Review: Looking for two Hermitian operators whose eigenvalues break degeneracy of hydrogen energy levels (\( \hat{H}, \hat{\ell} \))

\[ [\hat{H}, \hat{\ell}_z] = 0 \quad \text{(can thereby \( [\hat{\ell}_1, \hat{\ell}_j] = i \hbar \epsilon_{ijk} \hat{\ell}_k \))} \]
\[ \hat{H}, \hat{\ell}_z \]
\[ \hat{H}, \hat{\ell}_x \]
\[ \text{Hermitian, commuting set: } \hat{H}, \hat{\ell}_1, \hat{\ell}_2 \]

SHO: \( [\hat{H}, \hat{\rho}] \neq 0 \) \( \Rightarrow \hat{\rho}_z = \frac{i}{\hbar \omega} (\hbar \omega \hat{\ell}_z \hat{\rho}) \)
\[ \hat{\rho}, \hat{\ell}_1 \neq 0 \]
\[ \text{lowest eigenvalues of } \hat{H} \quad \hat{\rho}_z \rightarrow \rho_n \rightarrow \rho_{n+1} \]

\[ \text{Angular momentum: } [\hat{\ell}_z, \hat{\ell}_x] \neq 0 \]
\[ \hat{\ell}_z = i \hat{\ell}_x \pm i \hat{\ell}_y \]
\[ [\hat{\ell}_z, \hat{\ell}_y] \neq 0 \]
\[ \text{lowest eigenvalues of } \hat{\ell}_z \]

For SHO we set a lower bound \( \hat{\rho}_z \rho_n = 0 \)

For angular momentum we will set an upper and lower bound.
\[ L_{nf} = 0 \quad \text{or} \quad \{ L_{nf} = (4(z+1))F_{nw} \} \]

**Case:**
\[ L_{nf} = 0 \quad \text{or} \quad \{ L_{nf} = (4(z+1))F_{nw} \} \]
\[ L_{nf} = 0 \quad \text{or} \quad \{ L_{nf} = (4(z+1))F_{nw} \} \]

**Then:**
\[ L^2_{nw} = (L^2_{nf} + 4zF_{nw})(4(z+1)F_{nw}) = 4(z+1)F_{nw} \]

Equivalently, \( L^2_{nw} \) is how the variance of \( L_{nf} \).
\[ L^2_{nw} = (L^2_{nf} + 4zF_{nw})(4(z+1))F_{nw} \]
\[ L^2_{nw} = 4(z+1)F_{nw} \Rightarrow 4(z+1)F_{nw} \]
\[ \Rightarrow z + \frac{1}{4} \]

But \( L_{nf} \) also captures the variance of \( L_{nf} \) for any \( z \).
So, for any \( z \), we have \( L^2_{nw} = 4(z+1)F_{nw} \), not exactly.
Recall: $\hat{L}_2(\hat{L}_m^f) = (m \cdot k)(\hat{L}_m^f)$ when $\hat{L}_m^f = n^f$

In particular: $\hat{L}_2(\hat{L}_{m^p}^f) = (m^p \cdot k)(\hat{L}_{m^p}^f) = k(l-1)(\hat{L}_{m^p}^f)$

Now repeat $\hat{L}_2(\hat{L}_{m^p}^f)$, etc.

$f_m^p = \frac{k(l^l)}{l^l}
\hat{L}_{m^p} = k(l-1)
\hat{L}_{m'} = k(l-2)
\vdots
\vdots
f_m^p \geq \frac{k(l-N)}{l_N} \implies k(l-N) = -k l
\hat{L}_{m^p} = -k l \quad \exists \quad l = \frac{N}{2} \quad N = 0, 1, 2, \ldots

So what do have? $f$ are eigenvalues of $\hat{L}_2$ and $\hat{L}_m^f$ with

$\hat{L}_2 f_m^l = k l (l+1) f_m^l \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots
\hat{L}_m^f = k_m f_m^l \quad m = -l, \ldots, l$

Look's a lot like $Y_m^l(0, \pi)$ except for $l = \frac{N}{2}$-integers

We defined the spherical harmonics as solutions to the angular part of TISE for $\hat{L}_m^f$. 