n
\]

where \( q_n \) comes from \( \hat{Q}_n(x) = q_n(x) \).