A-1 Find polynomials \( f(x), g(x), \) and \( h(x), \) if they exist, such that for all \( x, \)
\[
|f(x)| - |g(x)| + h(x) = \begin{cases} 
-1 & \text{if } x < -1 \\
3x + 2 & \text{if } -1 \leq x \leq 0 \\
-2x + 2 & \text{if } x > 0.
\end{cases}
\]

A-2 Let \( p(x) \) be a polynomial that is nonnegative for all real \( x. \) Prove that for some \( k, \)
there are polynomials \( f_1(x), \ldots, f_k(x) \) such that
\[
p(x) = \sum_{j=1}^k (f_j(x))^2.
\]

A-3 Consider the power series expansion
\[
\frac{1}{1 - 2x - x^2} = \sum_{n=0}^\infty a_n x^n.
\]
Prove that, for each integer \( n \geq 0, \) there is an integer \( m \) such that
\[
a_n^2 + a_{n+1}^2 = a_m.
\]

A-4 Sum the series
\[
\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{m^2 n}{3^m (n3^m + m3^n)}.
\]

A-5 Prove that there is a constant \( C \) such that, if \( p(x) \) is a polynomial of degree 1999, then
\[
|p(0)| \leq C \int_{-1}^1 |p(x)| \, dx.
\]

A-6 The sequence \( (a_n)_{n \geq 1} \) is defined by \( a_1 = 1, a_2 = 2, a_3 = 24, \) and, for \( n \geq 4, \)
\[
a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.
\]
Show that, for all \( n, \) \( a_n \) is an integer multiple of \( n.\)

B-1 Right triangle \( ABC \) has right angle at \( C \) and \( \angle BAC = \theta; \) the point \( D \) is chosen on \( AB \) so that \( |AC| = |AD| = 1; \) the point \( E \) is chosen on \( BC \) so that \( \angle CDE = \theta. \) The perpendicular to \( BC \) at \( E \) meets \( AB \) at \( F. \) Evaluate \( \lim_{\theta \to 0} |EF|. \)
B-2 Let $P(x)$ be a polynomial of degree $n$ such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have $n$ distinct roots.

B-3 Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq m \leq n, \frac{1}{2} \leq n \leq 2} x^m y^n,$$

where the sum ranges over all pairs $(m, n)$ of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x, y) \to (1, 1), (x, y) \in A} (1 - xy^2)(1 - x^2y)S(x, y).$$

B-4 Let $f$ be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all $x$. Suppose that $f'''(x) \leq f(x)$ for all $x$. Show that $f'(x) < 2f(x)$ for all $x$.

B-5 For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where $I$ is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all $j, k$.

B-6 Let $S$ be a finite set of integers, each greater than 1. Suppose that for each integer $n$ there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.