

# Good Timing: The Economics of Optimal Stopping

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**Abstract:** This paper presents an economic interpretation of the optimal “stopping” of perpetual project opportunities under both certainty and uncertainty. Prior to stopping, the expected rate of return from delay exceeds the rate of interest. The expected rate of return from delay is the sum of the expected rate of change in project value and the expected rate of change in the option premium associated with waiting. At stopping the expected rate of return from delay has fallen to the rate of interest. Viewing stopping in this way unifies the theoretical and practical insights of the theory of stopping under certainty and uncertainty.

Keywords: investment timing,  $r$ -percent rule, real options, investment under uncertainty, Wicksell.

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## 1. Introduction

An important question in the economics of uncertainty is optimal investment “stopping” or timing, or more generally the optimal timing of any discrete action. When a profitable but irreversible opportunity is available, it may not be optimal to act immediately but to wait until a more advantageous time to proceed.

The mathematics of optimal stopping under uncertainty is unassailable. What is not well understood is the economics of stopping, especially for perpetual random processes. *Why* is it optimal to stop a perpetual random process at a given point? Two-period and other recursive stopping models based on finite decision horizons provide little guidance. Value-of-information explanations of benefits of waiting are ubiquitous and of some comfort, but why does waiting for more information at some point become ill advised? Sometimes, plausible explanations from stopping problems under certainty – forestry or wine storage – are evoked, though without a complete link to the costs and benefits of waiting under uncertainty.

Our aim is a rigorous clarification of the economics of optimal stopping under uncertainty. We bring out that stopping under both certainty and uncertainty has the same cost-benefit interpretation. The expected benefit to waiting is the rate of change of the project value plus the rate of change of an option premium associated with waiting. At stopping this sum falls to and equals the opportunity cost of waiting, the interest rate. This *r-percent rule* is similar to but more general than those for stopping under certainty found by Faustmann (1995), Fisher (1907), and Wicksell (1938). Unifying the economics of optimal stopping in this way yields a joint interpretational framework for the many different types of work in the field.

We begin with a new way of viewing stopping under certainty that provides a foundation for stopping under uncertainty.

## 2. Stopping under Certainty

The irreversible, lumpy economic actions or investments we consider throughout the paper involve an initial decision and a plan that specifies outputs in future time periods and possible other choices. We stress actions that can be delayed indefinitely at no explicit cost. The intensity of the action and the ensuing optimal production plan generally depend on the time of action,  $t_0$ , and the future equilibrium price path of outputs and inputs, among other things. The firm need not be a price taker. In limiting our attention to the more interesting case of an interior solution we assume that these equilibrium price and interest rate paths, along with the optimal production plan, yield a future stopping point. For ease of exposition we assume that it is unique.

Let the current time be  $t$ , and let  $Y(t_0) = W(t_0) - C(t_0)$  denote the *forward project value* received by irreversibly sinking a known discrete investment cost  $C(t_0) \geq 0$  at time  $t_0 \geq t$  in return for an incremental benefit from time  $t_0$  onward with time- $t_0$  present value  $W(t_0)$ . We assume that  $W(t_0)$  is generated by optimal actions subsequent to the initial action at  $t_0$ , which, as in compound option analysis, can include subsequent timing options. We also assume that  $W(t_0)$  is time-varying and differentiable. Following the discount-factor approach of Dixit et al. (1999), let

$D(t, t_0) = \exp\left[-\int_t^{t_0} r(s) ds\right] > 0$  denote the (riskless) discount factor, integrated over the short rates of interest  $r(s)$  that represent the required rate of return to all asset classes in the economy. Given our assumption of no holding costs, the present value of the investment initiated at time  $t_0$  is

$$\Pi(t, t_0) = D(t, t_0)Y(t_0). \quad (1)$$

This present value is rising at the rate of interest:

$$\Pi_t(t, t_0) = D_t(t, t_0)Y(t_0) = r(t)D(t, t_0)Y(t_0) = r(t)\Pi(t, t_0). \quad (2)$$

Marglin (1963) was among the first to point out that the correct algorithm for this problem, consistent with the work of Wicksell, Fisher, and Faustmann, is to maximize the present value of the investment opportunity by choosing the optimal stopping time  $\hat{t}_0$ . Dixit and Pindyk's (1994, 138-139) brief analysis of the stopping problem under certainty shows that in cases of perpetually static or declining project value investment is immediate when  $Y(t_0) > 0$ , but that interior solutions  $\hat{t}_0 > t_0$  are possible when  $Y(t_0)$  is growing in the long run.<sup>1</sup>

To find the optimal stopping time we differentiate  $\Pi(t, t_0)$  in (1) with respect to its second argument:

$$\begin{aligned}\Pi_{t_0}(t, t_0) &= D_{t_0}(t, t_0)Y(t_0) + D(t, t_0)Y'(t_0) \\ &= -r(t_0)D(t, t_0)Y(t_0) + D(t, t_0)Y'(t_0) \\ &= D(t, t_0)[Y'(t_0) - r(t_0)Y(t_0)].\end{aligned}\tag{3}$$

The solution, where  $\Pi_{t_0}(t, t_0) = 0$  and the second order condition for a maximum holds, yields  $\hat{t}_0$  and the critical forward project value  $Y(\hat{t}_0) > 0$ .<sup>2</sup> From (1) the current *market* or *option value* of the (optimally managed) investment opportunity is

$$\Pi(t, \hat{t}_0) = D(t, \hat{t}_0)Y(\hat{t}_0).\tag{4}$$

The first- and second-order conditions of this stopping problem imply what is known as *Wicksell's Rule*, that (a)  $Y'(t_0)/Y(t_0) > r(t_0)$  for  $t_0$  immediately prior to  $\hat{t}_0$ , and (b)  $Y'(\hat{t}_0)/Y(\hat{t}_0) = r(\hat{t}_0)$ .

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<sup>1</sup> Waiting (possibly forever) under certainty and uncertainty is obviously optimal when  $Y(t_0) < 0$ . We limit our discussion to the more interesting case of waiting when project value is positive. Fisher (1907) relates that Henry George and others thought that only organic forms of capital capable of reproducing and increasing with time were subject to interior stopping points. This is why Wicksell's analysis of wine, a manufactured capital good, was path breaking.

<sup>2</sup> If there are multiple times for which  $\Pi_{t_0}(t, t_0) = 0$ , the program is stopped at the one that yields the maximum value of  $\Pi(t, t_0)$ .

Condition (b) follows from equation (3) given  $D(\hat{t}_0, \hat{t}_0) = 1$ . Condition (a) follows from (3) given  $\Pi_{t_0}(t, t_0) > 0$  immediately prior to stopping. In keeping with Mensink and Requate's (2005) two-period analysis of stopping under uncertainty, in which they termed the value that comes from delaying a growing project as *pure postponement value*, we define  $\frac{Y'(t_0)}{Y(t_0)} - r(t_0)$  to be the rate of *pure postponement flow* that originates from waiting. Wicksell's rule is that the rate of pure postponement flow is strictly positive for  $t_0$  immediately prior to  $\hat{t}_0$  and falls to zero at  $\hat{t}_0$ .

The analysis can be carried further in order to generate additional insights for stopping under uncertainty. By the definition of an interior solution, at any trial stopping time prior to  $\hat{t}_0$ ,

$$\Pi(t_0, \hat{t}_0) > Y(t_0). \quad (5)$$

Consistently with the literature on stopping under uncertainty, we define the difference,

$$\Pi(t_0, \hat{t}_0) - Y(t_0) \triangleq O(t_0, \hat{t}_0) > 0, \quad (6)$$

as the *option premium* from waiting until  $\hat{t}_0$  to initiate the project. Accordingly, the value of the investment opportunity,  $\Pi(t_0, \hat{t}_0)$ , has two components, namely the forward project value  $Y(t_0)$  and the option premium  $O(t_0, \hat{t}_0)$  associated with optimal project timing.

Equation (5) reveals an intertemporal cost-benefit dynamic that can be brought out by multiplying both sides by  $r(t_0)$  and making the substitution  $\Pi_{t_0}(t_0, \hat{t}_0) = r(t_0)\Pi(t_0, \hat{t}_0)$  from (2):

$$\Pi_{t_0}(t_0, \hat{t}_0) > r(t_0)Y(t_0). \quad (7)$$

The left side of (7) is the benefit at time  $t_0$  from continuing to wait for an instant, the increase in the market value of the investment opportunity. The right side of (7) is the opportunity cost of waiting, the lost interest on proceeds from acting. Prior to stopping the former exceeds the later.

Converting (7) into a comparison of rates yields

$$\frac{\Pi_{t_0}(t_0, \hat{t}_0)}{Y(t_0)} > r(t_0). \quad (8)$$

From equation (6), the inequality in (8) can be expanded to

$$\frac{Y'(t_0)}{Y(t_0)} + \frac{O_{t_0}(t_0, \hat{t}_0)}{Y(t_0)} > r(t_0). \quad (9)$$

At all times prior to the optimal stopping time the *sum* of the rates of growth of the two components of the investment opportunity, expressed relative to the forward project value  $Y(t_0)$ , is greater than the short rate of interest. Waiting is optimal. A further interpretation of (9) is obtained as follows. By (2) and (6),  $O_{t_0}(t_0, \hat{t}_0) = r(t_0)\Pi(t_0, \hat{t}_0) - Y'(t_0)$ . Let

$$\alpha(t_0, \hat{t}_0) = \frac{O_{t_0}(t_0, \hat{t}_0)}{Y(t_0)} = \frac{r(t_0)\Pi(t_0, \hat{t}_0) - Y'(t_0)}{Y(t_0)} \quad (10)$$

denote the rate of capital gain or loss on the option premium as a fraction of the forward project value. The sign of  $\alpha(t_0, \hat{t}_0)$  depends on the value of  $\Pi(t_0, \hat{t}_0)$  and hence the entire program, rather than only local properties around the stopping point. If  $t_0$  is sufficiently close to  $\hat{t}_0$  it must be that  $\alpha(t_0, \hat{t}_0) < 0$  and  $Y'(t_0)/Y(t_0) > r(t_0)$  since  $O(t_0, \hat{t}_0)$  is continuous and falls to zero at stopping. But  $\alpha(t_0, \hat{t}_0) > 0$  when  $Y'(t_0)/Y(t_0) \leq r(t_0)$ , which can occur, temporarily, farther away from the stopping point.

Given  $Y'(\hat{t}_0) = r(\hat{t}_0)Y(\hat{t}_0)$ , at stopping (9) reduces to

$$\frac{Y'(\hat{t}_0)}{Y(\hat{t}_0)} + \frac{O_{t_0}(\hat{t}_0, \hat{t}_0)}{Y(\hat{t}_0)} = \frac{Y'(\hat{t}_0)}{Y(\hat{t}_0)} + 0 = r(\hat{t}_0) \quad (11)$$

At the optimal stopping time the total rate of capital gain of the two asset components has fallen to the rate of interest. The rate of gain on the option is equal to zero. The forward project value itself is rising at  $r$ .

Equations (9) and (11) constitute the main result of this section, an  $r$ -percent rule that is different from Wicksell's rule and that will inform the economics of stopping under uncertainty. Wicksell's rule focuses on the rate of pure postponement flow and for that reason is very suggestive. But it is only guaranteed to hold in the neighborhood of  $\hat{t}_0$ , and more importantly, neglects the option premium and its rate of change. Equations (9) and (11) hold over the entire candidate stopping domain where  $Y(t_0) > 0$ , and incorporate changes in both pure postponement flow and the option premium.

**Proposition 1. An  $r$ -percent rule under certainty.** *The benefit from delaying a profitable project is equal to the instantaneous rate of change of project value plus the instantaneous rate of change of the option premium. In an interior optimum this sum is greater than the short rate of interest,  $r$ , prior to stopping. Near stopping the rate of change of project value is greater than  $r$  (Wicksell's rule) and the rate of change in the option premium is less than zero. At stopping, (i) the rate of capital gain on the project value has fallen to  $r$ , (ii) the rate of capital loss on the option premium has risen to zero, such that (iii) the total rate of return on holding the investment opportunity has fallen to  $r$ .*

Proposition 1 is very general, holding for all perpetual opportunities with no holding costs, including those with subsequent timing options (compound options), and for all market structures. It

is a logical starting point from which to proceed to the search for a comparable  $r$ -percent rule associated with stopping under uncertainty.

### 3. Stopping under Uncertainty

The typical optimal stopping algorithm under uncertainty is to strike as soon as the (now) stochastic project payoff  $W$  reaches some endogenously determined trigger value or hitting boundary  $\hat{W}$  (Brock et al. 1989, Dixit and Pindyck 1994):

$$\hat{t}_0 = \inf \{t_0 | W(t_0) = \hat{W}\}. \quad (12)$$

The derivation of the stopping trigger is conducted in the value rather than the time domain. Even so, the optimal stopping literature has increasingly mentioned both the rate of growth of project value and opportunity cost as being of intuitive relevance in understanding stopping (e.g., Malchow-Møller and Thorsen 2005, Alvarez and Koskela 2007, Murto 2007). In this section we show that optimal stopping under uncertainty supports a rule comparable to the  $r$ -percent rule in Proposition 1. Our approach is to continue to focus on the time domain, even though the stopping algorithms under uncertainty are performed in the value domain since this is the source of randomness.

Let  $W$  be described by a density function of which the moments are assumed to be known. To facilitate closed-form solutions we represent changes in  $W$  as the one-dimensional, autonomous diffusion process in stochastic differential equation form,

$$dW = b(W(t_0))dt_0 + \sigma(W(t_0))dz, \quad (13)$$

where  $dz$  is a Wiener process.<sup>3</sup> As above,  $\tilde{Y}(W(t_0)) > 0$  is the (forward) value of a project if

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<sup>3</sup> The derivations that follow can also be conducted for the non-autonomous case. See, for example, the numerical example in Section 4, and in particular footnote 11.



irreversibly initiated at time  $t_0 \geq t$  for a known cost  $C(t_0) \geq 0$ .<sup>4</sup> Any subsequent options, including partial reversal of the stopping decision, are permitted and assumed to be priced into  $\tilde{Y}(W(t_0))$  and the program's solution  $\hat{W}$ . Once again, to simplify the derivations we assume that there are no holding costs or lost cash flows from delay. To make the problem more transparent we assume that the initiation cost is instantaneous and fixed at  $C \geq 0$ .<sup>5</sup>

Of most interest are situations that yield a random interior stopping time  $\hat{t}_0 > t$ . At candidate stopping time  $t_0$  the investment opportunity's market (option) value has the same discount factor form as the stopping problem under certainty (equation 4),

$$\tilde{\Pi}(W(t_0), \hat{W}) = E \left\{ \exp[-\rho(\hat{t}_0 - t_0)] \right\} \tilde{Y}(\hat{W}) \quad (14)$$

where  $\rho > 0$  is the appropriate constant risk-adjusted discount rate, and the expectation is taken over the uncertain time to stopping ( $\hat{t}_0 - t_0$ ) given a state value  $W(t_0)$  and the trigger value  $\hat{W}$  (Dixit et al. 1999). To reduce notational clutter we hereafter condense  $\tilde{\Pi}(W(t_0), \hat{W})$  to  $\tilde{\Pi}(W)$ ,  $\tilde{\Pi}(\hat{W}, \hat{W})$  to  $\tilde{\Pi}(\hat{W})$ ,  $\tilde{Y}(W(t_0))$  to  $\tilde{Y}(W)$ ,  $b(W(t_0))$  to  $b(W)$ ,  $\sigma(W(t_0))$  to  $\sigma(W)$ , and  $\tilde{\Pi}_{W(t_0)}(W(t_0), \hat{W})$  to  $\tilde{\Pi}'(W)$ .

Several alternative approaches to discounting the investment payoff  $\tilde{Y}(\hat{W})$  are employed in the literature. In an approximation used by practitioners and in most of the dynamic-programming-based literature,  $\rho$  is taken to be a constant risk-adjusted discount rate, as in (14) (Insley and Wirjanto

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<sup>4</sup> The notation now includes a tilde because of the change in domain from time to value.

<sup>5</sup> If investment is continuous over a finite interval,  $C$  represents the present value of the total investment if all investment must be spent once investment is irreversibly initiated, or it represents the present value of the minimum discrete lump of irreversible investment needed to initiate subsequent investment options.

2010).<sup>6</sup> Some authors either explicitly or implicitly assume risk neutrality on the part of the decision maker, in which case  $\rho = r$ , a constant risk-free rate. Others assume that the conditions for contingent claims analysis hold, using risk-adjusted expectations over the rate of drift, which can be denoted as  $b^*(W)dt_0$  in (13), and again setting  $\rho = r$ . Though our notation corresponds with the first approach, our analysis also allows for the other two approaches.<sup>7</sup>

As with the case of certainty, let

$$\tilde{\Pi}(W) - \tilde{Y}(W) \triangleq \tilde{O}(W) > 0 \quad (15)$$

be the option premium prior to stopping. For the broad class of functions  $\tilde{\Pi}(W)$  and  $\tilde{Y}(W)$  to which Ito's lemma can be applied, and given (13), the expected change in project value associated with delay is

$$\frac{E[d\tilde{Y}(W)]}{dt_0} = b(W)\tilde{Y}'(W) + \frac{1}{2}\sigma^2(W)\tilde{Y}''(W). \quad (16)$$

The expected change in option premium is

$$\frac{E[d\tilde{O}(W)]}{dt_0} = b(W)\tilde{O}'(W) + \frac{1}{2}\sigma^2(W)\tilde{O}''(W). \quad (17)$$

The fundamental valuation differential equation associated with this particular stopping problem is

$$b(W)\tilde{\Pi}'(W) + \frac{1}{2}\sigma^2(W)\tilde{\Pi}''(W) = \rho\tilde{\Pi}(W). \quad (18)$$

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<sup>6</sup> As noted in Insley and Wirjanto (2010), this assumption is likely to be correct only when  $W$  has a constant rate of volatility and where the investment opportunity has the form  $\tilde{\Pi}(W) = K_1W^{K_2}$ , where  $K_1$  and  $K_2$  are non-zero constants (see also Sick and Gamba 2010). This form is common in the stopping problem literature.

<sup>7</sup> If risk neutrality is assumed,  $\rho = r$  throughout the paper. If contingent claims analysis is assumed,  $\rho = r$  and  $b(W)$  is changed to the risk-adjusted drift  $b^*(W)$  throughout the paper.

Given

$$\frac{E[d\tilde{\Pi}(W)]}{dt_0} = b(W)\tilde{\Pi}'(W) + \frac{1}{2}\sigma^2(W)\tilde{\Pi}''(W), \quad (19)$$

equation (18) implies that

$$\frac{E[d\tilde{\Pi}(W)]}{dt_0} = \rho\tilde{\Pi}(W) : \quad (20)$$

As in the case of certainty, the market value of the investment opportunity is expected to rise at the rate of interest prior to stopping.

Benefits and costs of waiting can now be computed. Equations (15) and (20) imply that when waiting is optimal,

$$\frac{E[d\tilde{Y}(W)]}{dt_0} + \frac{E[d\tilde{O}(W)]}{dt_0} = \frac{E[d\tilde{\Pi}(W)]}{dt_0} = \rho\tilde{\Pi}(W) > \rho\tilde{Y}(W). \quad (21)$$

The benefit of delayed project initiation is expected capital gains, comprising changes in project value and the option premium. The opportunity cost of delayed initiation of the project is forgone proceeds on the investment of the realized project value  $\tilde{Y}(W)$ . For comparability across risky investment opportunities this investment opportunity must be in the same asset class as the stopping opportunity that is being delayed, which from (20) garners rate of return  $\rho$ . The total opportunity cost of waiting is then  $\rho\tilde{Y}(W)$ .

To facilitate the same comparison of costs and benefits as under certainty, let the normalized expected rate of change in the option premium be denoted by

$$\tilde{\alpha}(W) = \frac{E[d\tilde{O}(W)]}{\tilde{Y}(W)dt_0}. \quad (22)$$

From (21), (16), and (17),

$$\tilde{\alpha}(W) = \frac{E[d\tilde{\Pi}(W)]}{\tilde{Y}(W)dt_0} - \frac{E[d\tilde{Y}(W)]}{\tilde{Y}(W)dt_0} = \frac{-b(W)(\tilde{Y}'(W) - \tilde{\Pi}'(W)) + \frac{1}{2}\sigma^2(W)(\tilde{\Pi}''(W) - \tilde{Y}''(W))}{\tilde{Y}(W)}, \quad (23)$$

which, as in (10), is of indeterminate sign. Nevertheless, from (21) and (22),

$$\frac{E[d\tilde{Y}(W)]}{\tilde{Y}(W)dt_0} + \tilde{\alpha}(W) = \frac{\rho\tilde{\Pi}(W)}{\tilde{Y}(W)} > \rho. \quad (24)$$

As under certainty, prior to stopping the total expected rate of capital gain on the investment opportunity, consisting of the sum of the expected rates of change of the project value and of the option premium, exceeds the opportunity cost of waiting, the rate of interest.

At the interior free boundary the value matching and smooth pasting conditions are

$$\tilde{\Pi}(\hat{W}) = \tilde{Y}(\hat{W}) \quad (25)$$

and

$$\tilde{\Pi}'(\hat{W}) = \tilde{Y}'(\hat{W}). \quad (26)$$

From equations (24) and (25), at the stopping point  $\hat{W}$ ,

$$\frac{E[d\tilde{Y}(\hat{W})]}{\tilde{Y}(\hat{W})dt_0} + \tilde{\alpha}(\hat{W}) = \rho: \quad (27)$$

the expected rate of total capital gain on delay has fallen to the rate of interest, just as in the case under certainty. By equations (23) and (26),

$$\tilde{\alpha}(\hat{W}) = \frac{\frac{1}{2}\sigma^2(\hat{W})(\tilde{\Pi}''(\hat{W}) - \tilde{Y}''(\hat{W}))}{\tilde{Y}(\hat{W})}. \quad (28)$$

Unlike under certainty, there is no indication that  $\tilde{\alpha}(\hat{W}) = 0$  to yield Wicksell's rule in (27). In fact, analyses of stopping rules for some diffusion processes show that the expected rate of pure postponement flow,  $E[d\tilde{Y}(\hat{W})]/(\tilde{Y}(\hat{W})dt_0) - \rho$ , is negative at the stopping point (e.g., Brock et al. 1989, Mordecki 2002, Alvarez and Koskela 2007). In other cases it is zero at the stopping point (e.g.,

Clarke and Reed 1988, 1989, 1990). Hence,  $\tilde{\alpha}(\hat{W}) \geq 0$ , and under uncertainty Wicksell's rule, which relates only to pure postponement flow, can fail to explain waiting even near the stopping point.

**Numerical Example:** Consider an irreversible opportunity to invest in a stochastic stream of cash flows whose value follows a geometric Brownian motion with constant rate of drift  $b$ ,

$$dW = bWdt_0 + \sigma Wdz. \quad (13)'$$

To ensure an interior stopping point, let the required rate of return on the unlevered asset  $W$  be represented by  $u > b$ . Also let the risk-free rate be represented by  $r$ . With  $b$  constant the investment cost  $C$  must be positive to avoid bang-bang now or never stopping solutions (Brock et al. 1989). Let the forward project value be  $\tilde{Y}(W) = W - C$ .

The solution to the stopping problem yields investment opportunity value

$$\tilde{\Pi}(W) = \left(\frac{W}{\hat{W}}\right)^\beta \tilde{Y}(\hat{W}), \quad (14)'$$

$\hat{W} = (\beta / (\beta - 1))C$ , where  $\beta > 1$  is the positive root of the characteristic equation

$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + b\gamma - \rho = 0$ , and where  $\rho = r + \beta(u - r) > u$  (Dixit et al. 1999, McDonald and Siegel 1986).<sup>8</sup>

Let  $C = 1$ , and consider values of  $W > C$  where waiting is optimal. The expected rate of change in project value,  $bW / \tilde{Y}(W)$ , can be positive, negative, or zero depending on the drift parameter  $b$ . From (23) the expected rate of change in the value of the option premium is

$$\tilde{\alpha}(W) = \frac{-bW}{\tilde{Y}(W)} \left( 1 - \beta \left(\frac{W}{\hat{W}}\right)^{\beta-1} \left(\frac{\tilde{Y}(\hat{W})}{\hat{W}}\right) \right) + \frac{\frac{1}{2}\sigma^2 W^2 \beta(\beta-1)}{\tilde{Y}(W)} \left(\frac{W}{\hat{W}}\right)^{\beta-2} \left(\frac{\tilde{Y}(\hat{W})}{\hat{W}^2}\right). \quad (23)'$$

<sup>8</sup> The functional form of (14)' indicates that the use of a constant risk-adjusted discount rate is appropriate.

The total expected rate of return to delay is

$$\frac{E[d\tilde{Y}(W)]}{\tilde{Y}(W)dt_0} + \tilde{\alpha}(W) = \frac{bW}{\tilde{Y}(W)} - \frac{bW}{\tilde{Y}(W)} \left( 1 - \beta \left( \frac{W}{\hat{W}} \right)^{\beta-1} \left( \frac{\tilde{Y}(\hat{W})}{\hat{W}} \right) \right) + \frac{\frac{1}{2}\sigma^2 W^2 \beta(\beta-1)}{\tilde{Y}(W)} \left( \frac{W}{\hat{W}} \right)^{\beta-2} \left( \frac{\tilde{Y}(\hat{W})}{\hat{W}^2} \right) > \rho. \quad (24)'$$

At stopping this falls to

$$\frac{E[d\tilde{Y}(\hat{W})]}{\tilde{Y}(\hat{W})dt_0} + \tilde{\alpha}(\hat{W}) = b\beta + \frac{1}{2}\sigma^2 \beta(\beta-1) = \rho. \quad (27)'$$

Figure 1 depicts this stopping problem for specific parameter values that give  $\hat{W} = 2$ . At every point prior to stopping the expected rate of gain on the sum of the project value and option premium is greater than the discount rate. For low levels of  $W$  the expected rate of gain in the project value is greater than the discount rate, and waiting is optimal as per Wicksell's rule. However, the expected rate of change in project value is less than the discount rate for  $W > 1.56$ , and yet waiting is still optimal. More than offsetting this negative expected rate of pure postponement flow is a positive expected rate of change in the option premium, for a total expected rate of capital gain on waiting that exceeds the discount rate. The expected total rate of capital gain falls to the discount rate at  $\hat{W} = 2$ , and the program is stopped. At this point the rate of capital gain of the project value has fallen so far below the discount rate that the negative expected rate of pure postponement flow associated with additional waiting is no longer offset by the expected gain in the option premium. Given our parameterization of the problem the expected rate of pure postponement flow at stopping is

$$E[d\tilde{Y}(\hat{W})]/(\tilde{Y}(\hat{W})dt_0) - \rho = b\beta - \left( b\beta + \frac{1}{2}\sigma^2 \beta(\beta-1) \right) = -\frac{1}{2}\sigma^2 \beta(\beta-1) = -4.0\%, \text{ which is just offset}$$

by the expected rate of gain in the option premium,  $\tilde{\alpha}(\hat{W}) = 4.0\%$ .

#### 4. Comparisons of Stopping under Certainty and Uncertainty

Equations (24) and (27) present an  $r$ -percent rule that provides the intuition of stopping under uncertainty; while waiting the expected rate of gain of the two assets under the decision maker's command grows at a rate that exceeds the discount rate. The rule generalizes the rule under certainty; it can be shown that equations (24) and (27) reduce to equations (9) and (11) when  $\sigma(W) = 0$ . In each case when the program is stopped the sum of the rates of change of the values of the project and option premium has fallen to the opportunity cost of waiting, the rate of interest.

Even though the rule under uncertainty subsumes the rule under certainty, uncertainty introduces differences at and prior to stopping when  $\tilde{\alpha}(\hat{W}) > 0$ :

- 1) Under uncertainty the expected rate of change in project value at stopping is *less than* the rate of interest, whereas it is *equal to* the rate of interest under certainty;
- 2) By continuity (i.e., ruling out large jumps in the process for  $W$  in relation to  $dt_0$ ), the expected rate of change in the project value immediately prior to stopping is *less than* the rate of interest under uncertainty, whereas it is *greater than* the rate of interest under certainty; and
- 3) The expected rate of change of the option premium immediately prior to stopping is *positive* under uncertainty, whereas it is *negative* under certainty.

A positive value of  $\tilde{\alpha}(\hat{W})$  at stopping indicates that there is an additional gain to waiting under uncertainty that does not occur under certainty. In a two-period irreversible stopping problem for a diffusion process, the adjustment to the Wicksell rule has been defined as Arrow-Fisher-Hanemann-Henry (AFHH) quasi-option value (Hanemann 1989). We extend this AFHH tradition by referring to  $\tilde{\alpha}(\hat{W})$  as the rate of *quasi-option flow*. Conrad (1980) shows quasi-option value to be the expected value of information from delayed decision making given imperfect information updating in a discrete, stochastic environment. To the extent that the adjustment term  $\tilde{\alpha}(\hat{W})$  is non-zero only when

$\sigma^2(W) > 0$ , applying this same interpretation to quasi-option flow is appropriate;  $\tilde{\alpha}(\hat{W})$  is the instantaneous rate of information flow at stopping.

Fisher and Hanemann (1987) and Kennedy (1987) note that because quasi-option value cannot be estimated separately as an input to the analysis, it is not a tool of decision making. Their observation is also true of  $\tilde{\alpha}(W)$  in (24) and (27), since it is evaluated at a  $\hat{W}$  that is calculated using the usual stopping algorithm in the value domain. Nevertheless, the concept of declining expected pure postponement flow, which can be calculated in the time domain without knowing the value of  $\hat{W}$ , is useful in understanding why any perpetual stopping problem is eventually stopped.

In cases where  $\tilde{\alpha}(\hat{W}) = 0$  equations (24) and (27) replicate equations (9) and (11) even though  $\sigma(W) > 0$ . Ross (1971, Theorem 2.2a) has shown that for stopping problems that maximize the present value of a payoff contingent on a Markov process, such as the problems discussed in this paper, there is a set of sufficient conditions for which  $\tilde{\alpha}(\hat{W}) = 0$ . These conditions require that the process for  $W$  and the form of the value function  $\tilde{Y}(W)$  be such that once the expected rate of pure postponement flow,  $E[d\tilde{Y}(W)]/(\tilde{Y}(W)dt_0) - \rho$ , becomes non positive it remains non-positive thereafter. Brock et al. (1989) and Murto (2007) consider problems for which  $W$  is monotone in time (i.e., non-diffusion) and  $\tilde{Y}(W)$  is monotone in  $W$  and of a form that causes the condition to hold. Boyarchenko (2004), Clarke and Reed (1989), and Ross (1970, pp. 187-90, 1971) consider problems where the condition holds because  $\tilde{Y}(W)$  is monotone in time and of an accommodating functional form, even though  $W$  is a diffusion process. In these cases and in others where Ross's sufficiency conditions hold,

$$\frac{E[d\tilde{Y}(\hat{W}_M)]}{\tilde{Y}(\hat{W}_M)dt_0} = \rho \quad (29)$$



at stopping. Immediately prior to stopping,

$$\frac{E[d\tilde{Y}(W)]}{\tilde{Y}(W)dt_0} > \rho . \quad (30)$$

The pair (29) and (30) has been called an *infinitesimal look-ahead* stopping rule (Ross 1971), a *stochastic Wicksell* rule (Clarke and Reed 1988), and a *myopic-look-ahead* stopping rule (Clarke and Reed 1989, 1990). The subscript  $M$  in (29) and (30) denotes a ‘myopic’ stopping rule. It compares investment now with a commitment now to invest next period, as opposed to comparing investment now with an option to invest next period, as is done when Ross’s sufficiency conditions do not hold. The former comparison is optimal given that the sufficiency condition rules out the possibility of a regrettable commitment once expected pure postponement flow falls to zero. In other words, waiting beyond the stopping point produces no valuable information as to future levels of

$$E[d\tilde{Y}(W)] / (\tilde{Y}(W)dt_0) - \rho .$$

**Numerical Example:** The myopic rule is frequently mentioned in the forestry literature as the optimal harvesting rule under uncertainty. Clarke and Reed’s (1989) problem of costlessly and irreversibly harvesting a perpetually growing forest in a single rotation is an example where myopic stopping rule (29) and (30) obtains.<sup>9</sup> The example extends (13) to the non-autonomous case, where the logarithm of forest value (price times quantity) behaves according to the diffusion

$$dW = (b + g(t_0))dt_0 + \sigma dz . \quad (31)$$

Here  $b$  is the constant drift in the logarithm of the price of wood, which Clarke and Reed infer to be non-negative, and  $g$  is a deterministic time-dependent positive drift in the logarithm of forest size.

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<sup>9</sup> The main purpose of Clarke and Reed’s paper is to delimit the case where the myopic rule obtains. See Reed and Clarke (1990) for a harvesting case where the non-myopic rule obtains.

The state of the process at any time is given by the pair  $(W, t_0)$ . Growth in the forest size is decreasing in  $t_0$  and satisfies  $g(\infty) < r - b - \frac{1}{2}\sigma^2 < g(0)$ , where  $r$  is the constant risk-free discount rate.<sup>10</sup> Since project value is  $\tilde{Y}(W) = e^W$ , the expectation over project growth is  $E[d\tilde{Y}(W)]/(\tilde{Y}(W)dt_0) = b + g(t_0) + \frac{1}{2}\sigma^2$  by (31) and Ito's lemma.

The relationship  $r - b - \frac{1}{2}\sigma^2 < g(0)$  ensures an interior solution since the expected rate of growth of project value at  $t = 0$  is  $b + g(0) + \frac{1}{2}\sigma^2 > r$ . Waiting is optimal because of a positive expected rate of pure postponement flow. Since  $g(t_0)$  is monotonically declining,  $E[d\tilde{Y}(W)]/(\tilde{Y}(W)dt_0) - r$  will remain negative once it becomes negative, and stopping rule (29) and (30) applies. Waiting continues until  $E[d\tilde{Y}(W)]/(\tilde{Y}(W)dt_0)$  falls to  $r$ , at which point  $b + g(\hat{t}_0) + \frac{1}{2}\sigma^2 = r$ . This is solved for a deterministic  $\hat{t}_0$ .

To see that (27) reduces to (29) in this problem, the time  $t_0$  value of the investment opportunity is

$$\tilde{\Pi}(W, t_0) = e^{-r(\hat{t}_0 - t_0)} E_{t_0} \left[ e^{\hat{W}} \right] = e^{-r(\hat{t}_0 - t_0)} e^W \exp \left[ \int_{t_0}^{\hat{t}_0} \left[ b + g(s) + \frac{1}{2}\sigma^2 \right] ds \right]. \quad (14)''$$

At the optimal stopping time  $\hat{t}_0$ ,  $\tilde{\Pi}(W, \hat{t}_0) = e^W$ ,  $\tilde{\Pi}_{WW}(W, \hat{t}_0) = \tilde{Y}''(W)$ , and  $\tilde{\alpha}(W, \hat{t}_0) = 0$ .<sup>11</sup>

<sup>10</sup> Clarke and Reed conduct the analysis under risk-neutrality and under risk aversion with isoelastic utility of value. Both allow for a constant discount rate, as does the form of the option value, per Insley and Wirjanto (2010). We relate Clarke and Reed's risk-neutral analysis.

<sup>11</sup> In this case, with time being one of the arguments in the value function, the derivation of the rate of drift in the option premium leads to  $\tilde{\alpha}(W, t_0) = \frac{-(b + g(t_0))(\tilde{Y}'(W) - \tilde{\Pi}_W(W, t_0)) - \tilde{\Pi}_{t_0}(W, t_0) + \frac{1}{2}\sigma^2(W)(\tilde{\Pi}_{WW}(W, t_0) - \tilde{Y}''(W))}{\tilde{Y}(W)}$ . Time also

In this example uncertainty is not responsible for additional delay beyond that advised by positive expected pure postponement flow, and does not change the intuition of stopping that carries over from the case of certainty. It only influences the level of  $E[d\tilde{Y}(W)]/(\tilde{Y}(W)dt_0)$  and the time to stopping via the Ito adjustment  $\frac{1}{2}\sigma^2$ .

If the price of wood in this example is instead modeled as mean reverting, as is common in the forestry literature (e.g., Brazee and Mendelsohn 1988),  $b$  becomes state dependent (on price) and will be negative at high prices. Mean reverting processes yield the non-myopic stopping rule due to non-monotonicity. Though there are no closed-form solutions for  $\tilde{Y}$  and  $\tilde{\Pi}$  when price is mean reverting, the intuition of equations (24) and (27) yields the possibility that forests whose rate of growth in volume is greater than the discount rate could be optimally harvested immediately if prices are high and expected to revert sufficiently rapidly to a lower equilibrium. This is because expected growth in value would then be so negative as to outweigh expected growth in the option premium associated with uncertainty.

The discussion thus far supports the following proposition and corollary.

**Proposition 2: Non-myopic  $r$ -percent rule under uncertainty.** *For all stochastic processes defined in (13), all market structures, and all project values  $\tilde{Y}(W)$  and investment opportunity values  $\tilde{\Pi}(W)$  to which Ito's lemma applies, if the optimal stopping point is an interior solution the expected rate of return from waiting to invest is equal to the rate of interest at that stopping point.*

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introduces the additional optimality condition  $\tilde{\Pi}_{t_0}(W, \hat{t}_0) = 0$  (Clarke and Reed 1989, p. 579). This, along with smooth

pasting in value, gives  $\tilde{\alpha}(W, \hat{t}_0) = \frac{\frac{1}{2}\sigma^2(W)(\tilde{\Pi}_{ww}(W, \hat{t}_0) - \tilde{Y}''(W))}{\tilde{Y}(W)} = 0$ .

*The expected rate of return from waiting to invest is the sum of the expected rate of change in the project value and the expected rate of change in the option premium. The latter is a rate of quasi-option flow associated with irreversibility of the investment. Prior to the stopping point the total rate of return from waiting to invest, the sum of positive or negative capital gains on project value and positive or negative capital gains on the option premium, is expected to exceed the rate of interest.*

**Corollary 1. Myopic  $r$ -percent rule under uncertainty.** *When the expected rate of change of pure postponement flow remains non-positive once it becomes non-positive, at stopping the expected rate of change in the project value falls to the rate of interest and the expected rate of change in the option premium rises to zero, as in the case under certainty.*

In a unified theory of optimal stopping, the  $r$ -percent rule of Proposition 1 is a special case of the myopic  $r$ -percent rule of Corollary 1, which is a special case of the non-myopic  $r$ -percent rule of Proposition 2.

## **5. Discussion: Theoretical Issues**

There are theoretical and practical benefits to seeing stopping under uncertainty as a comparison of opportunity costs and benefits akin to the problem under certainty. We re-examine several common notions about stopping under uncertainty in the light of Propositions 1 and 2 and Corollary 1.

### *A. The Distinction Between Uncertainty and Pure Postponement Flow in Driving Delay*

Many analyses of optimal stopping in the academic and practitioner literature suggest that in the absence of uncertainty the traditional NPV rule, to stop whenever  $\tilde{Y}(W) > 0$ , is optimal. Section 2

shows that this rule is a corner solution, and that interior solutions are possible when there is positive pure postponement flow.

Another frequent claim is that delayed stopping is “driven by uncertainty.” If the myopic rule (29) and (30) is optimal, however, our findings for the case of certainty carry over: positive pure postponement flow, not uncertainty, is the reason for delay.

We return to the non-myopic problem depicted in Figure 1 to bring out a case where uncertainty does drive delay. Even here, though, positive pure postponement flow has a role in delaying stopping. Let  $\hat{W}_M < \hat{W}$  be the stopping trigger associated with stopping rule (29) and (30). In Figure 1  $\hat{W}_M = 1.56$  and  $\hat{W} = 2$ . On the interval  $[\hat{W}_M, \hat{W})$ , since the expected rate of pure postponement flow is zero or negative, uncertainty is the cause of delayed investment. There is a value  $\underline{W} < \hat{W}_M$ , however, such that on the interval  $(\underline{W}, \hat{W}_M)$  the expected rate of pure postponement flow is positive and waiting can be explained by the expected rate of gain in project value being greater than the discount rate. Hence, in non-myopic stopping problems uncertainty is the sole reason for delay only on a subset of value space near the stopping point. Just as with Wicksell’s analysis under certainty, a more complete  $r$ -percent rule is needed to explain waiting over the entire range of  $W$ .

### *B. The Impact of Increasing Uncertainty on the Stopping Trigger*

Does uncertainty always create a stricter investment hurdle  $\tilde{Y}(\hat{W})$ ? The myopic stopping rule (29) and (30) and non-myopic rule (24) and (27) show that the answer is “no” in risk-averse settings. Increasing uncertainty is usually held to increase  $E[d\tilde{Y}(W)]/dt_0$  and, where applicable,  $\tilde{\alpha}(\hat{W})$  via non-zero second order terms multiplying  $\sigma^2$  in (16) and (28). This will require an increase in  $\tilde{Y}(\hat{W})$  such that the stopping rule still holds. Yet increasing uncertainty can also increase  $\rho$ . Even where a

contingent claims analysis is warranted and the rate of interest is not affected by the increased uncertainty, there will be an adjustment to the risk-adjusted expectation over the rate of drift of  $W$ , impacting both  $E[d\tilde{Y}(\hat{W})]/dt_0$  and  $\tilde{\alpha}(\hat{W})$  in a way that results in an indeterminate adjustment to  $\tilde{Y}(\hat{W})$ .

For the investment option depicted in Figure 1, in the limit as uncertainty goes to zero the discount rate on the asset falls to the riskless rate, 6%;  $\hat{W}$  rises from 2 to 6 and  $\tilde{Y}(\hat{W})$  rises from 1 to 5. For  $2 \leq W < 6$  the program is continued under certainty, whereas under uncertainty it is stopped immediately. Sarkar (2003), Lund (2005), and Wong (2007) show for specific cases that uncertainty may increase or decrease the investment hurdle. The emphasis on the interest rate in the myopic and non-myopic  $r$ -percent rules shows that the indeterminacy is general, possibly explaining why empirical tests of irreversible investment behavior have not found a strong statistical relationship between the level of uncertainty and the stopping point (e.g., Hurn and Wright 1994, Holland et al. 2000, Moel and Tufano 2002).

### *C. Ranking of Projects*

Following the notion under certainty that higher discount rates are a result of higher opportunity costs of waiting (Stiglitz 1976), one might naturally propose an ordering of the timing of investment under uncertainty by a ranking of heterogeneous project discount rates. Given equal paths of expected project growth, those projects with the higher discount rate should be brought on line first, per Wicksell's rule. But (27) shows that heterogeneity in the rate of growth of the option premium,  $\tilde{\alpha}(\hat{W})$ , can also come into play in timing entry. It varies with the non-linearity of the underlying project and the nature of the options available to the project manager, as described by  $\tilde{Y}_i''(\hat{W})$  and  $\tilde{\Pi}_i''(\hat{W})$  for project  $i$ . The determination of investment timing can thus be the outcome of a complex

sectorial equilibrium involving price paths, interest rates, and the expected value of information flow, with expected rates of capital gain in project value from waiting being compared against  $(\rho_i - \tilde{\alpha}_i(\hat{W}))$  rather than  $\rho_i$ . Stopping condition (27) is endogenous to that equilibrium for certainty and uncertainty, perfect competition and imperfect competition, and indeed for all assets where the assumptions in Section 3 are satisfied.

#### *D. Backwardation*

Since  $E[d\tilde{Y}(W)] / (\tilde{Y}(W)dt_0) < \rho$  in some settings, stopping condition (27) supports the notion that irreversible projects must at some point exhibit a rate-of-return shortfall, or weak backwardation, to be forthcoming under uncertainty (Davis and Cairns 1999). Litzenberger and Rabinowitz (1995) find that weak backwardation in oil markets is an equilibrium condition that induces irreversible production from existing reservoirs. Our analysis complements theirs, showing that price backwardation serves to induce irreversible actions by sufficiently reducing the expected rate of pure postponement flow. During high prices there is greater price backwardation (Litzenberger and Rabinowitz 1995), which from (24) and (27) increases the incentive to produce due to the decreased expected rate of pure postponement flow. During low prices, price contango decreases the incentive to produce due to an increased expected rate of pure postponement flow.

The importance of backwardation is related to the increasing realization that mean reversion in asset values is a likely equilibrium process affecting real investment decisions (Tsekrekos 2010). The  $r$ -percent rules remind us that negative pure postponement flow in backwardated markets and positive pure postponement flow in contango markets are important factors in adjudicating the timing of investment.

## 6. Discussion: Issues in Practice.

Practitioners have been slow to adopt explicit optimal stopping algorithms when timing investment decisions (Triantis 2005). Timing triggers are presented in economics and finance as upper or lower boundaries on  $W$  or project value  $\tilde{Y}$  or as a present value index  $\tilde{Y}/C$  (e.g., Moore 2000). Copeland and Antikarov (2005, 33) note that relying on such presentations is unsatisfactory: “The academic literature about real options contains what, from a practitioner’s point of view, is some of the most outrageously obscure mathematics anywhere in finance. Who knows whether the conclusions are right or wrong? How does one explain them to the top management of a company?”

One does not use what one does not understand. The representation of stopping under uncertainty as an  $r$ -percent rule provides a link to the understanding that many practitioners already have from related timing rules under certainty. For example, the intuitive attractiveness of pure postponement flow is already leading to comparisons of expected change in project value with the opportunity cost of capital when deciding when to develop a mine or harvest a stand of trees (e.g., Torries 1998, pp. 44, 75, Yin 2001, pp. 480). Reed and Clarke (1990) suggest that such comparisons will lead to value destruction of no more than 2% in forestry applications. Empirically, land owners also appear to recognize and time development according to the rate of expected pure postponement flow (Arnott and Lewis 1979, Holland et al. 2000). While the mathematics of stopping under uncertainty will in most cases require that stopping be calculated in the value domain, the  $r$ -percent rules presented here show that it is not unreasonable for practitioners to compare expected rate of project growth with the discount rate when timing stopping under uncertainty; pure postponement flow is an integral part of the intuitions for stopping, and the time at which it falls to zero provides a lower bound on the optimal stopping time. They also show that a comparison of the expected rate of change in project value with the opportunity cost of capital will not always yield sufficient patience. The adjustment for



the rate of increase in the option premium, however, is a generalization of a rate-of-return rule involving opportunity cost of capital that is already widely used and understood, rather than a completely new way of viewing stopping.

## 7. Conclusions

While the mathematics of optimal stopping under uncertainty is well developed, the economic conceptualization of the stopping rule is not. In this paper we present the economics of optimal stopping under certainty and uncertainty for a common class of stopping problems. We use the concept of an “ $r$ -percent stopping rule” to show that a deferrable action is taken only once the total expected rate of return from waiting to act falls to the rate of interest. The first part of that total expected return is the expected rate of growth of project value. This tends to be underemphasized in explanations of waiting under uncertainty. The second part is an expected change in the option premium. This tends to be underemphasized in explanations of waiting under certainty.

Under uncertainty waiting can sometimes continue beyond the point where the expected rate of growth of the project falls to the rate of interest. The intuition here is that there is a rate of information flow associated with waiting. Building on notions from resource economics, we call this quasi-option flow.

Because the stopping rule under uncertainty is a generalization of the rule under certainty, the theory of investment under uncertainty is an incremental generalization of, not a qualitative break from, the traditional theory of investment under certainty. The weighing of opportunity costs and benefits of waiting, which include comparisons of capital gains against the force of interest, are as germane to stopping problems under uncertainty as they are under certainty. For the type of stopping

problem examined in this paper, the stopping rule under certainty is simply the limiting case of uncertainty as volatility goes to zero.

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## References

- Alvarez, L.H.R., Koskela, E., 2007. Optimal harvesting under resource stock and price uncertainty. *Journal of Economic Dynamics and Control* 31, 2461-2485.
- Arnott, R.J., Lewis, F.D., 1979. The transition of land to urban use. *Journal of Political Economy* 87:1, 161-169.
- Boyarchenko, S., 2004. Irreversible decisions and record-setting news principles. *American Economic Review* 94:3, 557-568.
- Brazeo, R., Mendelsohn, R., 1988. Timber harvesting with fluctuating prices. *Forest Science* 34:2, 359-372.
- Brock, W.A., Rothschild, M., Stiglitz, J.E., 1989. Stochastic capital theory, in: Feiwel, G.R. (Ed.), *Joan Robinson and Modern Economic Theory*. New York University Press, New York, pp. 591-622.

- Clarke, H.R., Reed, W.J., 1988. A stochastic analysis of land development timing and property valuation. *Regional Science and Urban Economics* 18, 357-381.
- \_\_\_\_\_, 1989. The tree-cutting problem in a stochastic environment: the case of age-dependent growth. *Journal of Economic Dynamics and Control* 13, 569-595.
- \_\_\_\_\_, 1990. Applications of optimal stopping in resource economics. *Economic Record* (September), 254-265.
- Conrad, J.M., 1980. Quasi-option value and the expected value of information. *Quarterly Journal of Economics* 94:4, 813-820.
- Copeland, T.E., Antikarov, V., 2005. Real options: meeting the Georgetown challenge. *Journal of Applied Corporate Finance* 17:2, 32-51.
- Davis, G.A., Cairns, R.D., 1999. Valuing petroleum reserves using current net price. *Economic Inquiry* 37:2, 295-311
- Dixit, A., Pindyck, R.S., 1994. *Investment Under Uncertainty*. Princeton University Press, Princeton.
- Dixit, A., Pindyck, R.S., Sødal, S., 1999. A markup interpretation of optimal investment rules. *Economic Journal* 109 (April), 179-189.
- Faustmann, M., 1995. Calculation of the value which forest land and immature stands possess for forestry. *Journal of Forest Economics* 1, 7-44 (originally published 1849).
- Fisher, A.C., Hanemann, W.M., 1987. Quasi-option value: some misconceptions dispelled. *Journal of Environmental Economics and Management* 14, 183-190.
- Fisher, I., 1907. *The Rate of Interest*, MacMillan, New York.
- Hanemann, W.M., 1989. Information and the concept of option value. *Journal of Environmental Economics and Management* 16, 23-27.

- Holland, S.A., Ott, S.H., Riddiough, T.J., 2000. The role of uncertainty in investment: an examination of competing investment models using commercial real estate data. *Real Estate Economics* 28, 33-64.
- Hurn, A.S., Wright, R.E., 1994. Geology or economics? Testing models of irreversible investment using North Sea oil data. *Economic Journal* 104, 363-371.
- Insley, M.C., Wirjanto, T.S., 2010. Contrasting two approaches in real options valuation: contingent claims versus dynamic programming. *Journal of Forest Economics* 16:2, 157-176.
- Kennedy, J.O.S., 1987. Uncertainty, irreversibility and the loss of agricultural land: a reconsideration. *Journal of Agricultural Economics* 38, 78-80.
- Litzenberger, R.H., Rabinowitz, N., 1995. Backwardation in oil futures markets: theory and empirical evidence. *Journal of Finance* 50:5, 1517-1545.
- Lund, D., 2005. How to analyze the investment-uncertainty relationship in real options models? *Review of Financial Economics* 14, 311-322.
- Malchow-Møller, N., Thorsen, B.J., 2005. Repeated real options: optimal investment behaviour and a good rule of thumb. *Journal of Economic Dynamics and Control* 29, 1025-1041, with Corrigendum, *Journal of Economic Dynamics and Control* 30, 889.
- Marglin, S.A., 1963. *Approaches to Dynamic Investment Planning*. North-Holland, Amsterdam.
- McDonald, R.L., Siegel, D., 1986. The value of waiting to invest. *Quarterly Journal of Economics* 101:4, 707-728.
- Mensink, P., Requate, T., 2005. The Dixit-Pindyck and the Arrow-Fisher-Hanemann-Henry option values are not equivalent: a note on Fisher (2000). *Resource and Energy Economics* 27, 83-88.

- Moel, A., Tufano, P., 2002. When are real options exercised? An empirical study of mine closings. *Review of Financial Studies* 15:1, 35-64.
- Moore, W.T., 2000. The present value index and optimal timing of investment. *Financial Practice and Education* 10:2, 115-120.
- Mordecki, E., 2002. Optimal stopping and perpetual options for Lévy processes. *Finance and Stochastics* 6, 473-493.
- Murto, P., 2007. Timing of investment under technological and revenue-related uncertainties. *Journal of Economic Dynamics and Control* 31, 1473-1497.
- Reed, W.J., Clarke, H.R., 1990. Harvest decisions and asset valuation for biological resources exhibiting size-dependent stochastic growth. *International Economic Review* 31:1, 147-169.
- Ross, S.M., 1970. *Applied Probability Models with Optimization Applications*. Holden-Day, San Francisco.
- \_\_\_\_\_, 1971. Infinitesimal look-ahead stopping rules. *Annals of Mathematical Statistics* 42:1, 297-303.
- Sarkar, S., 2003. The effect of mean reversion on investment under uncertainty. *Journal of Economic Dynamics and Control* 28, 377-396.
- Sick, G., Gamba, A., 2010. Some important issues involving real options: an overview. *Multinational Finance Journal* 14:3/4, 157-207.
- Stiglitz, J.E., 1976. Monopoly and the rate of extraction of exhaustible resources. *American Economic Review* 66:4, 655-661.
- Torries, T.F., 1998. *Evaluating Mineral Projects: Applications and Misconceptions*. Society for Mining, Metallurgy, and Exploration, Inc, Littleton, CO.

- Triantis, A., 2005. Realizing the potential of real options: does theory meet practice? *Journal of Applied Corporate Finance* 17:2, 8-15.
- Tsekrekos, A.E., 2010. The effect of mean reversion on entry and exit decisions under uncertainty. *Journal of Economic Dynamics and Control* 34, 725-742.
- Wicksell, K., 1938. *Lectures on Political Economy, Vol. 1*. Macmillan, New York (originally published 1901).
- Wong, K.P., 2007. The effect of uncertainty on investment timing in a real options model. *Journal of Economic Dynamics and Control* 31, 2152-2167.
- Yin, R., 2001. Combining forest-level analysis with options valuation approach: a new framework for assessing forestry investment. *Forest Science* 74:4, 475-483.

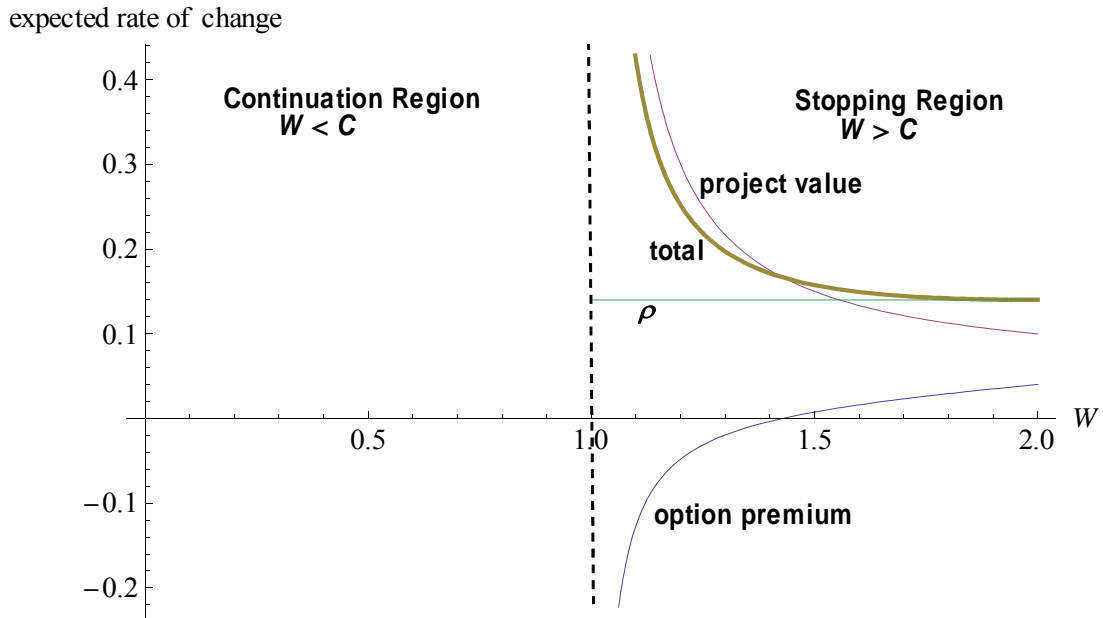


Figure 1: The  $r$ -percent rule for geometric Brownian motion, comparing the expected rates of growth of the project value, the option premium, and the total investment opportunity over the candidate stopping region  $\tilde{Y}(W) > 0$ . Investment cost  $C = 1$ . The risk-free rate  $r = 0.06$ , required rate of return  $u = 0.10$  on the unlevered asset  $W$ , drift parameter  $b = 0.05$ , and volatility parameter  $\sigma = 0.20$  give a risk-adjusted discount rate  $\rho = 0.14$ . In this example  $\hat{W} = 2$  and  $\hat{W}_M = 1.56$ .