Homework #3 Solution

The simplest form for waterflood of a linear core initially saturated with oil and at residual water saturation, without gravity or capillary effects, is represented by the 1-dimensional Buckley-Leverett material balance equation,

$$\frac{\partial S_w}{\partial t} = -u_t \frac{\partial f_w}{\partial x} \tag{1}$$
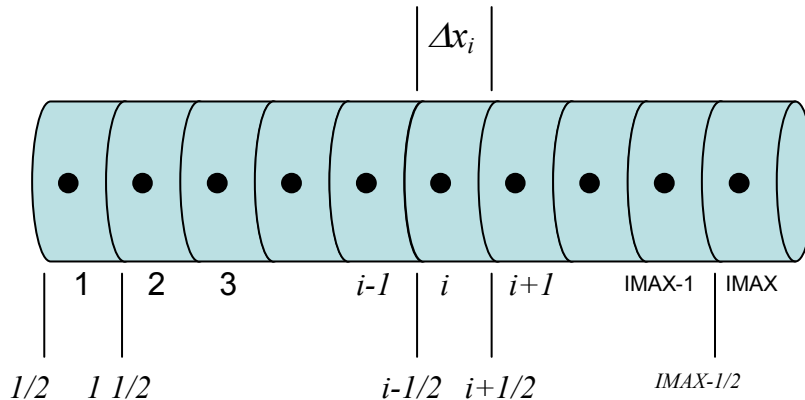
Explicit solution

Considering the space discretization shown in Figure 1, the explicit solution of this equation is given by,

$$\frac{S_{w,i}^{n+1} - S_{w,i}^{n}}{\Delta t} = -u_t \frac{f_{w,i}^{n} - f_{w,i-1}^{n}}{\Delta x} \tag{2}$$

To improve the computation process and avoid using an imaginary grid block at the left of the first node, the following equation is used for the first node,

$$\frac{S_{w,1}^{n+1} - S_{w,1}^{n}}{\Delta t} = -u_t \frac{f_{w,1}^{n} - f_{w,\frac{1}{2}}^{n}}{\Delta x/2} \tag{3}$$

These equations (Eqs. 2 and 3) can be solved explicitly for water saturation at the new time step (n+1).

Figure 1. 1-dimensional core space discretization

Implicit solution

Considering the space discretization shown in Figure 1, the implicit solution of equation (1) is given by,

$$\frac{S_{w,i}^{n+1} - S_{w,i}^{n+1}}{\Delta t} = -u_t \frac{f_{w,i}^{n+1} - f_{w,i-1}^{n+1}}{\Delta x} \tag{4}$$

The fractional flow at time $n+1$ can be related to the known fractional flow at time n using the Taylor polynomial of first order,

$$f_{w,i}^{n+1} = f_{w,i}^{n} + \left(\frac{\partial f_w}{\partial S_w}\right)_i^{n} \left(S_{w,i}^{n+1} - S_{w,i}^{n}\right) \tag{5}$$

Substituting (5) into (4) and reordering, an equation with two unknowns (Sw's at n+1) is obtained,

$$\left(\frac{\partial f_w}{\partial S_w}\right)_{i-1}^{n} S_{w,i-1}^{n+1} - \left[\left(\frac{\partial f_w}{\partial S_w}\right)_i^{n} + \frac{\Delta x_i}{u_t \Delta t}\right] S_{w,i}^{n+1} = -\left[\left(\frac{\partial f_w}{\partial S_w}\right)_i^{n} + \frac{\Delta x_i}{u_t \Delta t}\right] S_{w,i}^{n} + \left(\frac{\partial f_w}{\partial S_w}\right)_{i-1}^{n} S_{w,i-1}^{n} + f_{w,i}^{n} - f_{w,i-1}^{n}$$

$$\tag{6}$$

For node 1, this equation can be solve explicitly as follows,

$$S_{w,1}^{n+1} = \frac{\left\{-\left[\left(\frac{\partial f_w}{\partial S_w}\right)_1^{n} + \frac{\Delta x_1/2}{u_t \Delta t}\right] S_{w,1}^{n} + \cancel{\left(\frac{\partial f_w}{\partial S_w}\right)_{1/2}^{n} S_{w,1/2}^{n}} + f_{w,1}^{n} - f_{w,1/2}^{n} - \cancel{\left(\frac{\partial f_w}{\partial S_w}\right)_{1/2}^{n} S_{w,1/2}^{n+1}}\right\}}{-\left[\left(\frac{\partial f_w}{\partial S_w}\right)_1^{n} + \frac{\Delta x_1/2}{u_t \Delta t}\right]} \tag{7}$$

where,

$$S_{w,1/2}^{n} = S_{rw}$$

$$f_{w,1/2}^{n} = 1$$

$$\left(\frac{\partial f_w}{\partial S_w}\right)_{1/2}^{n} = 0$$

A back-substitution procedure can be applied to sequentially solve for the rest of the node saturations, using equation (6).

Both, the explicit and implicit formulations were coded in MATLAB to solve the 1-dimensional Buckley-Leverett problem. The procedure to solve analytically for the derivative of fractional flow with respect to water saturation is shown at the end of this report as an addendum.

Solution for Case 1: Explicit solution, Δt = 0.1 day, IMAX = 10, Δx = 10 ft

Table 1 shows the values of water saturation for the 10 grid blocks at 10 days. A plot of water saturation profile as a function of distance along path of flow at every five time steps is shown in Figure 2. The initial condition (at t = 0) corresponds to the horizontal line of constant water saturation equal 0.25 (i.e., S_{wr}). The figure shows an increase of water saturation with time and how the water injection front propagates along the 1-dimensional core. The spread of the curves is an indicative of numerical dispersion.

Table 1. Water saturation values for the 10 grid blocks at 10 days

| Grid block index	1	2	3	4	5	6	7	8	9	10
Sw (fraction)	0.68313	0.66975	0.66058	0.65302	0.64641	0.64045	0.63497	0.62986	0.62503	0.62045
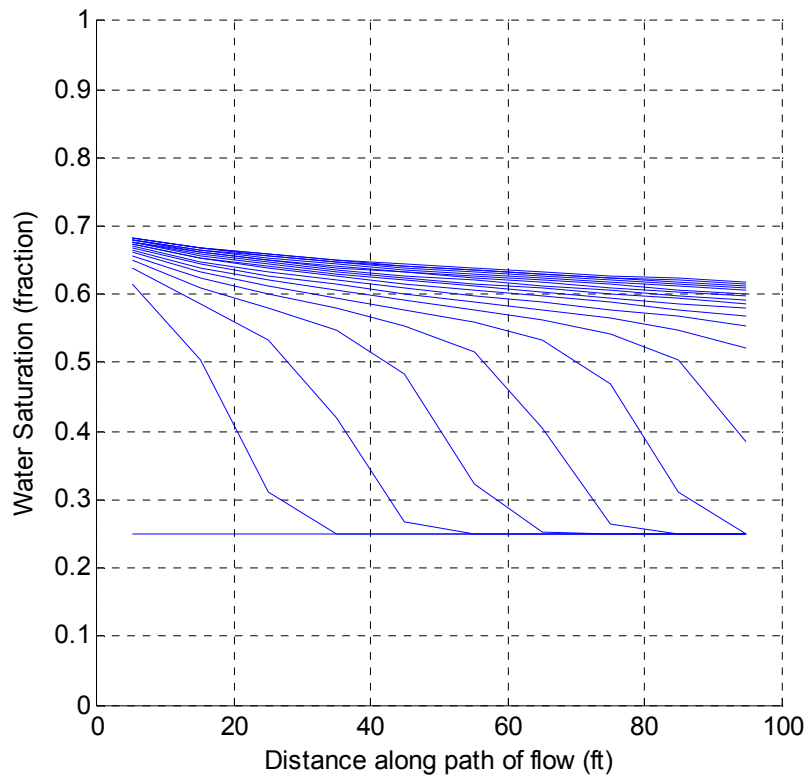
Figure 2. Water saturation profile along the path of flow at every five time steps

Solution for Case 2: Explicit solution, Δt = 0.1 day, IMAX = 20, Δx = 5 ft

Table 2 shows the values of water saturation for the 20 grid blocks at 10 days for Case 2. A plot of water saturation profile as a function of distance along path of flow at every five time steps is show in Figure 3. This case uses a small Δx than Case 1 and more nodes; it can be observe a sharp water injection front in comparison with the spread curves of Case 1, which means a decrease of numerical dispersion with a decrease of grid block size in the *x* direction.

Table 2. Water saturation values for the 10 grid blocks at 10 days. Case 2

Grid block index	1	2	3	4	5	6	7	8	9	10
Sw (fraction)	0.6894	0.6807	0.6746	0.6695	0.6650	0.6609	0.6572	0.6537	0.6504	0.6473

| Grid block index	11	12	13	14	15	16	17	18	19	20
Sw (fraction)	0.6444	0.6415	0.6388	0.6362	0.6336	0.6312	0.6288	0.6264	0.6242	0.6220
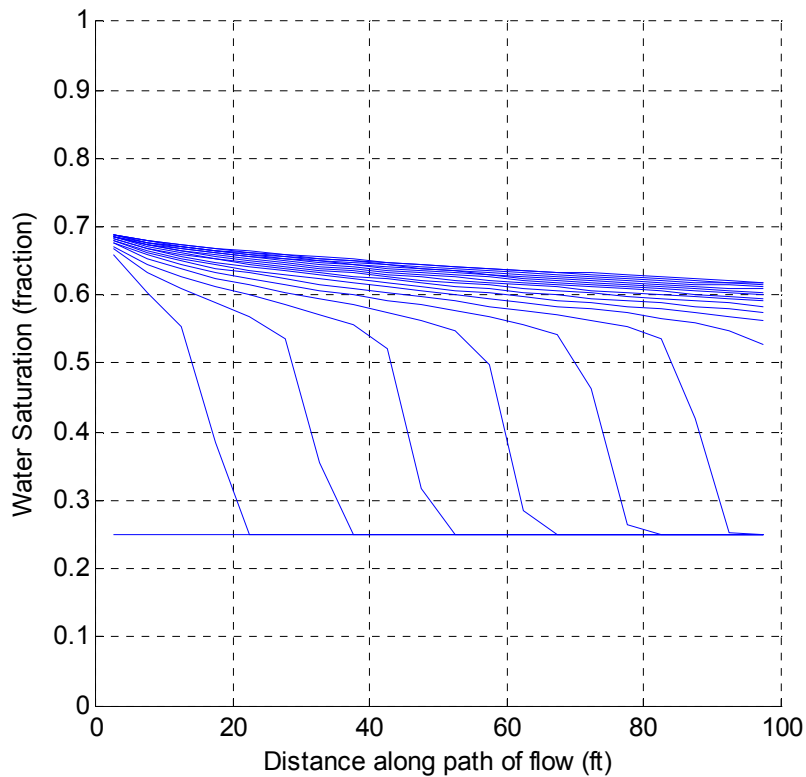
Figure 3. Water saturation profile along the path of flow at every five time steps. Case 2

Solution for Case 3: Implicit solution, Δt = 0.1 day, IMAX = 10, Δx = 10 ft

Table 3 shows the values of water saturation for the 10 grid blocks at 10 days of Case 3. A plot of water saturation profile as a function of distance along path of flow at every five time steps is show in Figure 4. It can be observe the water injection front propagation with time along the 1-dimensional core in the form of spread saturation curves, because of numerical dispersion.

Table 3. Water saturation values for the 10 grid blocks at 10 days. Case 3

| Grid block index	1	2	3	4	5	6	7	8	9	10
Sw (fraction)	0.6823	0.6684	0.6589	0.6511	0.6442	0.6381	0.6324	0.6271	0.6221	0.6173
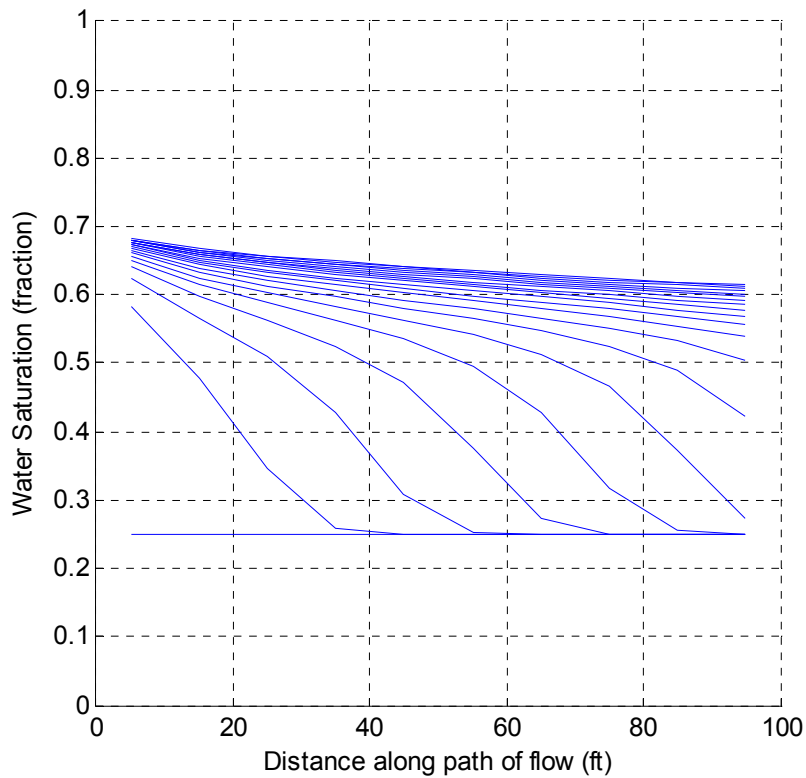
Figure 4. Water saturation profile along the path of flow at every five time steps. Case 3

Solution for Case 4: Implicit solution, Δt = 0.1 day, IMAX = 20, Δx = 5 ft

Table 4 shows the values of water saturation for the 20 grid blocks at 10 days for Case 4. A plot of water saturation profile as a function of distance along path of flow at every five time steps is show in Figure 5. This case uses a small Δx than Case 3 and more nodes, equal to Case 2; it can be observe a sharp water injection front, but not that sharp as in Case 3, which means that the decrease in numerical dispersion was less in the implicit case.

Table 4. Water saturation values for the 10 grid blocks at 10 days. Case 4

Grid block index	1	2	3	4	5	6	7	8	9	10
Sw (fraction)	0.6892	0.6800	0.6736	0.6683	0.6636	0.6593	0.6554	0.6518	0.6483	0.6451

Grid block index	11	12	13	14	15	16	17	18	19	20
Sw (fraction)	0.6420	0.6390	0.6362	0.6334	0.6307	0.6282	0.6257	0.6232	0.6208	0.6185
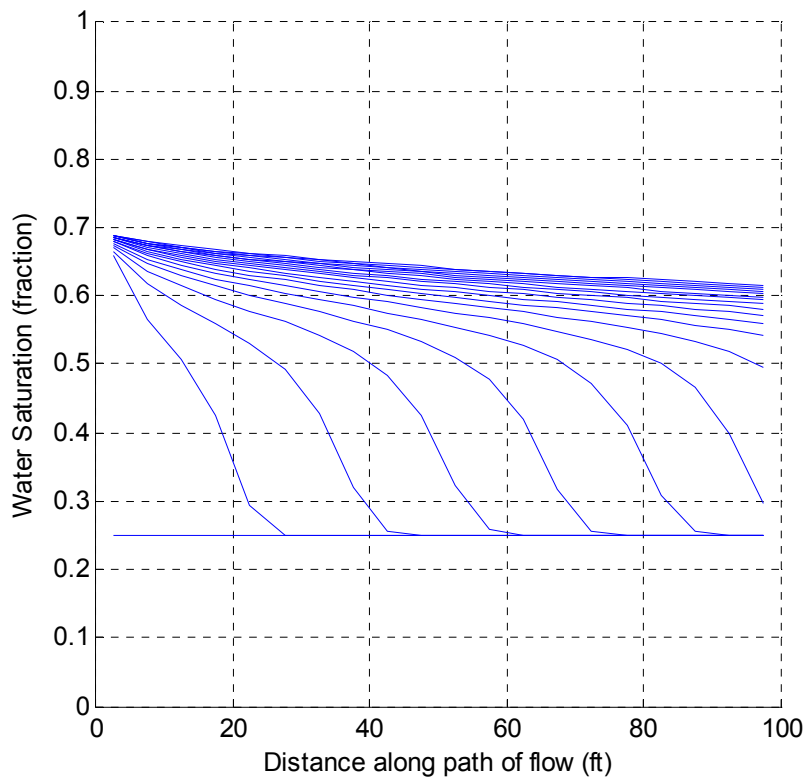
Figure 5. Water saturation profile along the path of flow at every five time steps. Case 4

Figure 6 shows the water saturation profile as a function of distance along path of flow at 1.5 days (15 time steps) for the four cases considered. The case that shows a greater numerical dispersion, manifested by a spreader saturation curve is Case 3, the implicit solution with coarse grid. For the numerical parameters (time step 0.1 day, and grid block size 10 ft) considered, the explicit Case 1 shows a lower numerical dispersion than the implicit Case 3. A reduction of the numerical dispersion is shown for the cases 2 and 4 which have a finer grid, with the explicit Case 2 showing the sharper water injection front (i.e., the lowest numerical dispersion). As can be seen, the explicit formulation offers a lower numerical dispersion than the implicit formulation when the same grid blocks size and time step is used.
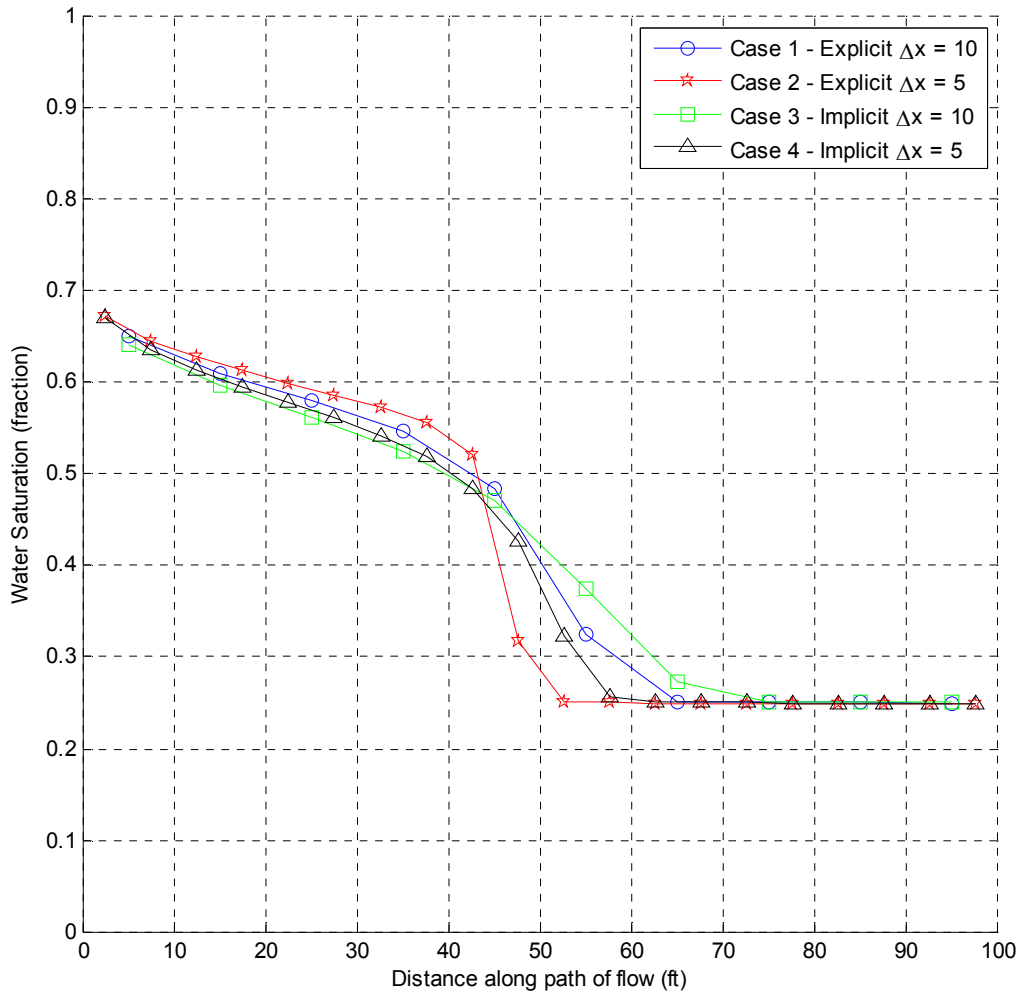
Figure 6. Comparison of the explicit and implicit Buckley-Leverett solution at t = 1.5 days, for the four cases studied

Addendum
Derivative of fractional flow with respect to water saturation

$$\frac{\partial f_w}{\partial S_w} = \frac{\partial}{\partial S_w}\left(\frac{\lambda_w}{\lambda_w + \lambda_o}\right) = \frac{\partial}{\partial S_w}\left(\frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{row}/\mu_o}\right) = \frac{\partial}{\partial S_w}\left(\frac{\mu_o k_{rw}}{\mu_o k_{rw} + \mu_w k_{row}}\right)$$

$$\frac{\partial f_w}{\partial S_w} = \frac{(\mu_o k_{rw} + \mu_w k_{row})\dfrac{\partial(\mu_o k_{rw})}{\partial S_w} - \mu_o k_{rw}\dfrac{\partial}{\partial S_w}(\mu_o k_{rw} + \mu_w k_{row})}{(\mu_o k_{rw} + \mu_w k_{row})^2}$$

Solving derivatives of relative permeability with respect to water saturation,

$$\frac{\partial(k_{rw})}{\partial S_w} = \frac{\partial}{\partial S_w}\left[k_{rw}^*\left(\frac{S_w - S_{wr}}{1 - S_{orw} - S_{wr}}\right)^{n_w}\right] = k_{rw}^* n_w\left(\frac{1}{1 - S_{orw} - S_{wr}}\right)\left(\frac{S_w - S_{wr}}{1 - S_{orw} - S_{wr}}\right)^{n_w - 1}$$

$$\frac{\partial(k_{row})}{\partial S_w} = \frac{\partial}{\partial S_w}\left[k_{orw}^*\left(\frac{1 - S_w - S_{orw}}{1 - S_{orw} - S_{wr}}\right)^{n_o}\right] = -k_{row}^* n_o\left(\frac{1}{1 - S_{orw} - S_{wr}}\right)\left(\frac{1 - S_w - S_{orw}}{1 - S_{orw} - S_{wr}}\right)^{n_o - 1}$$

Substituting,

$$\frac{\partial f_w}{\partial S_w} = \frac{1}{(\mu_o k_{rw} + \mu_w k_{row})^2}\left\{(\mu_o k_{rw} + \mu_w k_{row})\mu_o k_{rw}^* n_w\left(\frac{1}{1 - S_{orw} - S_{wr}}\right)\left(\frac{S_w - S_{wr}}{1 - S_{orw} - S_{wr}}\right)^{n_w - 1}\right.$$

$$\left.- \mu_o k_{rw}\left[\mu_o k_{rw}^* n_w\left(\frac{1}{1 - S_{orw} - S_{wr}}\right)\left(\frac{S_w - S_{wr}}{1 - S_{orw} - S_{wr}}\right)^{n_w - 1} - \mu_w k_{row}^* n_o\left(\frac{1}{1 - S_{orw} - S_{wr}}\right)\left(\frac{1 - S_w - S_{orw}}{1 - S_{orw} - S_{wr}}\right)^{n_o - 1}\right]\right\}$$