A–1 The possible values comprise the interval \((0, A^2)\).

To see that the values must lie in this interval, note that

\[
\left( \sum_{j=0}^{m} x_j \right)^2 = \sum_{j=0}^{m} x_j^2 + \sum_{0 \leq j < k \leq m} 2x_jx_k,
\]

so \(\sum_{j=0}^{m} x_j \leq A^2 - 2x_0x_1\). Letting \(m \to \infty\), we have \(\sum_{j=0}^{\infty} x_j^2 \leq A^2 - 2x_0x_1 < A^2\).

To show that all values in \((0, A^2)\) can be obtained, we use geometric progressions with \(x_1/x_0 = x_2/x_1 = \cdots = d\) for variable \(d\). Then \(\sum_{j=0}^{\infty} x_j = x_0/(1-d)\) and

\[
\sum_{j=0}^{\infty} x_j^2 = \frac{x_0^2}{1-d^2} = 1 - d \left( \sum_{j=0}^{\infty} x_j \right)^2.
\]

As \(d\) increases from 0 to 1, \((1-d)/(1+d)\) decreases from 1 to 0. Thus if we take geometric progressions with \(\sum_{j=0}^{\infty} x_j = A\), \(\sum_{j=0}^{\infty} x_j^2\) ranges from 0 to \(A^2\). Thus the possible values are indeed those in the interval \((0, A^2)\), as claimed.

A–2 First solution: Let \(a\) be an even integer such that \(a^2 + 1\) is not prime. (For example, choose \(a \equiv 2 \pmod{5}\), so that \(a^2 + 1\) is divisible by 5.) Then we can write \(a^2 + 1\) as a difference of squares \(x^2 - b^2\), by factoring \(a^2 + 1\) as \(rs\) with \(r \geq s > 1\), and setting \(x = (r + s)/2\), \(b = (r - s)/2\). Finally, put \(n = x^2 - 1\), so that \(n = a^2 + b^2\), \(n + 1 = x^2\), \(n + 2 = x^2 + 1\).

Second solution: It is well-known that the equation \(x^2 - 2y^2 = 1\) has infinitely many solutions (the so-called “Pell” equation). Thus setting \(n = 2y^2\) (so that \(n = y^2 + y^2, n + 1 = x^2 + 0^2, n + 2 = x^2 + 1^2\) yields infinitely many \(n\) with the desired property.

Third solution: As in the first solution, it suffices to exhibit \(x\) such that \(x^2 - 1\) is the sum of two squares. We will take \(x = 3^2\), and show that \(x^2 - 1\) is the sum of two squares by induction on \(n\): if \(3^{2^n} - 1 = a^2 + b^2\), then

\[
(3^{2^{n+1}} - 1) = (3^{2^n} - 1)(3^{2^n} + 1) = (3^{2^n} - a + b)(a - 3^{2^n} - b)^2.
\]

A–3 The maximum area is \(3\sqrt{5}\). We deduce from the area of \(P_1P_2P_3P_4\) that the radius of the circle is \(\sqrt{5}/2\). An easy calculation using the Pythagorean Theorem then shows that the rectangle \(P_2P_4P_6P_8\) has sides \(\sqrt{2}\) and \(2\sqrt{2}\). For notational ease, denote the area of the polygon by putting brackets around the name of the polygon.

By symmetry, the area of the octagon can be expressed as

\[
[P_2P_1P_6P_8] + 2[P_2P_3P_5] + 2[P_4P_5P_6].
\]

Note that \([P_2P_3P_4]\) is \(\sqrt{2}\) times the distance from \(P_3\) to \(P_2P_4\), which is maximized when \(P_3\) lies on the midpoint of arc \(P_2P_4\); similarly, \([P_4P_5P_6]\) is \(\sqrt{2}/2\) times the distance from \(P_5\) to \(P_4P_6\), which is maximized when \(P_5\) lies on the midpoint of arc \(P_4P_6\). Thus the area of the octagon is maximized when \(P_3\) is the midpoint of arc \(P_2P_4\) and \(P_5\) is the midpoint of arc \(P_4P_6\). In this case, it is easy to calculate that \([P_2P_3P_4] = \sqrt{5} - 1\) and \([P_4P_5P_6]\) = \(\sqrt{5}/2 - 1\), and so the area of the octagon is \(3\sqrt{5}\).

A–4 We use integration by parts:

\[
\int_0^B \sin x \sin x^2 \, dx = \int_0^B \sin x \sin x^2 (2x \, dx)
\]

\[
= -\frac{\sin x}{2x} \cos x^2 \bigg|_0^B
\]

\[
+ \int_0^B \left( \frac{\cos x}{2x} - \frac{\sin x}{2x^2} \right) \cos x^2 \, dx.
\]

Now \(\frac{\sin x}{2x} \cos x^2\) tends to 0 as \(B \to \infty\), and the integral of \(\frac{\sin x}{2x^2} \cos x^2\) converges absolutely by comparison with \(1/x^2\). Thus it suffices to note that

\[
\int_0^B \frac{\cos x}{2x} \cos x^2 \, dx = \frac{\cos x}{4x^2} \cos x^2 (2x \, dx)
\]

\[
= \frac{\cos x}{4x^2} \sin x^2 \bigg|_0^B
\]

\[
- \int_0^B 2x \cos x \sin x \sin x^2 \, dx,
\]

and that the final integral converges absolutely by comparison to \(1/x^3\).
An alternate approach is to first rewrite $\sin x \sin x^2$ as
\[
\frac{1}{2}(\cos(x^2 - x) - \cos(x^2 + x)).
\]
Then
\[
\int_0^B \cos(x^2 + x) \, dx = -\left. \frac{2x + 1}{\sin(x^2 + x)} \right|_0^B
\]
and
\[
\int_0^B 2 \sin(x^2 + x) \, dx \neq \int_0^B \frac{2 \sin(x^2 + x)}{(2x + 1)^2} \, dx
\]
converges absolutely, and $f_0^B \cos(x^2 - x)$ can be treated similarly.

A–5 Let $a, b, c$ be the distances between the points. Then the area of the triangle with the three points as vertices is $abc/4r$. On the other hand, the area of a triangle whose vertices have integer coordinates is at least $1/2$ (for example, by Pick’s Theorem). Thus $abc/4r \geq 1/2$, and so
\[
\max\{a, b, c\} \geq (abc)^{1/3} \geq (2r)^{1/3} > r^{1/3}.
\]

A–6 Recall that if $f(x)$ is a polynomial with integer coefficients, then $m - n$ divides $f(m) - f(n)$ for any integers $m$ and $n$. In particular, if we put $b_n = a_{n+1} - a_n$, then $b_n$ divides $b_{n+1}$ for all $n$. On the other hand, we are given that $a_0 = a_m = 0$, which implies that $a_1 = a_{m+1}$ and so $b_0 = b_m$. If $b_0 = 0$, then $a_0 = a_1 = \cdots = a_m$ and we are done. Otherwise, $|b_0| = |b_1| = |b_2| = \cdots$, so $b_n = \pm b_0$ for all $n$.

Now $b_0 + \cdots + b_{m-1} = a_m - a_0 = 0$, so half of the integers $b_0, \ldots, b_{m-1}$ are positive and half are negative. In particular, there exists an integer $0 < k < m$ such that $b_{k-1} = -b_k$, which is to say, $a_{k-1} = a_{k+1}$. From this it follows that $a_n = a_{n+2}$ for all $n \geq k - 1$; in particular, for $m = n$, we have
\[
a_0 = a_m = a_{m+2} = f(f(a_0)) = a_2.
\]

B–1 Consider the seven triples $(a, b, c)$ with $a, b, c \in \{0, 1\}$ not all zero. Notice that if $r_j, s_j, t_j$ are not all even, then four of the sums $ar_j + bs_j + ct_j$ with $a, b, c \in \{0, 1\}$ are even and four are odd. Of course the sum with $a = b = c = 0$ is even, so at least four of the seven triples with $a, b, c$ not all zero yield an odd sum. In other words, at least $4N$ of the tuples $(a, b, c, j)$ yield odd sums. By the pigeonhole principle, there is a triple $(a, b, c)$ for which at least $4N/7$ of the sums are odd.

B–2 Since $\gcd(m, n)$ is an integer linear combination of $m$ and $n$, it follows that
\[
\frac{\gcd(m, n)}{n} \binom{n}{m}
\]
is an integer linear combination of the integers
\[
\frac{m}{n} \binom{n}{m} = \binom{n-1}{m-1}
\]
and $\frac{n}{m} \binom{n}{m} = \binom{n}{m}$

and hence is itself an integer.

B–3 Put $f_k(t) = \frac{d^k}{dt^k}$. Recall Rolle’s theorem: if $f(t)$ is differentiable, then between any two zeroes of $f(t)$ there exists a zero of $f'(t)$. This also applies when the zeroes are not all distinct: if $f$ has a zero of multiplicity $m$ at $t = x$, then $f'$ has a zero of multiplicity at least $m - 1$ there.

Therefore, if $0 \leq a_0 \leq a_1 \leq \cdots \leq a_r < 1$ are the roots of $f_k$ in $[0, 1)$, then $f_{k+1}$ has a root in each of the intervals $(a_0, a_1), (a_1, a_2), \ldots, (a_r-1, a_r)$, so long as we adopt the convention that the empty interval $(t, t)$ actually contains the point $t$ itself. There is also a root in the “wraparound” interval $(a_r, a_0)$. Thus $N_{k+1} \geq N_k$.

Next, note that if we set $z = e^{2\pi it}$, then
\[
f_{4k}(t) = \frac{1}{2t} \sum_{j=1}^N j^{4k} a_j (z^j - z^{-j})
\]
is equal to $z^{-N}$ times a polynomial of degree $2N$. Hence as a function of $z$, it has at most $2N$ roots; therefore $f_k(t)$ has at most $2N$ roots in $[0, 1]$. That is, $N_k \leq 2N$ for all $N$.

To establish that $N_k \rightarrow 2N$, we make precise the observation that
\[
f_k(t) = \sum_{j=1}^N j^{4k} a_j \sin(2\pi jt)
\]
is dominated by the term with $j = N$. At the points $t = (2i + 1)/(2N)$ for $i = 0, 1, \ldots, N - 1$, we have $N^{4k} a_N \sin(2\pi N t) = \pm N^{4k} a_N$. If $k$ is chosen large enough so that
\[
|a_N|N^{4k} > |a_1|1^{4k} + \cdots + |a_{N-1}|(N-1)^{4k};
\]
then $f_k((2i + 1)/2N)$ has the same sign as $a_N \sin(2\pi N t)$, which is to say, the sequence $f_k(1/2N), f_k(3/2N), \ldots$ alternates in sign. Thus between these points (again including the “wraparound” interval) we find $2N$ sign changes of $f_k$. Therefore $\lim_{k \rightarrow \infty} N_k = 2N$.

B–4 For $t$ real and not a multiple of $\pi$, write $g(t) = \frac{f(\cos t)}{\sin t}$. Then $g(t + \pi) = g(t)$; furthermore, the given equation implies that
\[
g(2t) = \frac{f(2 \cos^2 t - 1)}{\sin(2t)} = \frac{2(\cos t f(\cos t)}{\sin(2t)} = g(t).
\]
In particular, for any integer $n$ and $k$, we have
\[
g(1 + n\pi/2^k) = g(2^k + n\pi) = g(2^k) = g(1).
\]
Since $f$ is continuous, $g$ is continuous where it is defined; but the set $\{1 + n\pi/2^k| n, k \in \mathbb{Z}\}$ is dense in the reals, and so $g$ must be constant on its domain. Since $g(-t) = -g(t)$ for all $t$, we must have $g(t) = 0$ when $t$ is not a multiple of $\pi$. Hence $f(x) = 0$ for $x \in (-1, 1)$. Finally, setting $x = 0$ and $x = 1$ in the given equation yields $f(-1) = f(1) = 0$.  

2
We claim that all integers $N$ of the form $2^k$, with $k$ a positive integer and $N > \max\{S_0\}$, satisfy the desired conditions.

It follows from the definition of $S_n$, and induction on $n$, that
\[
\sum_{j \in S_n} x^j \equiv (1 + x) \sum_{j \in S_{n-1}} x^j 
\equiv (1 + x)^n \sum_{j \in S_0} x^j \pmod{2}.
\]

From the identity $(x + y)^2 \equiv x^2 + y^2 \pmod{2}$ and induction on $n$, we have $(x + y)^{2^n} \equiv x^{2^n} + y^{2^n} \pmod{2}$. Hence if we choose $N$ to be a power of 2 greater than $\max\{S_0\}$, then
\[
\sum_{j \in S_n} x^j \equiv (1 + x^N) \sum_{j \in S_0} x^j
\]
and $S_N = S_0 \cup \{N + a : a \in S_0\}$, as desired.

For each point $P$ in $B$, let $S_P$ be the set of points with all coordinates equal to $\pm 1$ which differ from $P$ in exactly one coordinate. Since there are more than $2^{n+1}/n$ points in $B$, and each $S_P$ has $n$ elements, the cardinalities of the sets $S_P$ add up to more than $2^{n+1}$, which is to say, more than twice the total number of points. By the pigeonhole principle, there must be a point in three of the sets, say $S_P, S_Q, S_R$. But then any two of $P, Q, R$ differ in exactly two coordinates, so $PQR$ is an equilateral triangle, as desired.