The Economics of Optimal Stopping

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Abstract: This paper presents an economic interpretation of optimal stopping in a common class of perpetual decision problems under certainty and uncertainty. (a) Under certainty an *r-percent rule* reveals that at stopping the rate of return from delaying a project has fallen to the rate of interest. The return from delay is the sum of a capital gain on the project value and a capital loss on the option premium associated with waiting. The opportunity cost of delay is the short rate of interest. Prior to stopping the total rate of capital gain exceeds the short rate of interest. At stopping these forces are balanced. (b) Under uncertainty as well the expected rate of return from delaying a project exceeds the rate of interest prior to stopping. The return from delay is again the sum of a rate of change in project value and a rate of change in the option premium associated with waiting. At stopping the expected rate of capital gain from delay has fallen to the rate of interest. Viewing stopping in this unified way reveals additional theoretical and practical insights that have been obscured and even misinterpreted in conventional treatments of stopping under certainty and uncertainty. In particular, we review and reinterpret the roles of project capital gains, uncertainty, irreversibility, and repeatability in motivating delay.

Keywords: investment timing, stopping rules, investment under uncertainty, *r*-percent rule, real options, pure postponement value, quasi-option value, value of information, opportunity cost, Wicksell.

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1. Introduction

At any time in an industry certain distinct, irreversible economic actions may be perpetually available but not undertaken. The actions may be offensive (e.g., construction, expansion, acquisition, or re-opening) or defensive (e.g., abandonment, contraction, divestiture, or temporary closure). Some would be profitable (would produce a positive discounted payoff) if implemented at the time, but are instead held "on the shelf" until some optimal stopping time. An example is Total SA's April 2009 announcement that it is weighing delaying the development of its C$9 billion Joslyn oil sands mine in northern Alberta in anticipation of lower future capital and operating costs.[1]

The mathematics of optimal stopping under uncertainty is unassailable. What is not well understood is the economics of stopping rules. *Why* is it optimal to stop irreversibly a perpetual random process at a given point? Two-period and other recursive stopping models based on finite decision horizons provide little guidance. Value-of-information explanations are ubiquitous and of some comfort, but why does waiting for more information at some point become ill advised? Sometimes intuitively plausible explanations from stopping problems under certainty – forestry or wine storage – are evoked, as are notions of opportunity costs (e.g., Murto 2007). Our search is for a rigorous exposition of the economics of optimal stopping under uncertainty.

Continuing the "two-way street" between resource economics and the rest of economics (Heal 2007), we use insights from stopping problems in resource economics to show that stopping under uncertainty has the same economic interpretation as under certainty; stopping is optimal when the expected benefit to waiting, namely, the rate of capital gain on the project value plus the rate of capital gain on an option premium associated with waiting, falls to the

[1] *Wall Street Journal*, Tuesday April 7, 2009, p. B2.

opportunity cost of waiting, the interest rate. This "*r*-percent rule" was first recognized by Martin Faustmann in the 1840s for forestry harvesting decisions under certainty (Gane 1968). Modern resource economics provides the additional insights needed to extend the rule to the case of uncertainty. In developing this unified view of the economics of stopping problems, we bring out theoretical and practical insights about the roles of project appreciation, uncertainty, irreversibility, and repeatability in motivating delay that have been obscured and even misinterpreted in the conventional treatments of optimal stopping.

2. Stopping under Certainty

The results derived in this section provide a framework under which to investigate stopping under uncertainty. They also dispel some incorrect notions about waiting that are current in the profession. For example, in the absence of uncertainty it is often suggested that the simple NPV stopping rule applies. Fisher (2000, 203) unconditionally states that "…the option to postpone the investment has value only because the decision-maker is assumed to learn about future returns by waiting." Hurn and Wright (1994, p. 363) are equally unequivocal: "…in the absence of new information, waiting to invest has no value."[2] We will show that waiting to invest can have value under certainty; in an example in Appendix 1.A, waiting generates 99% of an asset's current value.

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[2] Similar statements abound in the literature. "The rule, 'invest if the net present value of investing exceeds zero' is only valid if the variance of the present value of future benefits and costs is zero…" (McDonald and Siegel 1986, 708). "The value of waiting is driven by uncertainty" (Amram and Kulatilaka 1999, 179). "…the simple NPV rule…is rarely optimal, since delaying can yield valuable information about prices and costs" (Moyen et al. 1996, 66). "The deferral option, or option of waiting to invest, derives its value from reducing uncertainty by delaying an investment decision until more information has arrived" (Brach 2003, p. 68).

The irreversible, lumpy economic actions we consider throughout the paper involve an initial decision as well as an optimal plan that specifies outputs in future time periods and possible other choices.[3] We stress actions that can be delayed indefinitely at no cost. The intensity of the action and the ensuing optimal production plan generally depend on the time of initial action, t_0, and the future equilibrium price path of outputs and inputs, among other things. The firm need not be a price taker. In limiting our attention to the more interesting case of an interior solution we assume that these equilibrium price and interest rate paths, along with the optimal production plan, initially yield a return to waiting that is at least equal to the rate of interest. For ease of exposition we assume that there is a unique strike point.

For concreteness we mainly consider projects involving offensive decisions, or call options. The discussion that follows could equally be applied to defensive decisions, or put options. We use the term "forward value" to describe the investment opportunity's net present value (NPV) if initiated at time $t_0 > t$, as differentiated from NPV if initiated at the current date t. Let $Y(W)$ be the (forward) value received by irreversibly sinking a known discrete investment cost $C(t_0) \geq 0$ at time $t_0 > t$ in return for a certain, incremental benefit from time t_0 onward with time-t_0 present value $W(t_0)$. We assume that $W(t_0)$ is generated by optimal actions subsequent to the initial action at t_0, which, as in compound option analysis, can include subsequent timing options. To emphasize that the irreversible economic action need not entail investment we assume that $C(t_0) = 0$, so that $Y(W) = W(t_0)$.[4] We also assume that $W(t_0)$ is time-varying and differentiable, and that $W(t_0) > 0$ for at least some non-

[3] Bar-Ilan and Strange (1999) have reviewed stopping when investment is incremental, identifying that this is not so much a stopping problem as an investment intensity problem, with intensity at times being zero. To isolate timing from intensity we focus on lumpy one-shot investments.

[4] The optimal timber harvesting literature often assumes costless harvesting. If $C(t_0) > 0$, the analysis applies to $Y(t_0) = W(t_0) - C(t_0)$.

degenerate interval of time. The discount-factor approach of Dixit et al. (1999) defines $D(t,t_0) = \exp[-\int_t^{t_0} r(s)ds] > 0$ to be the (riskless) deterministic discount factor, integrated over the short rates of interest $r(s)$ that represent the required rate of return to all asset classes in this economy. The current value of the investment opportunity at time t if initiated at time t_0 is

$$\Pi(t,t_0) = D(t,t_0)W(t_0)\,. \tag{1}$$

Traditional or "Marshallian" NPV analyses presume that the project should be initiated as soon as $W(t_0) > 0$ (Dixit and Pindyck 1994, 4-5, 145-47). Marglin (1963) was among the first to point out that this timing rule is not necessarily optimal. Maximizing the present value of the investment opportunity by choosing the optimal time of action $\hat{t}_0$, and assuming an interior solution $\hat{t}_0 \in (t,\infty)$,[5] we find from (1) that

$$\begin{aligned}\Pi_{t_0}(t,\hat{t}_0) &= D_{t_0}(t,\hat{t}_0)W(\hat{t}_0) + D(t,\hat{t}_0)W'(\hat{t}_0)\\ &= -r(\hat{t}_0)D(t,\hat{t}_0)W(\hat{t}_0) + D(t,\hat{t}_0)W'(\hat{t}_0)\\ &= D(t,\hat{t}_0)\left[W'(\hat{t}_0) - r(\hat{t}_0)W(\hat{t}_0)\right]\\ &= 0\end{aligned}\,. \tag{2}$$

The solution to (2) yields $\hat{t}_0$ and the critical forward value threshold $W(\hat{t}_0)$. The time t market or option value of the (optimally managed) investment opportunity is then

$$\Pi(t,\hat{t}_0) = D(t,\hat{t}_0)W(\hat{t}_0)\,. \tag{3}$$

The economics of optimal stopping can be gleaned from analysis of the solution mechanics. By equation (3), prior to stopping the market value of the opportunity is rising at the rate of interest:

$$\Pi_t(t,\hat{t}_0) = D_t(t,\hat{t}_0)W(\hat{t}_0) = r(t)D(t,\hat{t}_0)W(\hat{t}_0) = r(t)\Pi(t,\hat{t}_0)\,. \tag{4}$$

[5] For example, an interior solution is guaranteed if $W(t_0)$ is initially positive and rising at greater than the discount rate, but rises at less than the discount rate as of some future date.

Furthermore, by definition of an interior solution $\hat{t}_0 > t$ the market value $\Pi(t,\hat{t}_0)$ is greater than the project value $W(t)$. In keeping with the literature on stopping under uncertainty, we define the difference,

$$\Pi(t,\hat{t}_0) - W(t) \triangleq O(t,\hat{t}_0) > 0\,, \tag{5}$$

as the *option premium* at time t from waiting until $\hat{t}_0$ to initiate the project. Accordingly, the value of the investment opportunity, $\Pi(t,\hat{t}_0)$, can be described as having two components: the current, underlying project value $W(t)$ (identified simply as the "project value" hereafter); and an option premium $O(t,\hat{t}_0)$. Given (3) and $D(\hat{t}_0,\hat{t}_0) = 1$, equation (5) yields that, at stopping, $O(\hat{t}_0,\hat{t}_0) = 0$. Prior to stopping

$$O_t(t,\hat{t}_0) = r(t)\Pi(t,\hat{t}_0) - W'(t)\,, \tag{6}$$

which is of indeterminate sign in general. However, for t in some neighborhood $(\tau_1,\hat{t}_0)$, it must be negative as $O(t,\hat{t}_0)$ is continuous and falls to zero at stopping.

The most obvious rationale for delayed stopping is seen in equation (5), where the value of the investment opportunity, if kept alive, is greater than the value of the underlying project NPV. Another approach is to view the delay decision intertemporally, comparing the opportunity cost of waiting with the capital gains from waiting. This will reveal important similarities and differences between stopping under certainty and uncertainty that are not evident via comparisons of option value with underlying asset value. For $W(t) > 0$, which by continuity is guaranteed at least within a neighborhood $(\tau_2,\hat{t}_0)$, $\tau_2 < \tau_1$, the opportunity cost of waiting is the interest foregone by not immediately and costlessly realizing project value $W(t)$. In what follows we delimit the time domain to $(\tau_2,\hat{t}_0)$ and select $W(t)$ as the numeraire such that we can isolate $r(t)$ as the opportunity cost of waiting.

We now calculate the capital gains from waiting. Using (6), the option premium's rate of change, as a fraction of $W(t)$, while waiting is

$$\alpha(t,\hat{t}_0) = \frac{O_t(t,\hat{t}_0)}{W(t)} = \frac{r(t)\Pi(t,\hat{t}_0) - W'(t)}{W(t)}, \tag{7}$$

which is negative in the neighborhood $(\tau_1, \hat{t}_0)$. The rate of change in project value is $\frac{W'(t)}{W(t)}$.

Conditions (4), (5) and (7) imply that the total return to waiting is

$$\left[\frac{W'(t)}{W(t)} + \alpha(t,\hat{t}_0)\right] = \frac{\Pi_t(t,\hat{t}_0)}{W(t)} = \frac{r(t)\Pi(t,\hat{t}_0)}{W(t)} > r(t): \tag{8}$$

prior to stopping the rate of capital gain associated with the investment opportunity, which equals the rate of capital gain or loss on the project value plus the rate of capital gain or loss on the option premium, exceeds the opportunity cost of waiting, the short rate of interest. For the neighborhood $(\tau_1, \hat{t}_0)$ the project value appreciates at rate

$$\frac{W'(t)}{W(t)} > \left[\frac{W'(t)}{W(t)} + \alpha(t,\hat{t}_0)\right] > r(t). \tag{9}$$

Extending Mensink and Requate's (2005) two-period analysis of stopping to the continuous case, we define $\frac{W'(t)}{W(t)} - r(t)$ as the *rate of pure postponement flow*. It is positive immediately prior to stopping, though it may be negative at other times if $W(t)$ is not monotone.

What happens at stopping, where $t = \hat{t}_0$? From (3), since $D(\hat{t}_0,\hat{t}_0) = 1$, there is a value matching condition, $W(\hat{t}_0) = \Pi(\hat{t}_0,\hat{t}_0)$. Inserting this result into (3), differentiating, and substituting the result into (4) generates a smooth pasting condition, $W'(\hat{t}_0) = r(\hat{t}_0)\Pi(\hat{t}_0,\hat{t}_0) = \Pi_t(\hat{t}_0,\hat{t}_0)$. Applying these conditions to (6) and (7) yields $O_t(\hat{t}_0,\hat{t}_0) = 0 = \alpha(\hat{t}_0,\hat{t}_0)$. Therefore,

$$\frac{W'(\hat{t}_0)}{W(\hat{t}_0)} + \alpha(\hat{t}_0, \hat{t}_0) = \frac{W'(\hat{t}_0)}{W(\hat{t}_0)} = \frac{r(\hat{t}_0)\Pi(\hat{t}_0)}{\Pi(\hat{t}_0)} = r(\hat{t}_0)\,. \tag{10}$$

The total rate of rise of the two asset components has fallen to the rate of interest, and the rate of pure postponement flow has fallen to zero.

Equations (8) through (10) constitute the economics of optimal stopping under certainty. The following proposition is a generalization of Wicksell's analysis of serving wine to all stopping problems under certainty.

Proposition 1. Wicksell's *r*-percent rule under certainty. *The total, instantaneous rate of return from delaying investment is equal to the instantaneous rate of change of project value plus the instantaneous rate of change in the option premium. In an interior solution it is greater than the short rate of interest r(t) prior to stopping. Near stopping the rate of change of project value is greater than the short rate and the rate of change in the option premium is less than zero. At stopping,(i) the rate of capital gain on the project value has fallen to the short rate, (ii) the rate of capital loss on the option premium has risen to zero, such that (iii) the total rate of return on holding the investment opportunity has fallen to the short rate.*

Wicksell's "*r*-percent rule" is consistent with the ubiquity of such rules, often remarked upon, in optimal control problems involving present value maximization of natural resources. Indeed, we ought to be surprised if we did not find such a rule associated with optimal stopping. Proposition 1 is clearly very general, holding for all types of projects, including projects with subsequent timing options (compound options), and in all market structures, competitive and non-competitive. It provides a useful alternative way of understanding the general stopping problem under certainty.

Contrary to some interpretations, waiting is not valuable solely because of deferment of a fixed cost of investment; in the derivation of our stopping rule investment cost is zero. Nor is waiting only valuable under uncertainty, or at least "driven by uncertainty." In this model there is no uncertainty. Growth in project value at a rate greater than the rate of interest, a positive rate of pure postponement flow, induces waiting. Only if the growth rate of project value is forever less than *r*-percent does the traditional NPV "now or never" corner solution hold.

Proposition 1 is a logical starting point from which to proceed to the search for a comparable *r*-percent rule associated with stopping under uncertainty.

3. Stopping under Uncertainty

The typical optimal stopping rule under uncertainty is to strike as soon as the now stochastic project value W reaches some endogenously determined trigger value or hitting boundary $\hat{W}$ (Brock et al. 1989, Dixit and Pindyck 1994):

$$\hat{t}_0 = \inf\left\{ t_0 \middle| W(t_0) = \hat{W} \right\}. \tag{11}$$

The derivation of the stopping trigger is conducted in the value, rather than the time domain. Even so, the optimal stopping literature, and particularly that originating in resource economics problems, has increasingly mentioned both the rate of growth of project value and opportunity cost as being of relevance to the stopping calculation (e.g., Malchow-Møller and Thorsen 2005, Alvarez and Koskela 2007, Murto 2007). This no doubt comes from the experience that resource economists have with harvest timing problems under certainty. None, however, have derived an *r*-percent rule similar in both concept and form to the rule under certainty. In this section we reveal that the standard mathematical treatment of optimal stopping under uncertainty supports such a rule. Our approach is to continue to focus on the time domain, even though stopping calculations and valuations are perforce conducted in the value domain since this is the source of randomness.

Let W be described by a density function of which the moments are assumed to be known. To facilitate closed-form solutions we represent changes in W as the one-dimensional, autonomous diffusion process in stochastic differentiable equation form,

$$dW = b\left(W(t_0)\right)dt_0 + \sigma\left(W(t_0)\right)dz \tag{12}$$

over any short period of time dt_0, where dz is a Wiener process. As above, $Y(W)$ is the (forward) value of an offensive investment project if initiated at time ($t_0 > t$) $t_0 = t$ for a known investment $C(t_0)$ ≥ 0. Any subsequent options, including partial or total reversibility of the stopping decision, are permissible, and these are assumed to be priced into $Y(W)$. Initially, to make the problem more transparent we assume that the investment cost is instantaneous and fixed in scale at $C \geq 0$.[6]

Of most interest are situations that yield an interior stopping point $\hat{t}_0 > t$. At project starting time t_0, $t < t_0 \leq \hat{t}_0$, the investment opportunity's market (option) value has the same discount factor form as the stopping problem under certainty (equation 3),[7]

$$\tilde{\Pi}\left(W(t_0)\middle|W(\hat{t}_0)\right) = E\left[e^{-\rho\left(W(t_0)\right)\left(\hat{t}_0 - t_0\right)}\right]\tilde{\Pi}\left(W(\hat{t}_0)\middle|W(\hat{t}_0)\right),\;\; \tilde{\Pi}_W\left(W(t_0)\middle|W(\hat{t}_0)\right) > 0, \tag{13}$$

[6] If investment is continuous over a finite interval, C represents the present value of the total investment if all investment must be spent once investment is irreversibly initiated, or it represents the present value of the minimum discrete lump of irreversible investment needed to initiate subsequent investment options.

[7] The notation now includes a tilde because of the change in domain from time to value.

where $\tilde{\Pi}\left(W(\hat{t}_0)\middle|W(\hat{t}_0)\right) = Y\left(W(\hat{t}_0)\right) - C$ and $\rho(W) > 0$ is the appropriate, possibly state-dependent risk-adjusted discount rate. To reduce notational clutter we hereafter condense $\tilde{\Pi}\left(W(t_0)\middle|W(\hat{t}_0)\right)$ to $\tilde{\Pi}(W)$, $\tilde{\Pi}\left(W(\hat{t}_0)\middle|W(\hat{t}_0)\right)$ to $\tilde{\Pi}(\hat{W})$, $Y\left(W(\hat{t}_0)\right)$ to $Y(\hat{W})$, and $\tilde{\Pi}_W\left(W(t_0)\middle|W(\hat{t}_0)\right)$ to $\tilde{\Pi}'(W)$.

Since $\rho(W)$ is difficult to compute and even conceptualize, several alternative approaches to discounting the investment payoff are employed. In the first, which is an approximation used by practitioners and in most of the dynamic programming-based literature that we cite and build upon, ρ is taken to be a constant risk-adjusted discount rate. Some of the literature we cite either explicitly or implicitly assumes risk neutrality on the part of the decision maker, in which case $\rho = r$. Others perform the analysis via contingent claims, using risk-adjusted expectations over the date of drift of the stochastic process and again set $\rho = r$.

Given that our paper seeks to explain the real-world economic intuition of results derived in previous stopping models, our preference is to stay with the assumptions of these previous models. That means that we will for the moment assume that ρ is a known constant. As noted in Insley and Wirjanto (2008), this is likely to be a correct assumption only when W has a constant rate of volatility and where the investment option has the form $\tilde{\Pi}(W) = K_1 W^{K_2}$, where K_1 and K_2 are non-zero constants (see also Sick and Gamba 2005). As it happens, this is the solution to several of the examples that we examine.

As with the case of certainty, let

$$\tilde{\Pi}(W) - Y(W) \triangleq \tilde{O}(W) > 0 \tag{14}$$

be the option premium prior to stopping. Once again one could seek to explain the motive for waiting by noting via (14) that the value of the investment opportunity is greater than the value of the project if initiated immediately. We instead examine waiting in a dynamic context. For the broad class of functions $\tilde{\Pi}(W)$ and $Y(W)$ to which Ito's lemma can be applied, and given (12), the expected change in project value associated with delay is

$$\frac{E[dY(W)]}{dt_0} = b(W)Y'(W) + \tfrac{1}{2}\sigma^2(W)Y''(W)\,. \tag{15}$$

The expected change in option premium is

$$\frac{E[d\tilde{O}(W)]}{dt_0} = b(W)\tilde{O}'(W) + \tfrac{1}{2}\sigma^2(W)\tilde{O}''(W)\,. \tag{16}$$

The Hamilton-Jacobi-Bellman equation associated with this particular stopping problem is

$$b(W)\tilde{\Pi}'(W) + \tfrac{1}{2}\sigma^2(W)\tilde{\Pi}''(W) = \rho\tilde{\Pi}(W)\,. \tag{17}$$

It implies that

$$\frac{E[d\tilde{\Pi}(W)]}{dt_0} = \rho\tilde{\Pi}(W)\,. \tag{18}$$

As in the case of certainty, the market value of the investment opportunity is expected to rise at the rate of interest prior to stopping. Equations (14) and (18) imply that

$$\frac{E[d\tilde{\Pi}(W)]}{dt_0} = \frac{E[dY(W)]}{dt_0} + \frac{E[d\tilde{O}(W)]}{dt_0} = \rho\tilde{\Pi}(W)\,. \tag{19}$$

The benefit of delayed project initiation is expected capital gains comprising changes in project value and the option premium.

The opportunity cost of delayed initiation of the project is lost proceeds on the investment of realized project value $Y(W)$. For comparability across risky investment opportunities this investment opportunity must be in the same asset class as the stopping opportunity that is being delayed, which garners rate of return ρ. The total opportunity cost of waiting is then $\rho Y(W)$.

As under the case of certainty we normalize changes in asset values by Y to isolate ρ as the opportunity cost of delay. Let the normalized expected rate of change in the option premium be

$$\tilde{\alpha}(W) = \frac{E[d\tilde{O}(W)]}{Y(W)dt_0}, \tag{20}$$

for $Y(W) \neq 0$. From (19), (14), (15), and (16),

$$\tilde{\alpha}(W) = \frac{E[d\tilde{\Pi}(W)]}{Y(W)dt_0} - \frac{E[dY(W)]}{Y(W)dt_0} = \frac{-b(W)\left(Y'(W)-\tilde{\Pi}'(W)\right)+\frac{1}{2}\sigma^2(W)\left(\tilde{\Pi}''(W)-Y''(W)\right)}{Y(W)}, \tag{21}$$

which is again of indeterminate sign. Nevertheless, when (i) $Y(W) > 0$, which by continuity must hold in some neighborhood ($\underline{W}$, $\hat{W}$), and (ii) waiting is optimal, then $\tilde{\Pi}(W) > Y(W)$ and

$$\frac{E[dY(W)]}{Y(W)dt_0} + \tilde{\alpha}(W) = \frac{E[d\tilde{\Pi}(W)]}{Y(W)dt_0} = \frac{\rho\tilde{\Pi}(W)}{Y(W)} > \rho. \tag{22}$$

As under certainty, prior to stopping the total expected rate of capital gain on the investment opportunity, consisting of sum of the expected rates of change of the project value and of the option premium, exceeds the opportunity cost of waiting, the risk-adjusted rate of interest.

At the interior free boundary the value matching and smooth pasting conditions are

$$\tilde{\Pi}(\hat{W}) = Y(\hat{W}) > 0 \tag{23}$$

and

$$\tilde{\Pi}'(\hat{W}) = Y'(\hat{W}). \tag{24}$$

From equations (21), (23), and (24), at the stopping point $\hat{W}$,[8]

$$\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0} + \tilde{\alpha}(\hat{W}) = \frac{\rho\tilde{\Pi}(\hat{W})}{Y(\hat{W})} = \rho. \tag{25}$$

The expected rate of total capital gain on delay has fallen to the risk-adjusted rate of interest.[9]

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[8] Equation (25) can also be derived from equation (17) by imposing the value matching and smooth pasting conditions, equations (23) and (24), adding and subtracting $\frac{1}{2}\sigma^2(W)Y''(W)$ on the left-hand side of the equality, and rearranging to put ρ on the right-hand side.

Equations (22) and (25) constitute an *r*-percent rule that parallels the intuition of the rule under certainty, as expressed in equations (8) and (10). Since by equations (21), (23) and (24),

$$\tilde{\alpha}(\hat{W}) = \frac{\frac{1}{2}\sigma^2(\hat{W})\left(\tilde{\Pi}''(\hat{W}) - Y''(\hat{W})\right)}{Y(\hat{W})}, \tag{26}$$

this *r*-percent rule appropriately subsumes the rule under certainty when $\sigma = 0$.

With the aim of explaining stopping problems in terms of rates of asset growth, previous analyses of stopping rules for diffusion processes (including jump processes) have been able to determine that the *expected rate of pure postponement flow*, $\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0} - \rho = -\tilde{\alpha}(\hat{W})$, is negative at the stopping point (e.g., Brock et al. 1989, Mordecki 2002, Alvarez and Koskela 2007). By (26),

$$-\tilde{\alpha}(\hat{W}) = \frac{-\frac{1}{2}\sigma^2(\hat{W})\left(\tilde{\Pi}''(\hat{W}) - Y''(\hat{W})\right)}{Y(\hat{W})} < 0 \tag{27}$$

is an algebraic expression for the rate to which the expected rate of pure postponement flow, $\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0} - \rho$, falls at stopping.

While the *r*-percent rule under uncertainty parallels that under certainty, equations (25) and (27) reveal that there are differences in the rates of growth of the project value and option premium components of the investment opportunity between the two cases. i) The expected rate of change in

[9] It is important to distinguish this result from a result pointed out by Shackleton and Sødal (2005) in their effort to supplant traditional stopping rules with an additional stopping algorithm. They show that at stopping the *required* rate of return on the underlying project is equal to the expected rate of return on holding the call option. We are interested in the *expected* rate of return on holding the underlying project, $\frac{E[dY(W)]}{Y(W)dt_0}$, and investigate its properties prior to and at stopping in an effort to explain the economics of optimal stopping.

project value is *less than* the rate of interest at stopping under uncertainty, whereas it is *equal to* the rate of interest at stopping under certainty. ii) By continuity (i.e., ruling out large jumps in the process for W in relation to dt_0), the expected rate of change of the option premium, $\tilde{\alpha}(W)$, immediately prior to stopping is positive under uncertainty, whereas it is negative under certainty.

4. The Economics at Stopping Under Uncertainty

While stopping under certainty and uncertainty both obey r-percent rules, the expected rate of pure postponement flow, $\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0} - \rho$, is negative at the stopping point for diffusion processes because $\tilde{\alpha}(\hat{W}) > 0$, whereas it is zero at stopping under certainty. In this section we evaluate some possible interpretations of this difference.

One interpretation is that $\tilde{\alpha}(\hat{W})$ is an adjustment to the expected rate of return on the project due to uncertainty (as in equation 15).[10] This cannot be the case, as from (26) $\tilde{\alpha}(\hat{W}) > 0$ even when project value is linear in the stochastic variable and $Y''(W) = 0$. Another interpretation is that $\tilde{\alpha}(\hat{W})$ as a risk adjustment to the calculus of stopping under certainty, where $\frac{1}{2}\left(\tilde{\Pi}''(\hat{W}) - Y''(\hat{W})\right)\Big/ Y(\hat{W}) > 0$ is the per unit price of risk and $\sigma^2(\hat{W}) > 0$ is the quantity of risk. Arnott and Lewis (1979), for example, propose that the forward value $\hat{W}_M$ defined by the solution to

$$\frac{E[dY(\hat{W}_M)]}{Y(\hat{W}_M)dt_0} = \rho \tag{28}$$

[10] See Malchow-Møller and Thorsen (2005, pp. 1034-1035) and our discussion of their paper in Appendix 2.

is the appropriate stopping point for irreversible land development under uncertainty when the investor has rational expectations and is risk-neutral.[11] This, too, is incorrect. If the analysis is conducted in a risk-neutral context, equation (25) still holds, not equation (28), only with $\rho = r$.

Equation (28), which implicitly takes $\tilde{\alpha}(\hat{W}_M)$ to be zero, does hold in some circumstances. It has been called an *infinitesimal look-ahead* stopping rule (Ross 1970), a *stochastic Wicksell* rule (Clarke and Reed 1988), and a *myopic-look-ahead* stopping rule (Clarke and Reed 1989, 1990a). It is a rule for open-loop decision making, comparing investment now with a commitment now to invest next period. Stopping is proposed when the two choices create the same discounted payoff. Under irreversible investment strategies, open-loop decision making is optimal when the process for the state variable is monotone (Malliaris and Brock 1982, Brock et. al. 1989, Boyarchenko 2004, Murto 2007) or more generally when, once a stopping point is reached, the state variable cannot deviate back into the continuation region after a vanishingly small period of length *dt* (Ross 1970, pp. 188-190). When the stopping problem is framed as an absorbing barrier on rate of return rather than on value, via an *r*-percent rule, one can view $\frac{E[dY(W)]}{Y(W)dt_0}$ as the relevant state variable, and (28) is the stopping rule if $\frac{E[dY(W)]}{Y(W)dt_0}$ falls monotonically to ρ. Wicksell's rule also applies to this type of problem.

> **Proposition 2. The stochastic, myopic Wicksell *r*-percent rule.** *When the expected rate of change of project value is monotone, at stopping the expected rate of pure*

[11] Prior to stopping $\frac{E[dY(W)]}{Y(W)dt_0} > \rho$.

postponement flow is zero and the expected rate of capital gain on the project value falls to the instantaneous, risk-adjusted rate of interest.

When $\frac{E[dY(W)]}{Y(W)dt_0}$ is instead a diffusion process it is *not* optimal to irreversibly stop the process when (28) holds. Rather, the appropriate *closed-loop* decision is to compare investment now with *conditional* investment next period. This decision rule is clearly superior to an irreversible commitment strategy when the stochastic process can move disadvantageously prior to receiving the payoff from stopping. Stopping condition (25) reflects this decision rule: stop once $\frac{E[dY(W)]}{Y(W)dt_0}$ falls to $\left(\rho - \tilde{\alpha}(\hat{W})\right) < \rho$.

The derivation of (25) shows $\tilde{\alpha}(\hat{W})$ to be an additional expected return to holding the investment opportunity beyond the stopping point. In a two-period irreversible stopping problem for a diffusion process, the adjustment to the closed-loop decision rule has been defined as Arrow-Fisher-Hanemann-Henry (AFHH) quasi-option value (Hanemann 1989). Conrad (1980) shows quasi-option value to be the expected value of information from delayed decision making given imperfect information updating in a discrete, stochastic environment. We maintain the AFHH tradition and refer to $\tilde{\alpha}(\hat{W})$ as the rate of *quasi-option flow*. To the extent that the adjustment term $\tilde{\alpha}(\hat{W})$ is only effective when $\sigma^2(W) > 0$, applying this same interpretation to quasi-option flow is appropriate; $\tilde{\alpha}(\hat{W})$ is the instantaneous rate of information flow at stopping.

Fisher and Hanemann (1987) and Kennedy (1987) complain that because quasi-option value cannot be estimated separately as an input to the analysis, it is conceptual and not useful as a tool for inducing optimal stopping behavior in resource economics problems. This is also true for the calculation of the rate of quasi-option flow, $\tilde{\alpha}(\hat{W})$, since it evaluated at $\hat{W}$. Nevertheless, the concepts of expected pure postponement flow and quasi-option flow are useful in understanding the economics of any perpetual stopping problem under uncertainty, in resource economics or otherwise.

The discussion thus far supports the following proposition.

> **Proposition 3: The stochastic, non-myopic Wicksell *r*-percent rule.** *For all stochastic processes defined in (12), all market structures, and all project values Y(W) and investment opportunity values* $\tilde{\Pi}(W)$ *to which Ito's lemma applies, if the optimal stopping point is an interior solution the expected rate of return from waiting to invest is equal to the risk-adjusted rate of interest at that stopping point. The expected rate of return from waiting to invest is the sum of the expected rate of capital gain or loss on the project value and the expected rate of capital gain on the option premium. The later is a rate of quasi-option flow associated with irreversibility of the investment. For positive project values, prior to the stopping point the total rate of return from waiting to invest, the sum of positive or negative capital gains on project value and positive or negative capital gains on the option premium, is expected to exceed the risk-adjusted rate of interest.*

In a unified theory of optimal stopping, the myopic Wicksell *r*-percent stopping rule of Proposition 2 and Wicksell's *r*-percent rule under certainty, Proposition 1, are special cases of Proposition 3. Appendix 1 illustrates the applicability of Proposition 3 or its special cases to four canonical stopping problems.

The first example, costless stopping under certainty, is based on Wicksell's original insights for the simple point-input, point-output problem of serving wine. This example illustrates that Wicksell's *r*-percent rule under certainty is applicable to stopping involving any concept of value – consumption providing utility as well as action providing monetary gain. Figure 1 plots for a range of serving times the rate of change of the forward value of the wine and the rate of change of the option premium associated with waiting. In accordance with the *r*-percent rule the wine is served when the total rate of appreciation of the consumption opportunity falls to the rate of interest from above. Note that the rate of change of the option premium, $\alpha(t_0, \hat{t}_0)$, is initially positive but then becomes negative near the stopping point and rises to zero at the stopping point.

The second case illustrates the stochastic, non-myopic Wicksell *r*-percent rule when stopping a Brownian motion process. It is conducted in a contingent claims framework since this is a case where a constant discount rate ρ cannot be used. The third case illustrates the stochastic, non-myopic Wicksell *r*-percent rule when stopping a geometric Brownian motion, depicted in Figure 2. Here the program optimally continues for $W < \hat{W} = 2$, even in cases where the expected rate of pure postponement flow, $\frac{E[dY(W)]}{Y(W)dt_0} - \rho$, is negative. Waiting occurs because of the positive expected return on the option premium. Investment is made only once the rate of return on the sum of the project value and option premium components falls to the rate of interest. There is a strong similarity between the stopping rule depicted in Figure 2, in the value domain under uncertainty, and the stopping rule depicted in Figure 1, in the time domain under certainty. In each case the program is stopped once the rate of change of the value of the investment opportunity falls to the opportunity cost of waiting, the rate of interest.

The final example in the Appendix illustrates stopping a geometric Brownian motion with unsystematic jumps.

Our analysis and examples thus far have been for a specific single-factor stochastic process. More general single-factor stochastic processes allow time to enter as a variable, via a finite option termination date *T*, a time-dependent stopping cost $C(t_0)$ or discount rate $\rho(t_0)$, or a component of the drift or variance terms in the stochastic process for the underlying variable,

$$dW = b(W(t_0), t_0)dt_0 + \sigma(W(t_0), t_0)dz \,. \tag{29}$$

There may also be an exogenous flow $\Theta(W, t_0)$ associated with delay, such as costs paid ($\Theta(W, t_0)$ <0) or receipts accrued ($\Theta(W, t_0)$ >0) while keeping the option alive. Using the same derivation as previously, the stopping condition for an interior stopping point is now

$$\frac{E[dY(\hat{W}, t_0)]}{Y(\hat{W}, t_0)dt_0} + \frac{\left(\tilde{\Pi}_{t_0}(\hat{W}, t_0; T) - Y_{t_0}(\hat{W}, t_0)\right)}{Y(\hat{W}, t_0)} + \frac{\Theta(\hat{W}, t_0)}{Y(\hat{W}, t_0)} + \frac{\frac{1}{2}\sigma^2(\hat{W}, t_0)\left(\tilde{\Pi}_{WW}(\hat{W}, t_0; T) - Y_{WW}(\hat{W}, t_0)\right)}{Y(\hat{W}, t_0)} = \rho(t_0). \tag{30}$$

The partial derivatives $\tilde{\Pi}_{t_0}$ and Y_{t_0} are typically taken to be depreciation ($\tilde{\Pi}_{t_0}$, Y_{t_0} < 0) or appreciation ($\tilde{\Pi}_{t_0}$, Y_{t_0} > 0) due to worsening or improving project economics with the pure passage of time (Dixit and Pindyck 1994, 205-207). In the case of a finite-lived option, $\tilde{\Pi}_{t_0}$ also includes the decay in the value of the option as the time to expiry draws nearer. The first two terms in (30) are thus the total capital gains or losses from waiting to invest, coming from both changes in *W* and time. The third term is the net dividends received or costs paid by the option holder while waiting to invest. The fourth term is the information flow from waiting to invest. Equation (30) shows that at an interior stopping point, even for finite-lived options, the rate of return from delaying action, inclusive of expected capital gains or losses and any dividend flows, equals the rate of interest.

Corollary 1. *If the stopping problem is not autonomous in time, the stochastic Wicksell rule applies but with the sum of the rate of expected capital gains, dividend yield from net payments, rate of project appreciation with time and rate of quasi-option flow at stopping being equal to the risk-adjusted rate of interest.*

5. Discussion: Theoretical Issues

Under certainty Wicksell's *r*-percent stopping rule shows the decision maker waiting until the rate of rise of project value falls to the rate of interest before initiating the investment opportunity. An identical rule explains the economics of stopping monotone processes under uncertainty. In non-monotone processes a more general, non-myopic Wicksell *r*-percent rule shows the decision maker waiting until the total rate of return from delay, namely the expected rate of change of the project value plus the expected rate of change of the option premium, falls to the rate of interest. The rule applies to economic actions that can include any number of market structures and subsequent options to act.

There are theoretical and practical benefits to seeing stopping under uncertainty as a condition involving opportunity costs and benefits akin to the problem under certainty. We use comparisons of the simple NPV stopping rule, the stochastic, myopic Wicksell *r*-percent stopping rule, and the stochastic, non-myopic Wicksell *r*-percent rule to reexamine several common intuitive notions about stopping under uncertainty presented in the academic and practitioner literature.

A. The Distinction Between the Myopic and Non-myopic Stopping Rules

We first return to the often cited notions that uncertainty delays investment in stochastic timing problems, and clarify when this is indeed the case. In Section 2 we noted that uncertainty was not necessary for delayed stopping. Nor was it necessary for delayed stopping in Section 3. In some problems the myopic rule is the appropriate stopping rule, where stopping is warranted as soon as the expected rate of pure postponement flow falls to zero. Consequently, the rule is a simple generalization of the rule under certainty. Uncertainty plays no role in the delay of investment in these types of problems other than through its impacts on the calculation of the expected rate of change in project value and on the level of ρ.

For example, consider Clarke and Reed's (1989) well-known single-stage problem of costlessly and irreversibly harvesting a perpetually growing forest whose logarithm of value behaves according to

$$dW = \left(b + g(t_0)\right) dt_0 + \sigma dz \,. \tag{31}$$

In this problem b is the drift in the logarithm of the price of wood and g is a deterministic time-dependent drift in the logarithm of forest size. Growth in the forest size is decreasing in t_0 and satisfies $g(\infty) < r - b - \frac{1}{2}\sigma^2 < g(0)$, where r is the constant risk-free discount rate on the harvest payoff.[12] Since $Y(W) = e^W$, $\dfrac{E[dY(W)]}{Y(W)dt_0} = b + g(t_0) + \frac{1}{2}\sigma^2$ by Ito's lemma. It is non-stochastic and monotone.

The NPV rule would have the trees harvested immediately given $Y(W) > 0 \, \forall W$. But the expected rate of growth of project value is initially $b + g(0) + \frac{1}{2}\sigma^2 > r$, and waiting is optimal because of a positive expected rate of pure postponement flow. With the state variable $\dfrac{E[dY(W)]}{Y(W)dt_0}$ monotonically

[12] Clarke and Reed conduct the analysis under risk-neutrality and under risk aversion with isoelastic utility of value. Both allow for a constant discount rate, as does the form of the option value per Insley and Wirjanto (2008). We relate the risk-neutral analysis.

declining, the myopic Wicksell *r*-percent stopping rule applies, with $\hat{t}_0$ the solution to

$b + g(\hat{t}_0) + \frac{1}{2}\sigma^2 = r$.

The current value of the investment opportunity is

$$\tilde{\Pi}(W,t,\hat{t}_0) = e^{-r(\hat{t}_0 - t)} e^W \exp\left\{ \int_t^{\hat{t}_0} \left[b + g(s) + \tfrac{1}{2}\sigma^2 \right] ds \right\}. \tag{32}$$

Given the functional forms of *Y*(*W*) and $\tilde{\Pi}(W,t,\hat{t}_0)$, at the optimal stopping time $\hat{t}_0$, $\tilde{\Pi}_{WW}(W,\hat{t}_0,\hat{t}_0) = Y''(W)$, $\tilde{\alpha}(W,\hat{t}_0) = 0$, and stopping rule (25) collapses to (28).[13] Uncertainty is not crucial to delay, and has no impact on the structure of the problem other than through its influence on the level of $\frac{E[dY(W)]}{Y(W)dt_0}$. While this example is taken from a natural resource problem, the result is applicable to any costless stopping problem driven by the stochastic process in (31).

Now we turn to a problem that is non-myopic to bring out when uncertainty drives delay. Let the stochastic variable be *W* and the project value be *Y*(*W*), $Y'(W) > 0$. Assume that $\frac{E[dY(W)]}{Y(W)dt_0}$ is a diffusion process with optimal stopping point $\hat{W}$. Let $\hat{W}_M < \hat{W}$ correspond with myopic stopping condition (28). We select two intervals of *W* for analysis. The first is $[\hat{W}_M, \hat{W})$. In this interval, statements about uncertainty being the cause of delayed investment are correct, since the expected

[13] In this case, with time being one of the arguments in the value function, the derivation of the rate of drift in the option value leads to $\tilde{\alpha}(W,t_0) = \frac{-\left(b+g(t_0)\right)\left(Y'(W) - \tilde{\Pi}_W(W,t_0,\hat{t}_0)\right) - \tilde{\Pi}_{t_0}(W,t_0,\hat{t}_0) + \frac{1}{2}\sigma^2(W)\left(\tilde{\Pi}_{WW}(W,t_0,\hat{t}_0) - Y''(W)\right)}{Y(W)}$. Time also introduces the additional optimality condition $\tilde{\Pi}_{t_0}(W,\hat{t}_0,\hat{t}_0) = 0$. This gives

$\tilde{\alpha}(W,\hat{t}_0) = \frac{\frac{1}{2}\sigma^2(W)\left(\tilde{\Pi}_{WW}(W,\hat{t}_0,\hat{t}_0) - Y''(W)\right)}{Y(W)} = 0.$

rate of pure postponement flow is negative. The second interval is ($\underline{W}$, $\hat{W}_M$) where the expected rate of pure postponement flow is positive. Here, one needs observe nothing more than $\frac{E[dY(W)]}{Y(W)dt_0} > \rho$ to understand why waiting occurs. Figure 2 provides the results from the version of this model discussed in Appendix 1.C, where uncertainty drives delay in the interval [$\hat{W}_M$, $\hat{W}$) = [1.56, 2.00), and positive expected pure postponement flow drives delay in the interval ($\underline{W}$, $\hat{W}_M$) = (1, 1.56).

B. The Impact of Increasing Uncertainty on the Stopping Trigger

Another intuition is that uncertainty creates a stricter investment hurdle $\hat{W}$.[14] Stopping conditions (25) and (28) show that this is not so. While increasing uncertainty is usually held to increase $\frac{E[dY(\hat{W})]}{dt_0}$ and $\tilde{\alpha}(\hat{W})$ via the second order terms multiplying σ^2, it can also increase ρ. Even where a contingent claims analysis is warranted, though the rate of interest will not be affected there will be an adjustment to the risk-adjusted expectation, denoted by $\frac{E^*[dY(\hat{W})]}{dt_0}$. There may also be impacts on the level of $Y(W)$ where project value includes the present value of subsequent options. The net result is an indeterminate effect on $\hat{W}$ in both the myopic and non-myopic stopping rules.

For the call option example depicted in Figure 2, in the limit as uncertainty goes to zero the discount rate on the asset falls to the riskless rate, 6%, and $\hat{W}$ *rises* from 2 to 6 when all else is held constant. For $2 \leq W < 6$ the program is continued under certainty, whereas under uncertainty it is stopped immediately since $\hat{W} = 2$ under uncertainty. Sarkar (2003), Lund (2005), and Wong (2007) show for a specific case that uncertainty may increase or decrease the investment hurdle. Stopping

[14] Alvarez and Koskela (2007) and Wong (2007) are the latest in a recent surge in papers examining the uncertainty-stopping relationship.

conditions (25) and (28) show that the indeterminacy is general, possibly explaining why empirical tests of irreversible investment behavior fail to find a strong statistical relationship between the level of uncertainty and the stopping point (e.g., Hurn and Wright 1994, Holland, Ott, and Riddiough 2000, Moel and Tufano 2002).

C. Reversibility and the Optimality of the NPV Stopping Rule

Stopping conditions (25) and (28) also provide insights into the role of irreversibility in delaying stopping. Abel et al. (1996) have noted that the introduction of reversibility endows the investment problem with a put option that diminishes the incentive to wait. It is frequently stated that when investment is completely reversible the standard NPV investment timing rule is optimal (e.g., Dixit and Pindyck 1994, 6; Holland et al. 2000, 34; Adner and Levinthal 2004, 76; Demont, Wesseler, and Tollens 2005, 116). We show in Appendix 2 that this latter inference is unfounded. Define the degree of reversibility, ϕ, by the ability to undo any investment decision after period dt with the receipt of $k = \phi C(\hat{t}_0)$ upon reversal, where $-\infty < \phi \leq 1$ is the discount factor upon selling capital. For $C(\hat{t}_0) > 0$ complete irreversibility is induced via $\phi = -\infty$, and complete reversibility via $\phi = 1$.[15] Partial reversibility obtains for $-\infty < \phi < 1$. Appendix 2 shows that when a canonical stochastic investment problem is not completely reversible, there is a positive rate of quasi-option flow at the stopping point, $\tilde{\alpha}(\hat{W})$ (see Figure 3). As investment becomes more reversible $\tilde{\alpha}(\hat{W})$ falls. Under complete reversibility $\tilde{\Pi}''(\hat{W}) - Y''(\hat{W}) = 0$, $\tilde{\alpha}(\hat{W}) = 0$, and the myopic Wicksell *r*-percent stopping rule, not the standard NPV stopping rule, obtains.

[15] We note that there is no recovery of the opportunity cost of the invested capital, and as such the investments are not truly reversible.

This finding leads to two Corollaries to Propositions 2 and 3. They hold for certainty and uncertainty, monotone and diffusion processes, and reversible investment and irreversible investment:

> **Corollary 2:** *An investment timing problem will not be optimally stopped if the expected rate of capital gain on the project value while waiting exceeds the opportunity cost of waiting, ρ.*
>
> **Corollary 3:** *The traditional NPV investment timing trigger prematurely stops any uneconomic project whose expected rate of drift is always in a favorable direction.*

Corollary 2 implies that even under reversibility, the NPV stopping rule is optimal only when $\frac{E[dY(W)]}{Y(W)dt_0} < \rho \quad \forall \quad Y(W) > 0$. Dixit and Pindyck (1994, 27-28), for example, use this "pessimistic" case to argue that reversible investment should generally obey the NPV stopping rule. Corollary 3 comes from the fact that for any process where $\frac{E[dY(W)]}{dt_0} > 0$, $\lim_{Y \to 0^+} \frac{E[dY(W)]}{Y(W)dt_0} = \infty$.

D. Reversibility versus Repeated Options to Invest

Appendix 2 also shows that an infinitely repeatable option to invest can reduce the investment trigger compared with a single option to invest. Malchow-Møller and Thorsen (2005, 1036), on noting this, "intuitively" suggest that repeated options to invest have the same effect on an investment trigger as making the single-stage decision less irreversible. That is, repeated options and reversibility are suggested to be substitutes. Appendix 2 uses the non-myopic *r*-percent rule to show that this intuition is incorrect; the rate of quasi-option flow at stopping is the same under a single option to invest as it is under infinitely repeatable options to invest. Infinitely repeatable options lower the investment trigger due to reduced benefits to waiting, but this is because of increased forward project value, not decreased irreversibility.

E. Irreversibility and Equilibrium

Considerations of reversibility in optimal stopping have implications for intuitive perceptions of equilibrium. For example, a ranking of heterogeneous discount rates, ρ_i, where *i* indexes the project, has been proposed as being sufficient to order the timing of mineral production (Malliaris and Stefani 1994). The proposal is in line with the economic intuition under certainty that higher discount rates are a result of higher opportunity costs of waiting (Stiglitz 1976). But the *r*-percent rule (25) shows that heterogeneity in the expected value of information flow from delay, $\tilde{\alpha}_i(W)$, also comes into play in timing entry. Chavas (1994) has qualitatively noted that heterogeneity in irreversibility across firms affects information flows from waiting, and through these their entry and exit decisions. Equation (25) expresses the impact precisely, via the term $\tilde{\alpha}(W)$, which we show in Appendix 2 varies in a systematic way with the degree of irreversibility. Yet the expected value of information flow will also vary with the non-linearity of the underlying project and the nature of the options available to the project manager, as described by Y_i'' and $\tilde{\Pi}_i''$ respectively in (26). The determination of investment timing is thus the outcome of a complex sectorial equilibrium involving price paths, interest rates, and the expected value of information flow, with expected rates of capital gain from waiting being compared against (ρ_i - $\tilde{\alpha}_i(W)$) rather than ρ_i.[16] Stopping condition (25) is endogenous

[16] The true net opportunity cost of an irreversible decision, ρ_i - $\tilde{\alpha}_i(W)$, may induce substantially different entry timing decisions from those predicted by models in which investment is taken to be completely reversible and $\tilde{\alpha}_i(W)$ = 0 (e.g., Gaudet and Khadr 1991).

to that equilibrium for certainty and uncertainty, perfect competition and imperfect competition, and indeed all cases for all assets where the assumptions in Section 3 are satisfied.

Finally, since $\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0} < \rho$ in closed-loop settings with interior stopping points, stopping condition (25) supports the notion that irreversible projects must at some point exhibit a rate-of-return shortfall for new investment to be forthcoming under uncertainty (Davis and Cairns 1999). Litzenberger and Rabinowitz (1995) find that backwardation in oil markets is an equilibrium condition that induces irreversible production from existing reservoirs. Our analysis complements theirs, showing that price backwardation also serves to induce irreversible investments in new fields by reducing the expected rate of pure postponement flow. During high prices there is greater backwardation (Litzenberger and Rabinowitz 1995), which from (25) increases the incentive to irreversibly invest.[17] During low prices contango decreases the incentive to invest due to an increased expected rate of pure postponement flow.

F. What We Need to Know When Deciding When to Act

Luehrman's (1998) practical guide to investment timing under uncertainty suggests that investors should plot their investment opportunities in NPV space (or Y space in our notation) and volatility space (or σ^2 space). Provided that project NPV is positive, if project volatility is low investors should invest immediately, whereas if volatility is high they should wait. Adner and Levinthal (2004) suggest that the NPV rule is satisfactory as a stopping decision tool when there are low irreversibility and low volatility. Corollary 2 makes clear that current NPV and volatility are not sufficient statistics for timing investment, even when $\sigma^2 = 0$ and the investment is completely reversible.

[17] There is also more volatility during high oil prices (Litzenberger and Rabinowitz 1995), which, as we noted above, may either support or diminish this incentive for increased investment.

6. Discussion: Issues in Practice

The r-percent stopping rule and the intuition that it reveals may also influence practice. Practitioners have been slow to adopt explicit optimal stopping algorithms when timing their investment decisions (Triantis 2005). Timing rules are presented in the economics and finance literatures as an upper or lower boundary on project value or state price (e.g., Dixit and Pindyck 1994) or present value index (Moore 2000). Copeland and Antikarov (2005, 33) note, however, that relying on such presentations is unsatisfactory:

> The academic literature about real options contains what, from a practitioner's point of view, is some of the most outrageously obscure mathematics anywhere in finance. Who knows whether the conclusions are right or wrong? How does one explain them to the top management of a company?

One does not use what one does not understand. Practitioners themselves suggest that optimal stopping rules will only be adopted if they can be seen as a complement to, rather than a replacement of, traditional NPV analysis (e.g., Woolley and Cannizzo 2005). To this end, Berk (1999) derives an NPV-based optimal stopping rule for the special case of stochastic interest rates and cash flows that are riskless or where there is no resolution of uncertain cash flows by waiting. Boyarchenko (2004) derives an adjusted NPV stopping rule that applies when waiting updates information about future cash flows that are geometric Lévy processes.

The representation of stopping under uncertainty as an r-percent rule supports these advances by providing a link to the intuition many practitioners already have from related timing rules under certainty. For example, the intuitive attractiveness of pure postponement flow is already leading to comparisons of expected change in project NPV with the opportunity cost of capital when deciding when to develop a mine or harvest a stand of trees, as in myopic Wicksell rule (28) (e.g., Torries 1998, 44, 75; Yin 2001, 480). Empirically, land owners also appear to recognize and time development according to the rate of expected pure postponement flow (Arnott and Lewis 1979, Holland et al. 2000).

Stopping condition (25) shows that the expected value of information from waiting must also be taken into account, and that comparisons of expected changes in project value with the opportunity cost of capital will not always yield sufficient patience in cases where the action is irreversible. The adjustment for the rate of quasi-option flow, however, is a generalization of a rule involving opportunity cost of capital that is already widely used and understood, rather than a completely new way of viewing stopping.

The *r*-percent rule is also helpful in understanding why waiting should at some point stop. Explanations of scrapping options are particularly awkward in this regard, since under optimal stopping decision makers are expected to incur operating losses prior to stopping. The intuition given is that by waiting the payoff to scrapping, $Y(W)$, even if positive now, is expected to increase, and by the nature of the perpetual option the agent cannot be forced to accept a lower value of $Y(W)$ than its current value (McDonald and Siegel 1986). But why, then, ever scrap the project? In Appendix 3 we illustrate the intuition for scrapping provided by the stochastic *r*-percent rule using an oil well abandonment example. In that example the payoff to abandonment is indeed always expected to rise, yet at a slower and slower rate. Taking the opportunity cost of waiting into account reveals that the expected rate of pure postponement flow eventually becomes so negative as to overwhelm any information flow benefits from further waiting.

7. Conclusions

While the mathematics of optimal stopping under uncertainty is well developed, the economic conceptualization of the stopping rule is not. In this paper we present the economics of optimal stopping under certainty and uncertainty for a common class of problems. We use the concept of an "*r*-percent stopping rule" to show that a deferrable action is taken only once the expected rate of return from waiting to act falls to the rate of interest. Under certainty, and under uncertainty when the process is monotone or when the action is reversible, the return from waiting is capital gains on the project. In other cases the gains from waiting are augmented by a flow of information called quasi-option flow. Eventually, the expected rate of capital gain less the rate of interest, which we call the expected rate of pure postponement flow, becomes so negative as to more than outweigh the expected flow of information benefits from continued delay, and stopping is optimal.

Seeing the stopping condition under uncertainty as having close parallels to the case under certainty reveals that the theory of investment under uncertainty is an incremental generalization of, not a qualitative break from, the traditional theory of investment under certainty. The weighing of opportunity costs and benefits of acting are as germane to stopping problems under uncertainty as they are under certainty, a concept that is not surprising, but that has not been explicitly brought out in traditional analyses to date. For the type of stopping problem examined in this paper, the stopping rule under certainty is simply the limiting case of uncertainty as volatility goes to zero.

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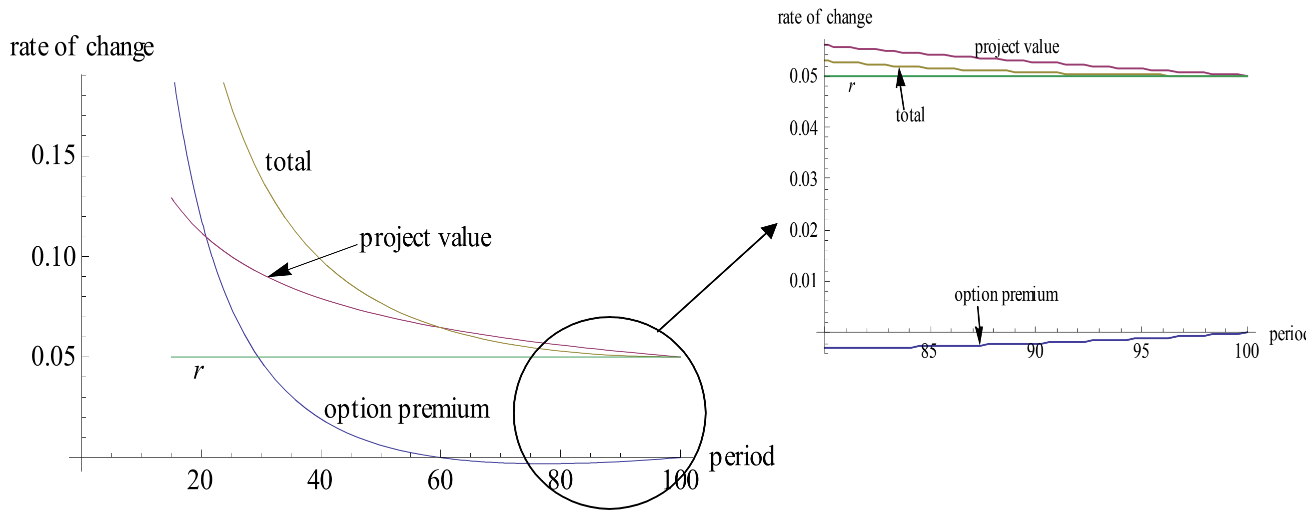

Figure 1: Wicksell's r-percent rule under certainty, Appendix 1.A, comparing the rates of growth of the project value, the option premium, and the total investment opportunity with the risk-free rate of interest $r = 0.05$. The stopping point is $\hat{t}_0 = 100$.

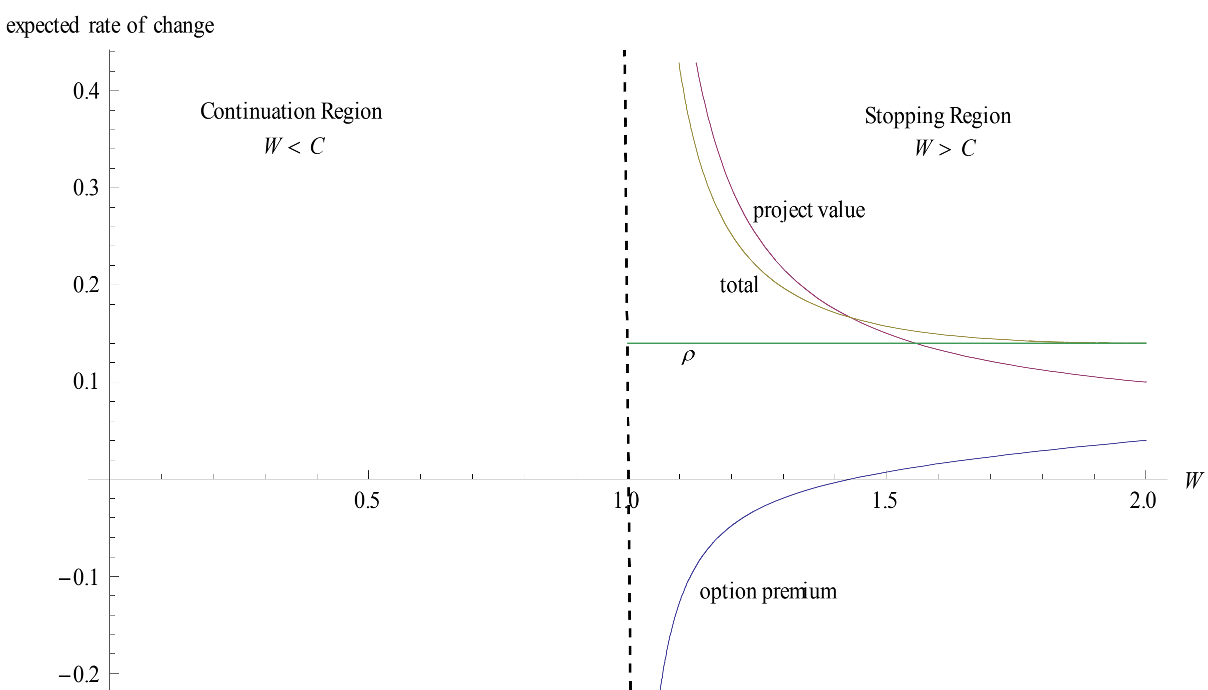

Figure 2: The non-myopic Wicksell r-percent rule for geometric Brownian motion, Appendix 1.C, comparing the expected rates of growth of the project value, the option premium, and the total investment opportunity, given a risk-adjusted rate of return $\rho = 0.14$ on the call option, investment cost $C = 1$, risk-free rate $r = 0.06$, required rate of return on the unlevered asset W, u, $= 0.10$, drift parameter $b = 0.05$, and volatility parameter $\sigma = 0.20$. In this example $\hat{W} = 2$ and $\hat{W}_M = 1.56$.

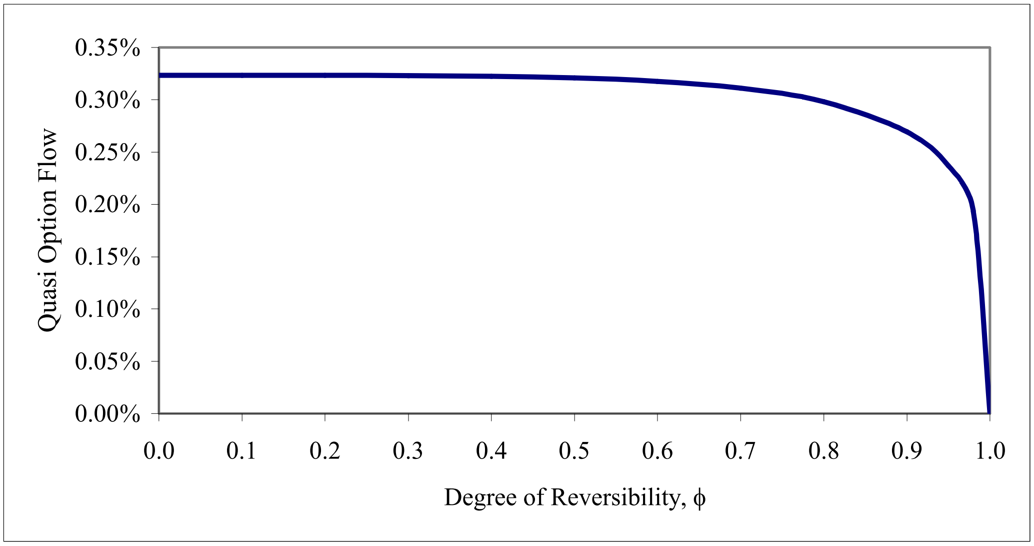

Figure 3: The rate of quasi-option flow $\tilde{\alpha}(\hat{s}_1, \hat{s}_2)$ for the Brennan and Schwartz model in Appendix 2, given start-up investment cost k_2 = \$1 million, shut-down receipt of $k_1 = \phi k_2$, output rate q = 10 million units/period, average production cost a = \$0, shut-down maintenance cost f = \$0, inflation rate π = 0%, commodity rate of return shortfall κ = 1%, commodity volatility σ^2 = 8%, no taxes, and risk-free rate of return ρ = 10%. The figure demonstrates that as reversibility becomes complete (ϕ = 1.0) the rate of quasi-option flow at stopping goes to zero and the myopic Wicksell r-percent stopping rule becomes applicable.

The Economics of Optimal Stopping

Appendices

Appendix 1. Four Illustrations of the *r*-percent Rule.

A. Serving Wine, Certainty. Suppose that a connoisseur has a bottle of wine that can provide one util if served immediately or can be stored costlessly and served at time $t_0 > t$ to yield $\exp\left(\sqrt{(t_0 - t)}\right)$ utils. Let the instantaneous utility discount rate be constant at r = 0.05 and let t = 0. The current value of the wine is

$$\Pi(0, t_0) = D(0, t_0) W(t_0) \; = \; \exp(-r t_0) \exp(\sqrt{t_0}) \; > 0. \qquad (1)'$$

Optimization via equation (2) implies that the connoisseur waits until period $\hat{t}_0$ = $1/(4r^2)$ = 100 to serve the wine, even though serving it now would provide a positive benefit of 1 util. Under optimal stopping the wine has a current value of

$$\Pi(0, \hat{t}_0) = D(0, \hat{t}_0) W(\hat{t}_0) \; = \; \exp(-r \hat{t}_0) \exp(\sqrt{\hat{t}_0}) = \exp\left(\frac{1}{4r}\right) = 148.4 \text{ utils.} \qquad (3)'$$

At time $t_0 < \hat{t}_0$ the option premium is

$$O(t_0, \hat{t}_0) = \Pi(t_0, \hat{t}_0) - W(t_0) = \exp\left(-r\left(\frac{1}{4r^2} - t_0\right)\right) \exp\left(\frac{1}{2r}\right) - \exp\left(\sqrt{t_0}\right) > 0\,. \qquad (5)'$$

At t_0 = t = 0 the option premium is 147.4 utils, over 99% of the wine's current value.

We confirm the r-percent rule in Proposition 1 by observing that prior to stopping the rate of change of the option premium is

$$\alpha(t_0, \hat{t}_0) = \frac{O_{t_0}(t_0, \hat{t}_0)}{W(t_0)} = \frac{r\Pi(t_0, \hat{t}_0) - W'(t_0)}{W(t_0)} = \frac{r \exp\left(-r\left(\frac{1}{4r^2} - t_0\right)\right) \exp\left(\frac{1}{2r}\right) - 0.5 (t_0)^{-0.5} \exp(\sqrt{t_0})}{\exp(\sqrt{t_0})}\,. \qquad (7)'$$

The rate of change in consumption utility is

$$\frac{W'(t_0)}{W(t_0)} = \frac{0.5(t_0)^{-0.5}\exp(\sqrt{t_0})}{\exp(\sqrt{t_0})} = 0.5(t_0)^{-0.5} > r \;\forall\; t_0 < \frac{1}{4r^2}, \tag{9)'$$

reflecting a positive rate of pure postponement flow over the life of the consumption option. Adding (7)′ and (9)′, the total rate of change of the utility of the consumption opportunity while waiting is

$$\left[\frac{W'(t_0)}{W(t_0)} + \alpha(t_0)\right] = \frac{\Pi_{t_0}(t_0,\hat{t}_0)}{W(t_0)} = \frac{r\Pi(t_0,\hat{t}_0)}{W(t_0)} = \frac{r\exp\left(-r\left(\frac{1}{4r^2} - t_0\right)\right)\exp\left(\frac{1}{2r}\right)}{\exp\left(\sqrt{t_0}\right)} > r. \tag{8)'$$

As the connoisseur waits the utility associated with the consumption opportunity is rising at a rate that is greater than the force of interest, the opportunity cost. At $\hat{t}_0$, the rate of growth of the utility of the consumption opportunity has fallen to r (See Figure 1).

B. Stopping an Arithmetic Brownian Motion. Consider a perpetual irreversible call option on W,

$$dW = b\,dt_0 + \sigma dz. \tag{12)'$$

Without loss of generality let stopping be costless ($C = 0$) and $Y(W) = W$. Also let r be the constant risk-free discount rate used to discount the payoff $\hat{W}$ in a risk-neutral contingent claims analysis.[18]

The market value of a perpetual opportunity to invest in W satisfies the second-order differential equation

$$r\tilde{\Pi}(W) - b\tilde{\Pi}'(W) - \tfrac{1}{2}\sigma^2\tilde{\Pi}''(W) = 0. \tag{17)'$$

The solution to (17)′ is of the form

$$\tilde{\Pi}(W) = A_1 e^{\lambda W} + A_2 e^{\mu W}, \tag{A1}$$

where $\lambda > 0$ and $\mu < 0$ are roots of the characteristic equation

[18] Dixit and Pindyck (1994) perform this same analysis using a constant rate of discount, ρ, without comment.

$$\tfrac{1}{2}\sigma^2\gamma^2 + b\gamma - r = 0\,. \tag{A2}$$

In the absence of any holding costs and coerced stopping at a finite lower bound, $\lim_{W \to -\infty} \tilde{\Pi}(W) = 0$.

From this, $A_2 = 0$. If the program is voluntarily stopped at $\hat{W}$, $\tilde{\Pi}(\hat{W}) = \hat{W}$ so that $A_1 = \hat{W}e^{-\lambda\hat{W}}$ and

$$\tilde{\Pi}(W) = e^{-\lambda(\hat{W} - W)}\hat{W}\,. \tag{13)$'$}$$

The smooth pasting condition,

$$\tilde{\Pi}'(\hat{W}) = \lambda e^{-\lambda(\hat{W} - \hat{W})}\hat{W} = \lambda\hat{W} = Y'(\hat{W}) = 1\,, \tag{24)$'$}$$

produces $\hat{W} = \lambda^{-1} = \left(\dfrac{-b + \sqrt{b^2 + 2\sigma^2 r}}{\sigma^2} \right)^{-1}$.

The economics of stopping are laid bare by the stochastic, non-myopic Wicksell *r*-percent rule in Proposition 3. Consider the domain where $W > 0$ and stopping is feasible. The expected rate of change in project value is $\dfrac{E[dY(W)]}{Y(W)dt_0} = \dfrac{b}{W}$, which could be positive, negative, or zero, depending on the parameterization of drift term *b*. Where the expected rate of change of project value is greater than *r*, the expected rate of pure postponement flow, $\dfrac{E[dY(W)]}{Y(W)dt_0} - r$, is positive and waiting is intuitive. Where the expected rate of pure postponement flow is negative, waiting is still optimal when the total rate of return on holding the investment opportunity is greater than the interest rate: $\dfrac{E[dY(W)]}{Y(W)dt_0} + \tilde{\alpha}(W) = \dfrac{b}{W} + \tilde{\alpha}(W) > r$. By (21),

$$\begin{aligned}\frac{b}{W}+\tilde{\alpha}(W) &= \frac{b}{W}-\frac{b}{W}(1-\lambda e^{-\lambda(\hat{W}-W)}\hat{W})+\tfrac{1}{2}\sigma^2\lambda^2 e^{-\lambda(\hat{W}-W)}\left(\frac{\hat{W}}{W}\right) \\ &= e^{-\lambda(\hat{W}-W)}\frac{\hat{W}}{W}\left(b\lambda+\tfrac{1}{2}\sigma^2\lambda^2\right) \\ &= e^{-\lambda(\hat{W}-W)}\frac{\hat{W}}{W}r \\ &> r\end{aligned} \qquad \text{(A3)}$$

for $\lambda W < 1 = \lambda\hat{W}$. If $b < 0$, for instance, the expected rate of pure postponement flow is negative for all $\hat{W} > W > 0$, but $\frac{b}{W}+\tilde{\alpha}(W) > r$, and waiting is still optimal.

The program is stopped when the expected return from delay falls to

$$\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0}+\tilde{\alpha}(\hat{W}) = \frac{b}{\hat{W}}+\tfrac{1}{2}\sigma^2\lambda^2 = r\,, \qquad (25)'$$

which happens when W rises to $1/\lambda$. At this point the expected rate of pure postponement flow is negative regardless of the parameterization of the rate of drift term b, having fallen to $-\tilde{\alpha}(\hat{W}) = -\tfrac{1}{2}\sigma^2\lambda^2 < 0$, where $\tilde{\alpha}(\hat{W})$ is the expected value of information about W from further delay.

C. Stopping a Geometric Brownian Motion. Consider an irreversible call option on a stochastic variable whose value follows a geometric Brownian motion with constant rate of drift b,

$$dW = bWdt_0 + \sigma W dz\,. \qquad (12)''$$

Let the required rate of return on the unlevered asset W be represented by u and be constant and greater than b. Also let the risk-free rate be represented by r. With b constant the investment cost C must be positive to avoid bang-bang now or never stopping solutions (Brock et al. 1989). Let the forward project value be

$$Y(W) = W - C\,. \tag{A4}$$

The solution to the stopping problem yields investment opportunity value

$$\tilde{\Pi}(W) = \left(\frac{W}{\hat{W}}\right)^{\beta} \hat{Y}\,, \tag*{(13)$''$}$$

$\hat{W} = \dfrac{\beta}{\beta - 1} C$, where $\beta > 1$ is the positive root of the characteristic equation

$$\tfrac{1}{2}\sigma^2 \gamma(\gamma - 1) + b\gamma - \rho = 0\,, \tag{A5}$$

and where $\rho = r + \beta(u - r) > u$ (Dixit et al. 1999, McDonald and Siegel 1986). In this case the use of a constant risk-adjusted discount rate is appropriate.

Again using the stochastic, non-myopic Wicksell *r*-percent stopping rule to reveal the economics of the problem, consider values of W for which $Y > 0$ and yet waiting is optimal. The expected rate of change in project value while waiting is $\dfrac{bW}{Y}$. This can be positive, negative, or zero, depending on the drift parameter b. From (21) the expected rate of change in the value of the option premium is

$$\tilde{\alpha}(W) = \frac{-bW}{Y}\left(1 - \beta\left(\frac{W}{\hat{W}}\right)^{\beta-1}\left(\frac{\hat{Y}}{\hat{W}}\right)\right) + \frac{\tfrac{1}{2}\sigma^2 W^2 \beta(\beta-1)}{Y}\left(\frac{W}{\hat{W}}\right)^{\beta-2}\left(\frac{\hat{Y}}{\hat{W}^2}\right). \tag*{(21)$'$}$$

The total return to waiting, $\dfrac{E[dY]}{Y(W)dt_0} + \tilde{\alpha}(W)$, is defined and greater than the discount rate for $C < W \leq \hat{W}$. The *r*-percent rule reveals that at stopping the expected return from delay has fallen to

$$\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0} + \tilde{\alpha}(\hat{W}) = \frac{b\hat{W}}{\hat{W} - C} + \tfrac{1}{2}\sigma^2\beta(\beta-1) = b\beta + \tfrac{1}{2}\sigma^2\beta(\beta-1) = \rho\,. \tag*{(25)$''$}$$

Figure 2 depicts this stopping problem for specific parameter values. Given these parameters the expected rate of change in project value is less than the interest rate for $W > 1.56$. More than offsetting this is a positive expected rate of change in the option premium, for a total rate of capital gain on waiting that exceeds the interest rate. Waiting is thus optimal despite the negative expected rate of pure postponement flow. The expected total rate of capital gain falls to the discount rate at $\hat{W} = 2$, at which point the program is stopped. The expected rate of pure postponement flow at stopping, equal to the negative of the rate of quasi-option flow from further waiting, is $-\frac{1}{2}\sigma^2\beta(\beta-1) = -4.0\%$.

D. Stopping a Combined Process. Extending Example C, if W follows a combined geometric Brownian motion with an independent downward Poisson jump of known percentage ψ and arrival rate η (Dixit and Pindyck 1994, pp. 167-173), the r-percent stopping condition can be shown to be

$$\frac{E[dY(\hat{W})]}{Y(\hat{W})dt_0} + \alpha(\hat{W}) = \frac{(b-\eta\psi)\hat{W}}{\hat{W}-C} + \tfrac{1}{2}\sigma^2\beta(\beta-1) = \rho\,, \tag{25$'''$}$$

where β is now the positive solution iteratively satisfying

$$\tfrac{1}{2}\sigma^2\gamma(\gamma-1) + b\gamma - (\rho+\eta) + \eta(1-\psi)^{\gamma} = 0 \tag{A6}$$

and $\tilde{\Pi}(0) = 0$. In the absence of a jump process ($\eta = 0$) equations (25)$'''$and (A6) revert to (25)$''$ and (A5). When $\eta > 0$ the jump process is seen to lower the expected rate of increase of project value in equation (25)$'''$ and thereby lower the expected rate of pure postponement flow. It also adjusts the value of β in the quasi option flow term. For example, for $\psi = 1$, $\eta > 0$ causes β to increase, and so causes the rate of quasi-option flow to rise. The reduced expected rate of pure postponement flow reduces the benefits of waiting, while the increased rate of quasi-option flow increases the benefits of waiting. In the examples given in Dixit and Pindyck (Table 5.1) the reduction in the expected rate of pure postponement flow wins out, and stopping is advanced as a result of the jump process.

Appendix 2. Optimal Stopping, Reversibility, and Infinitely Repeated Options

This appendix first demonstrates for a canonical stopping problem that in the limit as investment becomes completely reversible, $\tilde{\alpha}(\hat{W}) = 0$ and the myopic Wicksell r-percent stopping rule is optimal. It then shows that repeated options to invest and reversibility are not substitutes.

Consider the investment opportunity in Section II of Brennan and Schwartz (1985). The firm has an option to invest in (start up) and then disinvest in (close) an infinitely lived project. The problem is autonomous. For finite start-up cost $k_2 > 0$ the firm can enjoy a positive revenue stream with present value qs/κ, where q is the fixed (at capacity) per period output, s is the unit price, which follows a geometric Brownian motion, and κ is the constant rate of convenience on the unit price. The firm can reverse the initial timing decision dt periods later by paying k_1, re-start the project dt periods later by again paying k_2, and so on. The investment is completely expandable since k_2 is constant (Dixit and Pindyck 2000). It is completely irreversible when $k_1 = \infty$.

As in Dixit and Pindyck (2000) we eliminate the complication of operating options by assuming that there are no operating costs. We also set taxes and inflation to zero to simplify the analysis. Rather than paying k_1 to disinvest, let there be a receipt of $k_1 = -\phi k_2$ upon closing the project, where ϕ, $0 \leq \phi \leq 1$, is the price discount factor upon selling capital. Limiting ϕ to a lower bound of zero does not reduce the generality of the problem, as no firm would disinvest from a positive, perpetual income stream if payment were required; irreversibility is induced via $\phi = 0$, and complete reversibility via $\phi = 1$. Partial reversibility obtains for $0 < \phi < 1$.

Brennan and Schwartz conduct their analysis via contingent claims and a risk-adjusted rate of drift in the unit price. In Brennan and Schwartz's notation (though maintaining our caret symbol for the price at stopping), and given our assumptions about maintenance and operating costs, the functional form for the value of the option to start the project is

$$w(s) = \beta_1 s^{\gamma_1}, \quad \beta_1 > 0,\ \gamma_1 > 1. \tag{A7}$$

This functional form indicates that it would also have been valid to use a constant risk-adjusted rate of discount rather than contingent claims and a risk-free rate of discount.

The functional form for the forward project value at start-up is[19]

$$v(\hat{s}) = \beta_4 \hat{s}^{\gamma_2} + \frac{q\hat{s}}{\kappa} - k_2, \quad \beta_4 > 0,\ \gamma_2 < 0. \tag{A8}$$

The term $\beta_4 \hat{s}^{\gamma_2}$ is the value of any reversibility of the investment decision (the value of the put option that one obtains by investing), and $\beta_4 \hat{s}^{\gamma_2} > 0$ when $\phi > 0$.

We now analyze investment timing under various degrees of reversibility using the *r*-percent stopping rule. Define $\hat{s}_2$ as the optimal stopping point for start-up and $\hat{s}_1$ as the optimal stopping point for subsequent closure, $0 \le \hat{s}_1 \le \hat{s}_2$. When the start-up investment is completely irreversible it can be shown that $\hat{s}_2 = \frac{\kappa k_2}{q} \frac{\gamma_1}{(\gamma_1 - 1)} > \hat{s}_1 = 0$.[20] This is a stricter investment hurdle than the traditional NPV stopping hurdle, $\hat{s}_{2_{NPV}} = \frac{\kappa k_2}{q}$. When investment is completely reversible it can be shown that $\hat{s}_1 = \hat{s}_2 = \frac{\kappa k_2}{q} \frac{\gamma_1 \gamma_2}{(\gamma_1 - 1)(\gamma_2 - 1)} > \hat{s}_{2_{NPV}} > 0$.[21] Complete reversibility does not remove the

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[19] There is a typographical error in the Brennan and Schwartz paper; the correct expression for β_4 is $\frac{ds_1^*(\gamma_1 - 1) + e\gamma_1}{(\gamma_2 - \gamma_1) s_1^{*\gamma_2}}$.

[20] With complete irreversibility the invested capital is sunk and closure to avoid a positive perpetual income stream will not take place ($\hat{s}_1 = 0$).

[21] The shut-down price $\hat{s}_1$ is now positive because despite the absence of operating costs, the fixed output level means that at low prices it is optimal to close and recover the interest on invested capital rather than to produce.

consideration of expected pure postponement flow given that a bad decision still ties up capital for period dt, and thus the NPV stopping trigger is again premature.

The rate of quasi-option flow $\tilde{\alpha}(\hat{s}_1, \hat{s}_2)$ can be calculated from equations (26), (A7), and (A8) for varying degrees of reversibility, ϕ, given a riskless return $\rho = 0.10$ and other parameter values taken from Brennan and Schwartz (see Figure 3). There is no closed-form solution for $\hat{s}_2$ and $\hat{s}_1$ when $\hat{s}_2 > \hat{s}_1 > 0$ (i.e., when $0 < \phi < 1$ and investment is partially reversible). Brennan and Schwartz provide the algorithm for an iterated solution. As seen in Figure 3, the greater the degree of irreversibility, the greater the rate of quasi-option flow at stopping. Under complete reversibility it can be shown algebraically that $w''(\hat{s}_2) - v''(\hat{s}_2) = 0$, $\tilde{\alpha}(\hat{s}_1, \hat{s}_2) = 0$, and that the stopping trigger $\hat{s}_2$ under complete reversibility corresponds with the myopic Wicksell r-percent stopping rule, $\dfrac{E\left[dv(\hat{s}_2)\right]}{v(\hat{s}_2)dt_0} = \rho$.

We now show that even though repeated options to invest lower the investment trigger compared with a single option to invest, they do not reduce the irreversibility of investing. To illustrate this we use Malchow-Møller and Thorsen's (2005) (hereafter MT) single-factor model of the option to. repeatedly but irreversibly replace a stationary and certain existing productivity level, Θ, with the productivity level of a stochastic exogenous technology, θ. Using MT's notation, the exogenous technology evolves as a geometric Brownian motion $d\theta = \alpha_1 \theta dt + \sigma_1 \theta dz_1$ with rate of drift $\alpha_1 \in (0, r)$, where MT use a constant rate of discount r without comment. Choosing a constant discount rate is appropriate given the functional form of the option to invest in new technology (see equation A9 below). Technology replacement, at cost $c\theta$, $\frac{1}{r} > c > 0$, is optimal each time $\hat{\theta} = \lambda \Theta$, where $\lambda > 1$ is the normalized investment trigger. Let $w = \frac{\theta}{\Theta}$. The value of the ongoing program, the sum of the option to invest in new technology and the value of ongoing activities, is MT equation (14):

$$V(\theta) = \Theta A_1 w^{a_1} + \frac{\Theta}{r}, \tag{A9}$$

where $a_1 > 1$ is the positive root of the quadratic equation and $A_1 = -\frac{\lambda^{-\alpha_1}}{r(1-a_1)} > 0$.

MT discuss two cases. The first is a single option to irreversibly invest in the new technology. Here the forward project value is $Y(\theta) = \theta(\frac{1}{r} - c)$. The foregone flow upon investment is Θ, with present value Θ/r. In an interior solution the NPV rule, which is suboptimal, suggests investing the first time that $Y(\theta) = \frac{\Theta}{r}$, or when $\lambda \equiv \frac{\hat{\theta}}{\Theta} = (1-rc)^{-1}$. From equation (30), given these formulations of $V(\theta)$ and $Y(\theta)$, the (optimal) stochastic, non-myopic Wicksell r-percent rule is to invest when[22]

$$\alpha_1 + \frac{\Theta}{\hat{\theta}(\frac{1}{r} - c)} + \frac{1}{2}\sigma_1{}^2 \frac{a_1(a_1 - 1)A_1\hat{\theta}^{a_1}\Theta^{1-a_1}}{\hat{\theta}(\frac{1}{r} - c)} = r, \tag{30$'$}$$

or, after simplifying,

$$\alpha_1 + \frac{\Theta}{\hat{\theta}(\frac{1}{r} - c)} + \frac{1}{2}\sigma_1{}^2 \frac{\frac{a_1}{r}\Theta}{\hat{\theta}(\frac{1}{r} - c)} = r. \tag{30$''$}$$

The three terms on the left-hand side of (30)″ constitute the benefit of waiting: the expected rate of capital gain in project value plus the dividend yield from ongoing operations plus the rate of quasi-option flow. The right-hand side of (30)″ is the opportunity cost of waiting. Making use of the fundamental quadratic equation (equation 11 with MT Corrigendum) to find substitutions of σ_1^2 and other terms for α_1, (30)″ can be solved for investment trigger

$$\lambda = \frac{\hat{\theta}}{\Theta} = \left((1 - \tfrac{1}{a_1})(1 - rc)\right)^{-1}, \tag{A10}$$

[22] Equation (30) is used because the firm enjoys flow Θ while waiting to invest. By substituting (A10) into (30)″ it can be shown that (30)″ is the same as MT equation (24).

the solution obtained by MT (equation 23). Note that given $a_1 > 1$ the investment trigger is larger than that produced by the NPV rule.

Substituting (A10) into (30)″ yields

$$\alpha_1 + \frac{\Theta}{\hat{\theta}(\frac{1}{r} - c)} + \tfrac{1}{2}\sigma_1{}^2(a_1 - 1) = r\,. \tag{A11}$$

In their search for the intuition of this stopping problem MT alternatively express (A10) as $\frac{\Theta}{\hat{\theta}(\frac{1}{r} - c)} + \frac{r}{a_1} = r$ (MT equation 24). They then interpret $\frac{r}{a_1}$, which from (A10) is $\alpha_1 + \frac{1}{2}\sigma_1^2(a_1 - 1) > \alpha_1$, as the expected rate of capital gain in forward project value. They attribute the rate of growth beyond rate α_1 to impacts of uncertainty (i.e., $\sigma_1^2 > 0$) on the expected rate of growth of project value. Since $Y(\theta)$ is linear in θ this cannot be the case. By expressing the stopping condition as an *r*-percent rule, Equations (30)″ and (A11) reveal that the $\frac{r}{a_1}$ term instead includes expected capital gains in forward project value (3.00% using the base case parameter values in MT p. 1036) *and* a rate of quasi-option flow associated with the irreversibility of the investment at stopping (0.08%). The rate of dividend yield, at 1.92%, rounds out the benefits of waiting to invest given r = 5.00%.

The second case considered by MT involves infinitely repeated options to irreversibly invest in new levels of productivity. From MT equations (14) and (15) the forward value is now $Y(\theta) = \theta(\frac{1}{r} - c) + \theta A_1 w^{a_1}\big|_{w=1} = \theta(\frac{1}{r} - c) + \theta A_1$. The term $\theta(\frac{1}{r} - c)$ is the net value received upon adopting the new technology, as in the single-step option, and θA_1 is the value of the infinite set of options to switch technologies again given that the current switch resets w to 1. Note that the forward value is higher than in the single option case due to the compound options embedded in underlying project. The foregone flow upon investment is still Θ, and the functional form of the option value is still as in (A9).

MT show that the solution to this stopping problem gives an iterated stopping trigger, λ, that is less than $\left((1-\frac{1}{a_1})(1-rc)\right)^{-1}$, which was the stopping trigger without repeated options. We found above that reversibility lowers the investment trigger. Does this mean that repeated options are akin to reduced irreversibility of the single stage investment option, as MT suggest? The r-percent stopping equation (30) reveals that at the stopping point,[23]

$$\alpha_1 + \frac{\Theta}{\hat{\theta}(\frac{1}{r} - c + A_1)} + \frac{1}{2}\sigma_1^2 \frac{\frac{a_1}{r}\Theta}{\hat{\theta}(\frac{1}{r} - c + A_1)} = r \,. \qquad (30)'''$$

The expected rate of appreciation of project value is unchanged at α_1. Yet both the dividend yield from current cash flows and the rate of quasi-option flow are smaller for any given θ compared with (30)″. This is due to the strictly larger value of $Y(\theta)$. Stopping therefore occurs at a lower value of $\hat{\theta}$ because of the reduced benefits to waiting (or, equivalently, the increased opportunity cost of waiting). Repeated options do not, however, reduce the irreversibility of the investment, since they do not change the rate of quasi-option flow at stopping. Using the same baseline parameter as above and the resultant iterated solution for the stopping trigger, the rate of quasi-option flow is still 0.08% in (30)‴.

Appendix 3. The Economics of Optimal Scrapping

Here we demonstrate the economics behind optimal scrapping. We use Clarke and Reed's (1990b) model of the optimal time to irreversibly abandon a perpetually producing oil well with fixed

[23] Equation (30)‴ provides the same unique iterated solution for stopping as MT equation (25).

operating costs, c, but declining production. We use their notation. Both the oil price $P(t)$ and extraction rate $Q(t)$ evolve as geometric Brownian motions. Letting $\pi(t) = \log P(t)$ and $q(t) = \log Q(t)$,

$$d\pi = bdt + \sigma_\pi dw_\pi \,,\ b > 0 \qquad \text{(A12)}$$

$$dq = -\delta dt + \sigma_q dw_q \,,\ \delta > 0. \qquad \text{(A13)}$$

Let $z(t) \equiv \pi(t) + q(t)$ be the logarithm of revenue. Then, by Ito's lemma, $z(t)$ is a Brownian motion with drift $-d = b - \delta$ and variance $\sigma^2 = \sigma_\pi^2 + \sigma_q^2 + 2\sigma_{\pi q}$, where $\sigma_{\pi q}$ is the covariance between the logarithm of price and logarithm of the extraction rate. At abandonment time T the forward value of abandonment is

$$R\left(z(T)\right) \equiv A - Be^{z(T)} ,\ A > 0,\ B > 0 \qquad \text{(A14)}$$

where $A = \frac{c}{r} - c_0 > 0$ is the present value of perpetual operating costs avoided less the abandonment cost and $Be^{z(T)} = \frac{\gamma}{r+d-\sigma^2/2} e^{z(T)} > 0$ is the after-tax expected present value of revenues foregone given a gross proceeds tax rate of $(1 - \gamma)$. The goal is to determine the revenue level $e^{\bar{z}}$ that induces optimal abandonment timing given a risk-free rate of interest of r.[24]

Clarke and Reed show that the value of the option to abandon when $z(t) > \bar{z}$ is of the discount factor form

$$W\left(z(t)\right) = e^{\Theta\left(z(t)-\bar{z}\right)} R(\bar{z}) \,, \qquad \text{(A15)}$$

[24] Clarke and Reed implicitly assume risk-neutrality, even though the functional form of the option (see equation A15) allows for a constant risk-adjusted discount rate.

where $\Theta = [d - \sqrt{d^2 + 2r\sigma^2}\,]/\sigma^2 < 0$.[25] Invoking stopping condition (25) yields

$$\begin{aligned}\frac{E[dR(\bar{z})]}{R(\bar{z})dT} + \tilde{\alpha}(\bar{z}) &= \frac{-dR'(\bar{z}) + \frac{1}{2}\sigma^2 R''(\bar{z})}{R(\bar{z})} + \frac{\frac{1}{2}\sigma^2\left(W''(\bar{z}) - R''(\bar{z})\right)}{R(\bar{z})} \\ &= \frac{dBe^{\bar{z}} - \frac{1}{2}\sigma^2 Be^{\bar{z}}}{R(\bar{z})} + \frac{\frac{1}{2}\sigma^2\left(\Theta^2 e^{\Theta(\bar{z}-\bar{z})}R(\bar{z}) + Be^{\bar{z}}\right)}{R(\bar{z})} \\ &= \frac{dBe^{\bar{z}} - \frac{1}{2}\sigma^2 Be^{\bar{z}}}{R(\bar{z})} + \frac{\frac{1}{2}\sigma^2\left(\Theta^2 R(\bar{z}) + Be^{\bar{z}}\right)}{R(\bar{z})} = r.\end{aligned} \tag{25}''''$$

Solving,

$$e^{\bar{z}} = \frac{A(r - \frac{1}{2}\sigma^2\Theta^2)}{B(d + r - \frac{1}{2}\sigma^2\Theta^2)} = \frac{A\Theta}{B(\Theta - 1)}. \tag{A16}$$

This is equation (18) in Clarke and Reed, which they derive using the value-matching and smooth-pasting conditions $W(\bar{z}) = R(\bar{z})$ and $W_z(\bar{z}) = R_z(\bar{z})$.

We now demonstrate the economics revealed by the *r*-percent stopping rule in (25)′′′′. We use a gross proceeds tax rate (1 - γ) of 16.5%, an abandonment cost c_0 of 0, a fixed operating cost c of 33.53, a rate of drift of revenues of -2.5% (d = 0.025), a revenue volatility σ^2 of 0.03, and an interest rate, r, of 4.00%, all taken from Table 3 of Clarke and Reed. The rate of drift of revenues, being negative, means that the forward value of abandonment, $R(z)$, is expected to increase over time.

Table A3.1 summarizes three candidate stopping rules. The right-hand column shows that for any suboptimal stopping point the expected gains from waiting exceed the interest rate. The “now or never” NPV stopping rule, abandon as soon as it is profitable to do so, would have the project abandoned when the forward value of abandonment rises from a negative value to zero, at which

[25] The current value of the well is $V\left(z(t)\right) \equiv W\left(z(t)\right) - R\left(z(t)\right) - c_0$, and the HJB equation is $-dW_z + \frac{1}{2}\sigma^2 W_{zz} = rW$. The fact that the HJB equation operates on the option to abandon, rather than on the current value of the well, obviates the need to take into account current expected operating cash flows in the *r*-percent stopping rule.

Table A3.1: Stopping rules and their corresponding trigger values (in bold) for Clarke and Reed's (1990b) oil well abandonment problem, r = 4.00%.

	Economic variable at abandonment					
Stopping (abandonment) rule	Revenue e^z	After-tax income $\gamma e^z - c$	Forward value, option value of abandonment $R(z)$, $W(z)$	Expected rate of change in abandonment value $\frac{E[dR(z)]}{R(z)dT}$	Expected rate of change of option premium (eq. 21) $\tilde{\alpha}(z)$	Total expected rate of return to waiting $\frac{E[dR(z)]}{R(z)dT} + \tilde{\alpha}(z)$
Simple NPV "now or never" rule	50.2	8.4	**0.0, 209.6**	∞	∞	∞
Myopic r% rule	40.2	0.0	167.7, 262.0	**4.00%**	2.25%	6.25%
Non-myopic r% rule (optimal)	25.1	-12.6	419.1, 419.1	1.00%	3.00%*	**4.00%**

*quasi-option flow, equation (26).

point after-tax net income is still positive, at 8.4. At that point, the forward value of abandonment is expected to rise at an infinite rate. As stated in Corollary 3, this stopping rule is not optimal, and should readily be rejected by practitioners given that they would not intuitively abandon a project while it is still earning positive after-tax profits.

Myopic stopping rule (28) gives revenue level

$$e^{\bar{z}_M} = \frac{Ar}{B(d + r - \frac{1}{2}\sigma^2)} = 40.2 \tag{A17}$$

at stopping, at which point the forward value of abandonment is $R(\bar{z}_M)$ = 167.7. It is expected to rise at r-percent. The revenue level of 40.2 is also the point where after-tax net income becomes zero. The myopic stopping point is therefore a natural stopping point that practitioners would find more intuitive than the NPV stopping point. We alluded to practitioners' use of this myopic stopping rule earlier. But there remains a positive option premium of 94.3, and from (21) continued waiting at the myopic stopping point provides an expected rate of gain on the option premium of

$$\tilde{\alpha}(\overline{z}_M) = \frac{-(-d)\left(-Be^{\overline{z}_M} - \Theta e^{\Theta(\overline{z}_M - \overline{z})} R(\overline{z})\right) + \tfrac{1}{2}\sigma^2 \Theta^2 e^{\Theta(\overline{z}_M - \overline{z})} R(\overline{z}) + Be^{\overline{z}_M}}{R(\overline{z}_M)} = 2.25\% \,. \qquad (21)'$$

Even if one ignores notions of an option premium, the expected payoffs to abandonment are rising at 4% at this point, and this in itself has been offered as a reason to continue operating the well (McDonald and Siegel 1986, 714-15).

But that rate of increase is declining over time, and one must take into account the opportunity cost of waiting. From (25)′′′′, when the expected rate of rise of the forward value of abandoning falls to 1.00%, the expected rate of pure postponement flow, at -3.00%, becomes great enough to offset the quasi-option flow from further waiting. At this point the after-tax income level is -12.6, and the well is optimally abandoned.