Super-Resolution of Complex Exponentials from Modulations with Unknown Waveforms
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Super-Resolution

Enhancing the resolution limit of sensing systems
- single-molecule microscopy
- medical imaging
- radar imaging
- astronomy

Motivation
1. Super-resolution with unknown point spread functions
   \[ y(t) = \sum_{j=1}^{J} c_j \delta(t - \tau_j) * g_j(t) \]
2. Parameter estimation in radar imaging
   \[ y(t) = \sum_{j=1}^{J} c_j e^{2i\pi n_j t} g_j(t) \]
3. 3D single-molecule microscopy
4. Non-stationary blind deconvolution of seismic data

New Model
Consider the observation model:
\[ y(n) = \sum_{j=1}^{J} c_j e^{2i\pi n_j t_j} g_j(n), \quad n = -2M, \ldots, 2M. \]
Given samples \( \{y(n)\}, \) the goal is to
- super-resolve \( \{\tau_j\} \)
- recover \( \{c_j\} \)
- recover samples of the unknown waveforms \( \{g_j(n)\} \)

This problem is severely ill-posed
- number of samples \( N := 4M + 1 \)
- number of unknowns \( JN + 2J \)

Subspace Model and Atomic Norm Minimization
- A subspace model for \( g_j \)
  \[ g_j = B h_j, \quad B = [b_{-2M}, \ldots, b_{2M}]^H, \quad b_n \in \mathbb{C}^{K \times 1} \]
- Rewrite the observation
  \[ y(n) = \sum_{j=1}^{J} c_j a(\tau_j)^H e_n b_n^H h_j \]
  \[ = \langle \sum_{j=1}^{J} c_j h_j a(\tau_j)^H, b_ne_n^H \rangle \]
  \[ = \langle X_o, b_n^H e_n^H \rangle \]
where \( a(\tau) = [e^{2i\pi(-2M)t} \ldots 1 \ldots e^{2i\pi(2M)t}]^T. \)
- Lift the non-convex problem into a convex program
  Define the atomic norm associated with the set of atoms
  \( A = \{ h a(\tau)^H : \tau \in [0, 1], \|h\|_2 = 1, h \in \mathbb{C}^{K \times 1} \} \)
  \[ \|X\|_A = \inf \{ t > 0 : X \in tconv(A) \} \]
  \[ = \inf_{c_k, a_k, \|h\|_2 = 1} \left\{ \sum_k |c_k| : X = \sum_k c_k h_k a(\tau_k)^H \right\}. \]
We solve
\[
\begin{align*}
\text{minimize} & \quad \|X\|_A \\
\text{subject to} & \quad y(n) = \langle X, b_n^H e_n^H \rangle, \quad n = -2M, \ldots, 2M.
\end{align*}
\]
Denoting \( q(\tau) = \sum_{n=-2M}^{2M} \lambda(n) e^{2i\pi n \tau} h_n \) as the dual polynomial with \( \lambda \) being the dual optimizer, \( \{\tau_j\} \) are localized by selecting out the corresponding values of \( \tau \) such that \( \|q(\tau)\|_2 = 1. \)

Main Result
If the following conditions are satisfied,
\[ 1. \Delta_2 = \min_{k \neq j} |\tau_k - \tau_j| \geq \frac{1}{3\pi}, \quad M \geq 64, \]
2. \( b_n \) are i.i.d. samples from a distribution \( F \) satisfying
   i) \( \mathbb{E}[b b^H] = I_K; \)
   ii) \( \max_{1 \leq p \leq K} |b(p)|^2 \leq \mu \) for \( b \in F, \)
3. \( h_j \) drawn i.i.d. from the uniform distribution on the complex unit sphere \( \mathbb{C}^{K-1}, \)
are satisfied, then there exists some some \( C \) such that
\[ M \geq C \mu K \log \left( \frac{MJK}{\delta} \right) \frac{\log^2 \left( \frac{MK}{\delta} \right)}{\delta} \]
is sufficient to guarantee that we can recover \( X_o \) with probability at least \( 1 - \delta. \)

Numerical Simulations
1. A simple example
   - we use CVX to solve the optimization problem (SDP)
   - set \( N = 64, \quad J = 3 \) and \( K = 4, \)
   - randomly generate the locations of \( J \) spikes on \([0, 1]\) under the minimum separation condition \( \Delta_2 = \frac{1}{3\pi} \)
   - build \( B \) with entries generated randomly from the standard Gaussian distribution
   - \( h_j \) is also generated using i.i.d. real Gaussian random variables and is then normalized

2. Phase transition

3. A practical example
   - set \( J = 3 \) and generate the locations of \( \{\tau_j\} \) uniformly at random between 0 and 1 under the minimum separation \( \Delta_2 = \frac{1}{3\pi} \)
   - \( g_j(n) \) are samples of the Gaussian waveform \( g_j(t) = \frac{1}{\sqrt{2\pi\sigma_j^2}} e^{-\frac{t^2}{2\sigma_j^2}} \)
   - with unknown variance \( \sigma_j^2 \in [0, 1] \)
   - \( B \) is a rank-5 approximation of the dictionary \( D_g, \)

\[ D_g = [g_{\sigma_j=0.1}, g_{\sigma_j=0.11}, g_{\sigma_j=0.12}, \ldots, g_{\sigma_j=0.1}] \]

Reference