Spherical Principal Component Analysis*  

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Abstract  
Principal Component Analysis (PCA) is one of the most broadly used methods to analyze high-dimensional data. However, most existing studies on PCA aim to minimize the reconstruction error measured by the Euclidean distance, although in some fields, such as text analysis in information retrieval, analysis using the angle distance is known to be more effective. In this paper, we propose a novel PCA formulation by adding a constraint on the factors to unify the Euclidean distance and the angle distance. Because the objective and constraints are nonconvex, the optimization problem is difficult to solve in general. To tackle the optimization problem, we propose an alternating linearized minimization method with guaranteed convergence and provable convergence rate. Experiments on synthetic data and real-world data sets have validated the effectiveness of our new method and demonstrated its advantages over state-of-art competing methods.

1 Introduction  
In many real-world applications such as text categorization and face recognition, the dimensionality of the input data is usually very high. Because dealing with high-dimensional data is computationally expensive and noise or outliers in the data can increase dramatically as the dimension increases, dimension reduction plays a critical role when one analyzes high-dimensional data [4,17,20]. Among many dimension reduction methods, Principal Component Analysis (PCA) is one of the most broadly used one due to its simplicity and effectiveness.

PCA is a statistical procedure that uses an orthogonal transformation to convert a set of correlated variables into a set of linearly uncorrelated principal directions. Usually the number of principal directions is less than or equal to the number of original variables. This transformation is defined in such a way that the first principal direction has the largest possible variance (that is, accounts for as much of the variability in the data as possible), and each succeeding direction has the highest variance under the constraint that it is orthogonal to the preceding directions. The resulting vectors are an uncorrelated orthogonal basis set.

When data points lie in a low-dimensional manifold and the manifold is linear or nearly-linear, the low-dimensional structure of data can be effectively captured by a linear subspace spanned by the principal PCA directions. More specifically, let \( X = (x_1, \ldots, x_n) \) be \( n \) data points in \( m \)-dimensional space while \( U = (u_1, \ldots, u_r) \) contains the principal directions and \( V = (v_1, \ldots, v_k) \) contains the principal components (projected data along the principal directions). There exist two broadly used formulations for PCA:

- **Covariance-based approach** computes the covariance matrix \( C = \sum(x_i - \bar{x})(x_i - \bar{x})^T = XX^T \). By assuming that the data are centered, \( i.e., \bar{x} = 0 \), we can drop the factor \( \frac{1}{n-1} \) that does not affect \( U \). The principal directions are obtained by:

  \[
  (1.1) \quad \max_{U^T U = I} \text{trace}(U^T XX^T U),
  \]

- **Low-rank approximation-based approach** solves the following optimization problem:

  \[
  (1.2) \quad \min_{U^T U = I} J = \|X - UV\|_F^2 = \sum_{i,j} [X_{ij} - (UV)_{ij}]^2,
  \]

where we approximate \( X \) by \( UV \).

Taking the derivative w.r.t. \( V \) and setting it to zero, we have \( V = X^T U \), by which Eq. (1.2) reduces to Eq. (1.1). Therefore, the solutions to these two approaches are identical. In our paper, we will focus on the second formulation.

2 Motivation  
In Eq. (1.2), the objective function measures the difference between the original data \( X \) and its approximation of \( UV \) in the projected space, which is measured by the squared Euclidean distances and uses each feature with equal importance. However, in the real-world applications, there exist data sets which are preprocessed to be normalized and different features may have varied significances. Thus distance-based measurement method may
yield poor results. On the other side, similarity-based measurement methods, such as the cosine similarity derived from the angle distance, have been proved to be more effective in some applications, such as information retrieval [18], signal processing [8], metric learning [15], etc. Thus, deriving new methods that can directly measure the angle distance from PCA is of vital importance to solve real-world problems. However, this problem has not been well studied in literature yet.

Motivated by the above observations and a previous work [19], in this paper we propose a spherical PCA model that can unify the Euclidean distance and the angle distance. By noticing that larger angle in the sphere in Fig. 1 also has larger Euclidean distance, we can add the normalization constraint to the component matrix, where the norm of each column in \( V \) is restricted to be 1 to guarantee the spherical distribution of components in the projected space:

\[
(3.3) \quad \min_{U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r}} J = \|X - UV\|_F^2 = \sum_{i,j} |X_{ij} - (UV)_{ij}|^2,
\]

\[s.t. \quad U \in \mathbb{R}^m, V \in \mathbb{R}^n, \]

where we define:

\[
(4.4) \quad \mathbb{U} := \{ U : U^TU = I \}, \quad \mathbb{V} := \{ V : \|V(:,j)\| = 1 \forall j \in [0, n] \}.
\]

Suppose the component is spherically distributed, then the Euclidean distance between \( v_i \) and \( v_j \) is:

\[
(3.5) \quad \|v_i - v_j\|^2 = |v_i|^2 + |v_j|^2 - 2\langle v_i, v_j \rangle = 2 - 2 \cos(\theta), \quad \theta \in [0, \pi].
\]

which is equivalent to angle distance such that a bigger angle \( \theta \) will result in a larger Euclidean distance, and vice versa.

Remark 1. In traditional PCA, without the normalization constraint on each column of \( V \), the optimized solution to Eq. (1.2) can barely satisfy the spherical distribution. Since \( r \) is usually less than \( m \), PCA will lose some component more or less, thus \( x_i \neq Uv_i \) and usually \( \|x_i\| \neq \|Uv_i\| \) (they may be equal, but it barely happens). We have \( \|X_i\|^2 = 1 \) for normalized data and if \( \|v_i\|^2 = 1 \) then \( \|Uv_i\|^2 = tr(v_i^T U^T U v_i) = tr(v_i^T v_i) = \|v_i\|^2 = 1 \), which leads a contradiction. Thus we have to additionally enforce the constraint on \( V \) to guarantee our motivation.

3 Formulation and Algorithm
3.1 Objective Function with Proximal Term
We first denote:

\[
(3.6) \quad h(U,V) = \|X - UV\|_F^2, \quad s.t. \quad U \in \mathbb{U}, V \in \mathbb{V}.
\]

By noting the nonconvexity of Eq. (2.3), where no closed solution exists, we propose an alternating minimization method to get the optimized solution as:

\[
(3.7) \quad U_{k+1} = \|X - UV_k\|_F^2, \quad U \in \mathbb{U},
\]

\[
v_{k+1} = \|x - U_{k+1} v\|_F^2, \quad V \in \mathbb{V}.
\]

where \( v \) denotes the column of \( V \) and \( V \) in Eq. (2.3) can be decoupled into column-wise for optimization.

Note that because of the constraints \( U \in \mathbb{U}, V \in \mathbb{V} \), the problem (3.6) is known as the nonconvex matrix factorization problems, which was previously studied in [12, 25]. In this work we will focus on develop an efficient algorithm with provable convergence to solve (3.6). Note that the proximal algorithm recently has been successfully applied to a wide variety of situations: convex optimization, nonmonotone operators [6, 10] with various applications to nonconvex programming. It was first introduced by Rockafellar [16] as an approximation regularization method in convex optimization and in the study of variational inequalities associated to maximal monotone operators.

Considering the fact that the objective function in Eq. (2.3) is nonconvex w.r.t. \( U \) and \( V \), and the constraint on \( U \) and \( V \) are also nonconvex, we add the proximal term, which leads to the following alternating linearized minimization solutions:

\[
(3.8) \quad U_{k+1} = \arg \min_{U \in \mathbb{U}} \langle U - U_k, \nabla h(U_k) \rangle + \frac{\mu}{2}||U - U_k||_F^2,
\]

\[
v_{k+1} = \arg \min_{v \in \mathbb{V}} \langle v - v_k, \nabla h(v_k) \rangle + \frac{\lambda}{2}||v - v_k||_F^2.
\]

Remark 2. We add the proximal term to ensure that the updated solution will not go apart too far away.
from the previous step to avoid drastic changes. One

can see that when the proximal term regularization

parameters \( \mu, \lambda \) are sufficiently large, they will dominate

the objective function. Moreover, we can exploit the

linearized minimization that minimizes the objective

with Taylor expansion by making use of first order

(linear) information.

3.2 Proposed Algorithm Given the alternating

minimization objective in Eq. (3.8), now we turn to de-

rive the detailed updating algorithm with closed-form

solutions in every step.

First, we derive the solution for \( U \). Before the

derivation, we prove the following useful lemma that is

similar to [23, Theorem 1] and [22, Theorem 1]:

**Lemma 1.** \( \max_{X,T} \text{tr}(X^T B) \) is given by \( X = UV^T \),

where \([U, \Sigma, V] = \text{svd}(B)\).

**Proof.** On one hand, we have:

\[
\text{tr}(X^T B) = \text{tr}(X^T U \Sigma V^T) = \text{tr}(PS).
\]

where \( P = V^T X^T U \) is an orthonormal matrix since \( PP^T = V^T X^T U U^T X V = I \). Thus every element

including the diagonal of \( P \) is less than or equal to 1.

Thus we have:

\[
\text{tr}(PS) \leq \text{tr}(\Sigma).
\]

On the other hand, when \( X = UV^T \), we have

\[
\text{tr}(X^T B) = \text{tr}(VU^T \Sigma V^T) = \Sigma. \text{ Thus } X = UV^T\)

is the optimal solution that maximizes the ob-

jective. \( \square \)

Accordingly, we have:

\[
U_{k+1} = \arg \min_{U \in \mathbb{R}^{m \times n}} \langle U - U_k, \nabla h(U_k) \rangle + \frac{\mu}{2} \| U - U_k \|_F^2
\]

\[
= \arg \max_{U \in \mathbb{R}^{m \times n}} \text{trace}(U^T M) = YZ^T.
\]

where \( M = 2(X - U_k V_k) V_k^T + \mu U_k \) and \( Y, Z \) is obtained from \([Y, \Sigma, Z] = \text{svd}(M)\).

Second, we optimize \( v_{k+1} \) given that \( U_{k+1} \) is:

\[
v_{k+1} = \arg \min_{v \in \mathbb{R}^n} \langle v - v_k, \nabla h(v_k) \rangle + \frac{\lambda}{2} \| v - v_k \|_F^2
\]

\[
= \arg \max_{v \in \mathbb{R}^n} \langle v, q \rangle \text{ subject to } \| v \|_2^2 = 1
\]

where \( q = 2U_{k+1} x + (\lambda - 2)v_k \).

### Algorithm 1 Alternating Linearized Minimization for Problem Eq. (3.6)

**Input:** data \( X \in \mathbb{R}^{m \times n} \), rank of factors \( r \), regularization parameters \( \lambda, \mu \), number of iterations \( K \)

**Initialization:** \( U_0 \in \mathbb{R}^{m \times r}, V_0 \in \mathbb{R}^{r \times n} \)

while \( k < K \) do

optimize \( U_{k+1} \) as Eq. (3.11)

optimize each \( v_{k+1} \) as Eq. (3.12)

end while

**Output:** \( U_K \) and \( V_K \)

4 Convergence Analysis

In the following case, we let \( U \) and \( V \) be as defined in

Eq. (2.4), and show the convergence of our proposed

algorithm in the last section.

To begin with, we first show that \( h(U, V) \) has

Lipschitz continuous gradient at \( U \in U, V \in V \),

which will be very useful for the following convergence

analysis.

**Proposition 1.** \( h(U, V) \) has Lipschitz continuous

gradient at \( U \in U, V \in V \), where \( U \) and \( V \) are defined in

Eq. (2.4). That is, there exists a constant \( L_c \) such that

\[
\| \nabla h(U, V) - \nabla h(U', V') \|_F \leq L_c \| (U, V) - (U', V') \|_F
\]

for all \( U, U' \in U \) and \( V, V' \in V \). Here \( L_c > 0 \) is referred
to as the Lipschitz constant.

**Proof.** Proof of Proposition 1 is equivalent to show

\[
\| \nabla^2 h(U, V) \|_F \leq L_c \|
\]

for all \( U, V \in U \). Standard computations give the Hessian quadrature form

\[
[C^2 h(U, V)](\Delta, \Delta) = \begin{bmatrix} \Delta_U & \Delta_V \end{bmatrix} \in \mathbb{R}^{(n+m) \times r}
\]

where \( \Delta_U \in \mathbb{R}^{m \times r} \) and \( \Delta_V \in \mathbb{R}^{r \times n} \) as

\[
\| \nabla^2 h(U, V) \|_F \leq L_c \|
\]

which gives:

\[
\| \nabla^2 h(U, V) \|_F \leq 2 \| U \|_F^2 + \| V \|_F^2 + \| U \|_F \| V \|_F + \| X \|_F^2 \leq L_c
\]

where the inequality follows from \( \langle A, B \rangle \leq \| A \|_F \| B \|_F \)

and \( \| CD \|_F \leq \| C \|_F \| D \|_F \). Due to the constraints on \( U \)

and \( V \), we have \( \| U \|_F^2 = \text{tr}(U^T U) = \text{tr}(I) = r, \| V \|_F^2 = \sum_{j=1}^n \| v \|_2^2 = n \).

To analyse the convergence, we rewrite Eq. (3.6) as

\[
\min_{U, V} f(U, V) = h(U, V) + \delta_U(U) + \delta_V(V),
\]
where \( \delta_U(U) = \begin{cases} 0, & U \in U \\ \infty, & U \notin U \end{cases} \) is the indicator function of the set \( U \) and therefore nonsmooth, so is \( \partial \varphi(V) \).

The following result establishes that the subsequence convergence property of the proposed algorithm, i.e., the sequence generated by Algorithm 1 is bounded and any of its limit point is a critical point of Eq. (4.16).

**Theorem 1. (Subsequence Convergence)** Let \( \{W_k\}_{k \geq 0} = \{(U_k, V_k)\}_{k \geq 0} \) be the sequence generated by Algorithm 1 with constant step size \( \lambda, \mu > L_c \). Then the sequence \( \{W_k\}_{k \geq 0} \) is bounded and obeys the following properties:

1. **(P1)** sufficient decrease:
   \[
   f(W_{k-1}) - f(W_k) \geq \frac{\min(\lambda, \mu) - L_c}{2} \|W_k - W_{k-1}\|_F^2,
   \]
   which implies that
   \[
   \lim_{k \to \infty} \|W^{k-1} - W^k\|_F = 0.
   \]
2. **(P2) the sequence \( \{f(W_k)\}_{k \geq 0} \) is convergent.**

   \[
   \lim_{k \to \infty} f(W_{k'}) = \lim_{k \to \infty} f(W_k) = f(W^*).
   \]

Before proving Theorem 1, we give out some necessary definition.

**Definition 1.** [3] Let \( f : \mathbb{R}^d \to (-\infty, \infty] \) be a proper and lower semi-continuous function, whose domain is defined as

\[
\text{dom } f := \{u \in \mathbb{R}^n : f(u) < \infty \}.
\]

The (Fréchet) subdifferential \( \partial f \) of \( f \) at \( u \) is defined by

\[
\partial f(u) = \left\{ z : \lim_{v \to u} \frac{f(v) - f(u) - \langle z, v - u \rangle}{\|v - u\|} \geq 0 \right\}
\]

for any \( u \in \text{dom } h \) and \( \partial f(u) = \emptyset \) if \( u \notin \text{dom } f \).

We say \( u \) is a limiting critical point, or simply a critical point of \( f \) if

\[
0 \in \partial f.
\]

We now turn to prove Theorem 1.

**Proof.** [Proof of Theorem 1] (P1): First note that for all \( k \), according to our alternating minimization method, we always have \( \delta_U(U_k) = \delta_V(V_k) = 0 \) and thus \( f(W_k) = h(U_k) \).

Since \( h(U, V) \) has Lipschitz continuous gradient at \( U \in \mathbb{U}, V \in \mathbb{V} \) with Lipschitz gradient \( L_c \) and \( \lambda > L_c \), as proved in Proposition 1, we define \( h(U, V', V) \) as proximal regularization of \( h(U, V) \) linearized at \( U', V' \):

\[
h(U', V) + \langle \nabla_U h(U', V), U - U' \rangle + \frac{L_c}{2} \|U - U'\|_F^2,
\]

By the definition of Lipschitz continuous gradient and Taylor expansion, we have

\[
(4.20) \quad h(U, V) \leq h_{L_c}(U, U', V).
\]

Also by the definition of proximal map, we get:

\[
(4.21) \quad U_k = \arg \min_U \delta_U(U) + \frac{\mu}{2} \|U - U_{k-1}\|_F^2 + \langle \nabla_U h(U_{k-1}, V_{k-1}), U - U_{k-1} \rangle.
\]

Hence we take \( U_k = U \), which implies that

\[
(4.22) \quad \delta_U(U_k) + \frac{\mu}{2} \|U_k - U_{k-1}\|_F^2 + \langle \nabla_U h(U_{k-1}, V_{k-1}), U_k - U_{k-1} \rangle \leq \delta_U(U_{k-1}).
\]

Combining Eq. (4.20) to Eq. (4.22), we have:

\[
(4.23) \quad h(U_k, V_{k-1}) + \delta_U(U_k)
\]

\[
\leq h(U_{k-1}, V_{k-1}) + \langle \nabla_U h(U_{k-1}, V_{k-1}), U_k - U_{k-1} \rangle
\]

\[
+ \frac{L_c}{2} \|U_k - U_{k-1}\|_F^2 + \delta_U(U_k)
\]

\[
\leq h(U_{k-1}, V_{k-1}) + \frac{L_c}{2} \|U_k - U_{k-1}\|_F^2
\]

\[
+ \delta_U(U_{k-1}) - \frac{\mu}{2} \|U_k - U_{k-1}\|_F^2.
\]

Similarly, we have

\[
(4.24) \quad h(U_k, V_k) - h(U_k, V_{k-1}) + \delta_V(V_k) - \delta_V(V_{k-1})
\]

\[
\leq -\frac{\lambda - L_c}{2} \|V_k - V_{k-1}\|_F^2,
\]

which together with the above equation gives Eq. (4.17). Now repeating Eq. (4.17) for all \( k \) will give

\[
(4.25) \quad (\min(\lambda, \mu) - L_c) \sum_{k=1}^{\infty} \|W_k - W_{k-1}\|_F^2 \leq f(W_0),
\]

which gives Eq. (4.18).
Owing to the closedness properties of $\partial f$, we conclude that $(4.31)$ dist$(0, \partial f)$ is convergent.

(P3) Since $U_k' \subseteq U, V_k' \subseteq V$ for all $k'$ and both of the sets $U$ and $V$ are closed, we have $U^* \subseteq U, V^* \subseteq V$. Since $h$ is continuous, we have

$$
\lim_{k' \to \infty} f(W_{k'}) = \lim_{k' \to \infty} h(U_{k'}, V_{k'}) + \delta_U(U_{k'}) + \delta_V(V_{k'}) = f(W^*),
$$

which together with the fact that $\{f(W_k)\}_{k \geq 0}$ is convergent gives Eq. (4.18).

To show $W^*$ is a critical point, we first consider Eq. (2) and the optimality condition yields:

$$
(4.26) \nabla_U h(U_{k-1}, V_{k-1}) + \mu(U_k - U_{k-1}) + \partial \delta_U(U_k) = 0.
$$

Similarly, we have

$$
(4.27) \nabla_V h(U_k, V_{k-1}) + \lambda(V_k - V_{k-1}) + \partial \delta_V(V_k) = 0.
$$

Now, define

$$
\nabla_U h(U_k, V_k) + \partial \delta_U(U_k),
$$

$$
\nabla_V h(U_k, V_k) + \partial \delta_V(V_k).
$$

Thus, we have

$$
(4.28) A_k \in \partial U f(U_k, V_k), B_k \in \partial V f(U_k, V_k).
$$

It follows from the above that

$$
\lim_{k \to \infty} A_k = \lim_{k \to \infty} \nabla_U h(U_k, V_k) = \nabla_U h(U_{k-1}, V_{k-1}) + \mu(U_k - U_{k-1}) + \partial \delta_U(U_k)
$$

$$
\leq \lim_{k \to \infty} \| \nabla_U h(U_k, V_k) - \nabla_U h(U_{k-1}, V_{k-1}) \|_F + \mu \| U_k - U_{k-1} \|_F
$$

$$
\leq \lim_{k \to \infty} \| L_c + \mu \| W_k - W_{k-1} \|_F = 0.
$$

Similarly, we have:

$$
(4.30) \lim_{k \to \infty} B_k = \lim_{k \to \infty} \| L_c + \lambda \| W_k - W_{k-1} \|_F = 0.
$$

Then we have:

$$
(4.31) \text{dist}(0, \partial f(W_k)) \leq (2L_c + \mu + \lambda) \| W_k - W_{k-1} \|
$$

Owing to the closedness properties of $\partial f(W_k)$, we finally obtain $0 \in \partial f(W^*)$. Thus, $W^*$ is a critical point of $f$. \hfill \Box

**Theorem 2. (Sequence Convergence)** The sequence $\{W_k\}_{k \geq 0}$ generated by Algorithm 1 with a constant step size $\lambda, \mu > L_c$ is global-sequence convergence.

Before proving Theorem 2, we give out another important definition.

**Definition 2. (Kurdyka-Lojasiewicz (KL) property)** We say a proper semi-continuous function $h(u)$ satisfies Kurdyka-Lojasiewicz (KL) property, if $u$ is a critical point of $h(u)$, then there exist $\delta > 0$, $\theta \in [0, 1)$, $C_1 > 0$, s.t.

$$
|h(u) - h(\bar{u})|^\theta \leq C_1 \text{dist}(0, \partial h(u)), \forall u \in B(\bar{u}, \delta).
$$

We mention that the above KL property(also known as KL inequality) states the regularity of $h(u)$ around its critical point $u$ and the KL inequality trivially holds at non-critical point. There are a very large set of functions satisfying the KL inequality including any semi-algebraic functions [3]. Clearly, the objective function $f$ is semi-algebraic as both $h$, $\delta_U$ and $\delta_V$ are semi-algebraic.

**Lemma 2. (Uniform KL property)** There exist $\delta_0 > 0$, $\theta_{KL} \in [0, 1)$, $C_{KL} > 0$ such that for all $W$ s.t. dist$(W, C(W_0)) \leq \delta_0$:

$$
(4.32) \| f(W) - \bar{f} \|^\theta_{KL} \leq C_{KL} \text{dist}(0, \partial f(W))
$$

with $\bar{f}$ denoting the limiting function value defined in $P$ (2) of Theorem 1.

**Proof.** First we recognize the union $\bigcup_i B(W_i^*, \delta_i)$ forms an open cover of $C(W_0)$ with $W_i^*$ representing all points in $C(W_0)$ and $\delta_i$ to be chosen so that the following KL property of $f$ at $W_i^* \in C(W_0)$ holds:

$$
\| f(W) - \bar{f} \|^\theta_{KL} \leq C_{KL} \text{dist}(0, \partial f(W))
$$

where we have used all $f(W_i^*) = \bar{f}$ by assertion (P3) of Theorem 1. Then due to the compactness of the set $C(W_0)$, it has a finite subcover $\bigcup_{i=1}^p B(W_i^*, \delta_{k_i})$ for some positive integer $p$. Now combining all, we have for all $W \in \bigcup_{i=1}^p B(W_i^*, \delta_{k_i})$:

$$
(4.33) \| f(W) - \bar{f} \|^\theta_{KL} \leq C_{KL} \text{dist}(0, \partial f(W))
$$

with $\theta_{KL} = \max_{k=1}^p \{ \theta_{k_i} \}$ and $C_{KL} = \max_{k=1}^p \{ C_{k_i} \}$. Finally, since $\bigcup_{i=1}^p B(W_i^*, \delta_{k_i})$ is an open cover of $C(W_0)$, there exists a sufficiently small number $\delta_0$ so that

$$
\{(W) : \text{dist}(W, C(W_0)) \leq \delta_0 \} \subseteq \bigcup_{i=1}^p B(W_i^*, \delta_{k_i}).
$$

Therefore, eq. (4.33) holds whenever dist$(W, C(W_0)) \leq \delta_0$. \hfill \Box
We now turn to prove Theorem 2.

Proof. [Proof of Theorem 2] According to Definition 2, there exists a sufficiently large \( k_0 \) satisfying:

\[
(4.34) \quad \left[ f(W_k) - f(W^*) \right]^2 \leq C_2 \text{dist}(0, \partial f(W_k)), \quad \forall k \geq k_0.
\]

In the subsequent analysis, we restrict to \( k \geq k_0 \). Construct a concave function \( x^{\lambda} \) for some \( \lambda \in [0, 1) \) with domain \( x > 0 \). Obviously, by the concavity, we have

\[
x_1^{1^{\lambda}} - x_2^{1^{\lambda}} \geq (1 - \theta)x_2^{1^{\lambda}}(x_2 - x_1), \forall x_1 > 0, x_2 > 0
\]

Replacing \( x_1 \) by \( f(W_{k+1}) - f(W^*) \) and \( x_2 \) by \( f(W_k) - f(W^*) \) and using the sufficient decrease property, we have

\[
(4.35) \quad 2\|W_k - W_{k+1}\|_F^2 - \|W_k - W_{k-1}\|_F^2 - \kappa \|W_k - W_{k-1}\|_F
\]

\[
\geq \kappa (2\|W_k - W_{k+1}\|_F - \|W_k - W_{k-1}\|_F).
\]

And accordingly, we have:

\[
(4.36) \quad \sum_{k=\tilde{k}}^\infty \|W_k - W_{k+1}\|_F < \infty.
\]

Following some standard arguments one can see that

\[
\limsup_{t \to \infty, t_1, t_2 \geq t} \|W_{t_1} - W_{t_2}\|_F = 0,
\]

which implies that the sequence \( \{W_k\} \) is Cauchy, and hence convergent. Hence, the limit point set \( C(W_0) \) is singleton \( W^* \).

\[\square\]

Theorem 3. (Convergence Rate) The convergence rate is at least sub-linear.

Towards that end, we first know from the above argument that \( \{W_k\} \) converges to some point \( W^* \), i.e., \( \lim_{k \to \infty} W_k = W^* \). Then using Equation (4.36) and the triangle inequality, we obtain

\[
(4.37) \quad \|W_k - W^*\|_F \leq \sum_{k=\tilde{k}}^\infty \|W_k - W_{k+1}\|_F 
\]

\[
\leq \|W_k - W_{k-1}\|_F + \beta[f(W_k) - f(W^*)]^{1-\theta},
\]

which indicates the convergence rate of \( W_k \) is at least as fast as the rate that \( \|W_k - W_{k-1}\|_F + \beta[f(W_k) - f(W^*)]^{1-\theta} \) converges to 0. In particular, the second term \( \beta[f(W_k) - f(W^*)]^{1-\theta} \) can be controlled:

\[
(4.38) \quad \leq \beta C_2 (2B_0 + \lambda + \|X\|_F) \|W_k - W_{k-1}\|_F.
\]

Plugging (4.38) back to (4.37), we then have

\[
\sum_{k=\tilde{k}}^\infty \|W_k - W_{k+1}\|_F \leq \|W_k - W_{k-1}\|_F + \alpha \|W_k - W_{k-1}\|_F^{1-\theta}.
\]

We divide the following analysis into two cases based on the value of the KL exponent \( \theta \).

- Case I: If \( \theta = 0 \), we set \( Q := \{k \in N : W_{k+1} \neq W_k\} \) and take \( k \) in \( Q \). When \( k \) is sufficiently large, then we have:

\[
(4.39) \quad \|W_k - W_{k+1}\|_F^2 = C_4 > 0.
\]

On the other hand,

\[
(4.40) \quad f(W_{k+1}) - f(W_k) \geq \frac{\min(\lambda, \mu) - L_c}{2}\|W_{k+1} - W_k\|_F^2.
\]

Since \( f(W_k) \) is known to be convergent to 0, Eq. (4.40) implies that \( Q \) is finite and sequence \( W_k \) converges in a finite number of steps.

- Case II: \( \theta \in (0, \frac{1}{2}] \). This case means \( \frac{1-\theta}{\theta} \geq 1 \). We define \( P_k = \sum_{k=\tilde{k}}^\infty W_{k+1} - W_k \|F \|

\[
(4.41) \quad P_k \leq P_{k-1} - P_k + \alpha \left[ P_{k-1} - P_k \right]^{1-\theta}.
\]
Since $P_{k-1} - P_k \to 0$, there exists a positive integer $k_1$ such that $P_{k-1} - P_k < 1$, $\forall k \geq k_1$. Thus,

$$P_k \leq (1 + \alpha) (P_{k-1} - P_k), \quad \forall k \geq \max\{k_0, k_1\},$$

which implies that

$$P_k \leq \rho \cdot P_{k-1}, \quad \forall k \geq \max\{k_0, k_1\},$$

where $\rho = \frac{1 + \alpha}{2 + \alpha} \in (0, 1)$. This together with (4.37) gives the linear convergence rate

$$\|W_k - W^*\|_F \leq O(\rho^{k-k}), \quad \forall k \geq \bar{k},$$

where $\bar{k} = \max\{k_0, k_1\}$.

- **Case III:** $\theta \in (1/2, 1)$. This case means $1 - \theta \leq 1$. Based on the former results, we have

$$P_k \leq (1 + \alpha) \left( P_{k-1} - P_k \right)^{\frac{1 - \theta}{2 + \alpha}}, \quad \forall k \geq \max\{k_0, k_1\}.$$

We now run into the same situation as in [2]. Hence following a similar argument gives

$$P_k^{\frac{1-2\theta}{2-\alpha}} - P_k^{\frac{1-2\theta}{2-\alpha}} \geq \zeta, \quad \forall k \geq \bar{k}$$

for some $\zeta > 0$. Then repeating and summing up the above inequality from $\bar{k} = \max\{k_0, k_1\}$ to any $k > \bar{k}$, we can conclude

$$P_k \leq \left[ P_{k-1}^{\frac{1-2\theta}{2-\alpha}} + \zeta (k - \bar{k}) \right]^{\frac{-1-\theta}{2+\alpha}} = O \left( (k - \bar{k})^{-\frac{1-\theta}{2+\alpha}} \right).$$

Finally, the following sublinear convergence holds

$$\|W_k - W^*\|_F \leq O \left( (k - \bar{k})^{-\frac{1-\theta}{2+\alpha}} \right), \quad \forall k > \bar{k}.$$
Figure 2: **Top**: two groups of data generated from two angles. **Middle**: clustering result with distance-based method $K$-means. **Bottom**: clustering result with our method. Blue and red represent different clusters.

Figure 3: **Left**: $\|U_{k+1} - U_k\|_F$ with updates. **Center**: $\|V_{k+1} - V_k\|_F$ with updates. Both converge to 0 after several iterations. **Right**: Objective converges at sub-linear rate. All validate our analysis.

Table 1: Clustering performance of different algorithms on 20-newsgroup dataset

<table>
<thead>
<tr>
<th>Methods</th>
<th>$K$-means</th>
<th>MUA</th>
<th>PCA</th>
<th>R1-PCA</th>
<th>K-SVD</th>
<th>Spherical PCA</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Groups</td>
<td>Acc</td>
<td>NMI</td>
<td>Acc</td>
<td>NMI</td>
<td>Acc</td>
<td>NMI</td>
</tr>
<tr>
<td>5</td>
<td>0.651</td>
<td>0.621</td>
<td>0.674</td>
<td>0.614</td>
<td>0.703</td>
<td>0.628</td>
</tr>
<tr>
<td>10</td>
<td>0.487</td>
<td>0.316</td>
<td>0.478</td>
<td>0.320</td>
<td>0.502</td>
<td>0.383</td>
</tr>
<tr>
<td>15</td>
<td>0.398</td>
<td>0.307</td>
<td>0.387</td>
<td>0.301</td>
<td>0.412</td>
<td>0.319</td>
</tr>
<tr>
<td>20</td>
<td>0.315</td>
<td>0.242</td>
<td>0.314</td>
<td>0.221</td>
<td>0.362</td>
<td>0.248</td>
</tr>
</tbody>
</table>

Table 2: Clustering performance of different algorithms on four UCI data sets

<table>
<thead>
<tr>
<th>Data (#class)</th>
<th>$K$-means</th>
<th>MUA</th>
<th>PCA</th>
<th>R1-PCA</th>
<th>K-SVD</th>
<th>Spherical PCA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Acc</td>
<td>NMI</td>
<td>Acc</td>
<td>NMI</td>
<td>Acc</td>
<td>NMI</td>
</tr>
<tr>
<td>glass (6)</td>
<td>0.687</td>
<td>0.566</td>
<td>0.692</td>
<td>0.574</td>
<td>0.732</td>
<td>0.608</td>
</tr>
<tr>
<td>diabetes (2)</td>
<td>0.775</td>
<td>0.632</td>
<td>0.788</td>
<td>0.654</td>
<td>0.761</td>
<td>0.613</td>
</tr>
<tr>
<td>mfeat (10)</td>
<td>0.365</td>
<td>0.223</td>
<td>0.358</td>
<td>0.211</td>
<td>0.371</td>
<td>0.225</td>
</tr>
<tr>
<td>isolet (26)</td>
<td>0.267</td>
<td>0.198</td>
<td>0.253</td>
<td>0.181</td>
<td>0.262</td>
<td>0.182</td>
</tr>
</tbody>
</table>

5. **$K$-means** [9].

We vary the number of clusters from 5 to 10, 15 and 20. In each newsgroup, 200 documents are randomly sampled, and we repeat for 10 times by taking the average and report the clustering result as Table 1 demonstrates.

We see that our proposed method Spherical PCA can always achieve both higher clustering accuracy and normalized mutual information in text analysis.

We also compare our method with other methods on UCI data sets including: glass, diabetes, mfeat and isolet. Table 2 illustrates the results. We see that
though our method doesn’t show the absolute advantage as on text, still the result is considerably good. All the experiments indicate that our method can achieve good performance on both text and non-text data sets, showing its potential for broader application.

6 Conclusion

In this paper, we study spherical PCA where the direction matrix is orthonormal and the component vectors are assumed to lie in the unitary sphere. The benefit is obvious that it can make the angle distance equivalent to Euclidean distance. Due to the nonconvexity of objective function and constraints on the factors which are difficult to tackle, we propose an alternating linearized minimization method to derive the solution, which is proved to be sequence convergent. Moreover, we analyze the convergence rate which is validated by our experiments. The results on real-world datasets and synthetic data illustrate the superiority of our method.

References


