A1 Basketball star Shanille O’Keal’s team statistician keeps track of the number, \( S(N) \), of successful free throws she has made in her first \( N \) attempts of the season. Early in the season, \( S(N) \) was less than 80% of \( N \), but by the end of the season, \( S(N) \) was more than 80% of \( N \). Was there necessarily a moment in between when \( S(N) \) was exactly 80% of \( N \)?

A2 For \( i = 1, 2 \) let \( T_i \) be a triangle with side lengths \( a_i, b_i, c_i \), and area \( A_i \). Suppose that \( a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2 \), and that \( T_2 \) is an acute triangle. Does it follow that \( A_1 \leq A_2 \)?

A3 Define a sequence \( \{u_n\}_{n=0}^{\infty} \) by \( u_0 = u_1 = u_2 = 1 \), and thereafter by the condition that

\[
\begin{align*}
\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} &= n!
\end{align*}
\]

for all \( n \geq 0 \). Show that \( u_n \) is an integer for all \( n \). (By convention, \( 0! = 1 \).)

A4 Show that for any positive integer \( n \), there is an integer \( N \) such that the product \( x_1x_2\cdots x_n \) can be expressed identically in the form

\[
x_1x_2\cdots x_n = \sum_{i=1}^{N} c_i(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n
\]

where the \( c_i \) are rational numbers and each \( a_{ij} \) is one of the numbers \(-1, 0, 1\).

A5 An \( m \times n \) checkerboard is colored randomly: each square is independently assigned red or black with probability \( 1/2 \). We say that two squares, \( p \) and \( q \), are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at \( p \) and ending at \( q \), in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than \( mn/8 \).

A6 Suppose that \( f(x, y) \) is a continuous real-valued function on the unit square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). Show that

\[
\begin{align*}
\int_0^1 \left( \int_0^1 f(x, y) \, dx \right)^2 \, dy + \int_0^1 \left( \int_0^1 f(x, y) \, dy \right)^2 \, dx \\
\leq \left( \int_0^1 \int_0^1 f(x, y) \, dx \, dy \right)^2 + \int_0^1 \int_0^1 (f(x, y))^2 \, dx \, dy.
\end{align*}
\]

B1 Let \( P(x) = c_0x^n + c_{n-1}x^{n-1} + \cdots + c_0 \) be a polynomial with integer coefficients. Suppose that \( r \) is a rational number such that \( P(r) = 0 \). Show that the \( n \) numbers

\[
c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \\
\ldots, c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r
\]

are integers.

B2 Let \( m \) and \( n \) be positive integers. Show that

\[
\frac{(m + n)!}{(m + n)^{m+n}} < \frac{m!}{m^n} \frac{n!}{n^n}.
\]

B3 Determine all real numbers \( a > 0 \) for which there exists a nonnegative continuous function \( f(x) \) defined on \([0, a]\) with the property that the region

\[
R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}
\]

has perimeter \( k \) units and area \( k \) square units for some real number \( k \).

B4 Let \( n \) be a positive integer, \( n \geq 2 \), and put \( \theta = 2\pi/n \). Define points \( P_k = (k, 0) \) in the \( xy \)-plane, for \( k = 1, 2, \ldots, n \). Let \( R_k \) denote the map that rotates the plane counterclockwise by the angle \( \theta \) about the point \( P_k \). Let \( R \) denote the map obtained by applying, in order, \( R_1 \), then \( R_2 \), then \( R_n \). For an arbitrary point \( (x, y) \), find, and simplify, the coordinates of \( R(x, y) \).

B5 Evaluate

\[
\lim_{x \to 1} \prod_{n=0}^{\infty} \left( \frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.
\]

B6 Let \( A \) be a non-empty set of positive integers, and let \( N(x) \) denote the number of elements of \( A \) not exceeding \( x \). Let \( B \) denote the set of positive integers \( b \) that can be written in the form \( b = a - a' \) with \( a \in A \) and \( a' \in A \). Let \( b_1 < b_2 < \cdots \) be the members of \( B \), listed in increasing order. Show that if the sequence \( b_{i+1} - b_i \) is unbounded, then

\[
\lim_{x \to \infty} N(x)/x = 0.
\]