Review: Free particle in 3D

\[ \hat{H}^4 = E^4 \]

\[ \frac{\hat{P}^4}{\hat{M}^4} = E^4 \]

\[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) = E \Rightarrow 4 = XYZ \]

\[ -\frac{\hbar^2}{4m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) = E \]

\[ -\frac{\hbar^2}{4m} \left( -k_x^2 - k_y^2 - k_z^2 \right) = E \]

\[ \frac{\partial^2}{\partial x^2} - k_x^2 X = X = A_x e^{i k_x x} \]

\[ Y = A_y e^{i k_y y} \]

\[ Z = A_z e^{i k_z z} \]

\[ 4 = A e^{i k x} \]

No boundary conditions.

Highly degenerate! \[ k_x^2 + k_y^2 + k_z^2 = \frac{2\hbar}{m} E \]

Knowing only one gives \[ k_x, k_y, k_z \]

Consider:

\[ [\hat{H}, \hat{P}_x] = [\hat{P}_x, \hat{P}_y] = [\hat{P}_y, \hat{P}_z] = 0 \]

\[ \hat{P}_x \hat{P}_y = -i \hbar \frac{\partial}{\partial z} = -i \hbar k_x A e^{i k_x x} = i \hbar k_x \hat{P}_y \]

\[ \hat{P}_y \hat{P}_z = t \hbar k_y \]

So knowing eigenvalues of \( [\hat{H}, \hat{P}_x, \hat{P}_y] \) "maximally contains set" fully specifies the state (gives all separation constants).

Note: Other sets are possible! Demix work for 0-cube!

Hydrogen: We know \( \hat{H} \) is degenerate w.r.t. \( \hat{P}_x, \hat{P}_y \), and \( \hat{P}_z \) is deg. w.r.t. \( \hat{H}, \hat{P}_x, \hat{P}_y \), so we need to know all of \( n, l, m \) to specify state.

\[ \uparrow \uparrow \uparrow \]

\[ \hat{F} \hat{L} \hat{M} \]

we know \( \hat{F}, \hat{L}, \hat{M} \) work!

\[ \hat{F} = \hat{P}_x + V(x, y, z) \]

Q. What hermitian operators commute w/ hydrogen \( \hat{H} \)?

Need 3
six 3 sets.

Constants!

This is a rotationally symmetric problem, so you may expect \( \hat{L} \) to play a role.

So let's look at \( \hat{L} \) and determine what commutes and what doesn't.

Note: More complicated than \( \hat{P}_z \) when \( [\hat{P}_x, \hat{P}_y] = 0 \)
Angular Momentum
\[ L = \hat{r} \times \hat{p} \]
\[ L_x = y p_z - z p_y \]
\[ L_y = z p_x - x p_z \]
\[ L_z = x p_y - y p_x \]

Does \( [L_x, L_y] \neq 0 \) (like \( [\hat{r}_x, \hat{p}_y] = 0 \)) I am going to drop hats to save time.

\[ [\hat{L}_x, \hat{L}_y] = [\hat{y} \hat{p}_z - \hat{z} \hat{p}_y, \hat{z} \hat{p}_x - \hat{x} \hat{p}_z] \]
Recall: \( \hat{E}_x \hat{y} \hat{z} = \hat{E}_x \hat{z} \hat{y} = - \hat{E}_x \hat{z} \hat{y} \)
all other = 0

\[ = (y \hat{p}_z - \hat{z} \hat{p}_y)(2 \hat{p}_x - \hat{x} \hat{p}_z) - (\hat{z} \hat{p}_x - \hat{x} \hat{p}_z)(y \hat{p}_z - \hat{z} \hat{p}_y) \]
\[ = y \hat{p}_z \hat{p}_x - y \hat{p}_z \hat{p}_x - \hat{z} \hat{p}_x \hat{p}_z + \hat{x} \hat{p}_z \hat{p}_x + \hat{z} \hat{p}_x \hat{p}_z - \hat{x} \hat{p}_x \hat{p}_z + y \hat{p}_z \hat{p}_x - y \hat{p}_z \hat{p}_x \]
\[ = y \hat{p}_x (\hat{p}_z \hat{z} - \hat{z} \hat{p}_z) + x \hat{p}_y (\hat{p}_z \hat{z} - \hat{z} \hat{p}_z) \]
\[ = \frac{\hbar}{\hbar} \]

\[ = i \hbar (x \hat{p}_y - y \hat{p}_x) \]

\[ [\hat{L}_x, \hat{L}_y] = i \hbar L_z \]
\[ [\hat{L}_y, \hat{L}_z] = i \hbar L_x \]
\[ [\hat{L}_z, \hat{L}_x] = i \hbar L_y \]

This actually stems in part from classical noncommutativity of rotations:

\[ R_{90} \quad R_{90} \quad R_{90} \quad 90 \]

This means we can't have a simultaneous basis from \( \hat{L}_x, \hat{L}_y \), and \( \hat{L}_z \).

You will prove in the HW that \( [\hat{H}, \hat{L}] = 0 \) so we can make use of one component, but at most one since \( [\hat{L}_i, \hat{L}_j] \neq 0 \).

Conventional choice is \( \hat{L}_z \).

What else satisfies \( [\hat{H}, \hat{Q}] = [\hat{L}_z, \hat{Q}] = 0 \)?
Consider \( L^1 = L_x + L_y + L_z \)

\[
[L^1, L_2] = [L_x, L_2] + [L_y, L_2] + [L_z, L_2]
\]

\[
= L_x L_2 x - L_2 L_x x + L_y L_2 y - L_2 L_y y
\]

\[
= L_x L_2 x - i k L_2 y y - L_2 L_y y - i k L_y y
\]

\[
= - L_x L_2 x - i k L_2 y y + L_2 L_y y + i k L_y y
\]

\[
= 0
\]

In HW you will prove \([\hat{A}, \hat{L}_z^2] = 0\).

Okay, so we have \([\hat{A}, \hat{L}_2] = [\hat{A}, \hat{L}_z^2] = [\hat{L}_2, \hat{L}_z^2] = 0\)

but do these give rise to sep. constants?

Need eigenvalues of \( \hat{L}_1 \) and \( \hat{L}_2 \).

Recall SHO: \( \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \rightarrow [\hat{H}, \hat{p}_x] \neq 0 \) \( \Rightarrow \hat{A} \pm \frac{i}{\hbar} \frac{1}{2m \omega} (m \omega x + \hat{p}) \)

\( \hat{A}_+ \) raised eigenvalue of \( \hat{A} \), \( \hat{A}_+ \rightarrow \hat{A} \rightarrow \hat{A}_{+1} \)

\( \hat{A}_- \) lowered eigenvalue of \( \hat{A} \), \( \hat{A}_- \rightarrow \hat{A} \rightarrow \hat{A}_{-1} \)

For angular momentum: \([\hat{L}_2, \hat{L}_x] \neq 0 \) \( \Rightarrow \hat{L}_+ \equiv \hat{L}_x \pm i \hat{L}_y \)

\([\hat{L}_2, \hat{L}_y] \neq 0 \)

expect \( \hat{L}_+ \) to raise eigenvalue of \( \hat{L}_2 \)

\( \hat{L}_- \) to lower eigenvalue of \( \hat{L}_2 \)

Note: For SHO \( \hat{A}_- \) had lower bound, i.e. \( \hat{A}_+ \rightarrow 0 \). For \( \hat{L}_2 \) we need upper and lower.