1. (20) Maxwell “Quickies”:

a. Write down Maxwell’s equations in \( \mathbf{E}, \mathbf{B} \) form.

Solution:

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.
\]

b. Define \( \mathbf{P} \) and \( \mathbf{M} \), and give the definitions of \( \mathbf{D} \) and \( \mathbf{H} \) in terms of them.

Solution: \( \mathbf{P} \) (\( \mathbf{M} \)) is the electric (magnetic) dipole moment per unit volume. \( \mathbf{D} \) and \( \mathbf{H} \) are defined:

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}.
\]

c. Define the bound charge density and bound current density in terms of \( \mathbf{P} \) and \( \mathbf{M} \).

Solution:

\[
\nabla \cdot \mathbf{P} = -\rho_b \quad \nabla \times \mathbf{M} = \mathbf{J}_b.
\]

d. Write down Maxwell’s equations in \( \mathbf{D}, \mathbf{H} \) form.

Solution:

\[
\nabla \cdot \mathbf{D} = \rho_f \quad \nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.
\]

e. What does it mean for \( \mathbf{P} \) and \( \mathbf{M} \) if the material is linear?

Solution: If the material is linear, then the response of the material is proportional to the applied field. Specifically, \( \mathbf{P} = X_e \mathbf{E} \) and \( \mathbf{M} = X_m \mathbf{H} \). Thus, if the material is linear, then \( \mathbf{D} = \epsilon \mathbf{E} \), where \( \epsilon = \epsilon_0 (1 + X_e) \), and \( \mathbf{H} = \mathbf{B}/\mu \), where \( \mu = \mu_0 (1 + X_m) \).

f. Write down Maxwell’s equations in \( \mathbf{E}, \mathbf{B} \) form for a linear material.

Solution: Under the conditions in part e, Maxwell’s equations in part d become:

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \quad \nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{E}}{\partial t}.
\]
2. (15) EM Wave “Quickies”:

a. Derive the wave equation for either $\mathbf{E}$ or $\mathbf{B}$ in free space from Maxwell’s equations.

(Useful vector identity: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}((\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A})$

Solution: Take the curl of the left hand side of Maxwell’s third equation and use the vector identity: $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}((\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E})$. The divergence term vanishes by Maxwell’s first equation where for free space, $\rho = 0$. Take the curl of the right hand side of Maxwell’s third equation, and then use Maxwell’s fourth equation with $\vec{J} = 0$ (free space), to find $-\frac{1}{\mu_0 \varepsilon_0} \frac{\partial^2 \vec{E}}{\partial t^2}$. Thus,

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (0)$$

b. From your wave equation, identify the speed of the wave.

Solution: By dimensional arguments, the speed of the wave is given by $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

c. Show that $\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$ is a solution to the wave equation.

Solution: Use $\vec{\nabla} f(\vec{k} \cdot \vec{r}) = \vec{k} f'(\vec{k} \cdot \vec{r})$ to find: $\nabla^2 \vec{E} = -k^2 \vec{E}$. The time derivatives give: $\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = -\mu_0 \varepsilon_0 \omega^2 \vec{E}$. Using part b, the wave equation is satisfied if the parameters, $k$ and $\omega$ satisfy $c = \frac{\omega}{k}$.

d. Using Maxwell’s equations, from the electric field of part c, find the magnetic field, $\vec{B}$ and show that $\vec{E}$ and $\vec{B}$ are perpendicular.

Solution: Using Maxwell’s third equation, we have: $\vec{\nabla} \times \vec{E} = -(\vec{k} \times \vec{E}_0) \sin((\vec{k} \cdot \vec{r} - \omega t) = -\frac{\partial \vec{B}}{\partial t}$, thus

$$\vec{B} = \int dt (\vec{k} \times \vec{E}_0) \sin((\vec{k} \cdot \vec{r} - \omega t) = \frac{(\vec{k} \times \vec{E}_0)}{\omega} \cos(\vec{k} \cdot \vec{r} - \omega t). \quad (0)$$

Due to the property of the cross product, $\vec{B}$ is perpendicular to $\vec{E}$. 

3. (Fresnel Formula basics) Consider the interface between two linear materials with electromagnetic parameters \(\{\epsilon_1, \mu_1\}\), and \(\{\epsilon_2, \mu_2\}\) which yield indices of refraction, \(n_1\) and \(n_2\), respectively. A plane electromagnetic wave of wavenumber, \(k_I\), is incident from the left (region 1). (Measure all angles with respect to the z-axis.)

a. For the case where the magnetic field is parallel to the interface (\(\pm \hat{y}\)), draw on the figure the \(\{\vec{E}, \vec{B}, \vec{k}\}\) vector triads for the incident, reflected and transmitted waves. Solution shown on the figure above.

b. Write down the general solutions for the electric and magnetic fields, rewriting the magnetic fields in terms of the appropriate electric field amplitude. Solution: The general solutions in each region are:

\[
\vec{E}_I = \vec{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}, \quad \vec{B}_I = \frac{n_1}{c} (\vec{k}_I \times \vec{E}_{0I}) e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}
\]
\[
\vec{E}_R = \vec{E}_{0R} e^{i(k_R \cdot \vec{r} - \omega t)}, \quad \vec{B}_R = \frac{n_1}{c} (\vec{k}_R \times \vec{E}_{0R}) e^{i(k_R \cdot \vec{r} - \omega t)}
\]
\[
\vec{E}_T = \vec{E}_{0T} e^{i(k_T \cdot \vec{r} - \omega t)}, \quad \vec{B}_T = \frac{n_2}{c} (\vec{k}_T \times \vec{E}_{0T}) e^{i(k_T \cdot \vec{r} - \omega t)}
\]

Referring to the figure, \(\vec{k}_I = k_I(\sin \theta_I \hat{x} + \cos \theta_I \hat{z})\), \(\vec{k}_R = k_R(\sin \theta_I \hat{x} - \cos \theta_I \hat{z})\), and \(\vec{k}_T = k_T(\sin \theta_T \hat{x} + \cos \theta_T \hat{z})\), while \(\vec{E}_{0I} = E_{0I}(\cos \theta_I \hat{x} - \sin \theta_I \hat{z})\), \(\vec{E}_{0R} = E_{0R}(\cos \theta_I \hat{x} + \sin \theta_I \hat{z})\), \(\vec{E}_{0T} = E_{0T}(\cos \theta_T \hat{x} - \sin \theta_T \hat{z})\), where \(\theta_I = \theta_R\) was used. The fields in region 1 are the sum of the incident and reflected fields.

c. Referring to your drawing, write down the boundary conditions in sufficient detail such that you could solve for the transmission and reflection coefficients.

Solution: We need two independent conditions to solve for the ratios, \(\mathcal{E}_R = E_{0R}/E_{0I}\) and \(\mathcal{E}_T = E_{0T}/E_{0I}\). Using Maxwell’s first equation for linear material, we have \(\epsilon_1 E_I^+ = \epsilon_2 E_T^+\) which here gives: \(\epsilon_1 (E_{0I} + E_{0R}) \sin \theta_I = -\epsilon_2 E_{0T} \sin \theta_T\). We can rewrite this using the law of refraction to find: \(1 - \mathcal{E}_R = \frac{n_1 \epsilon_1}{n_2 \epsilon_2} \mathcal{E}_T\).

For the second equation, we need to use Maxwell’s third equation (Maxwell’s second equation is trivially satisfied since the magnetic field has no perpendicular component in this problem, and Maxwell’s fourth equation gives the same condition as Maxwell’s first equation.) Applying Maxwell’s third equation to the boundary gives \(E_{1z} = E_{2z}\), which here gives: \((E_{0I} + E_{0R}) \cos \theta_I = E_{0T} \cos \theta_T\). Dividing by the incident amplitude gives: \(1 + \mathcal{E}_R = \frac{\cos \theta_I}{\cos \theta_T} \mathcal{E}_T\).

These two can be solved to give (solution not asked for):

\[
\mathcal{E}_T = \frac{2n_2 \cos \theta_I \epsilon_1}{n_2 \cos \theta_2 \epsilon_1 + n_1 \cos \theta_I \epsilon_2}
\]
\[
\mathcal{E}_R = \frac{n_2 \cos \theta_2 \epsilon_1 - n_1 \cos \theta_I \epsilon_2}{n_2 \cos \theta_2 \epsilon_1 + n_1 \cos \theta_I \epsilon_2}.
\]
4. (35) A loop of radius, $R$, is centered at the origin and lies in the xy-plane. The loop consists of two oppositely-charged half loops of uniform linear charge density, ±$\lambda_0$. Let $\beta(t)$ be the angle with respect to the x-axis to one of the joints where the half-loops are joined (as shown). Initially, $\beta(t=0) = 0$; so at this time the loop is oriented with the positive side centered on the positive y-axis, that is,

$$\lambda(\theta, t=0) = \begin{cases} 
+\lambda_0, & 0 < \theta < \pi \\
-\lambda_0, & \pi < \theta < 2\pi.
\end{cases}$$

where $\theta$ identifies an element of the loop. The loop rotates in the +$\hat{z}$ direction with an angular velocity which increases linearly in time according to, $\omega(t) = d\beta/dt = \alpha t$. The general formula for the retarded vector potential, $\vec{A}$, is:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_R)d^3r'}{|\vec{r}' - \vec{r}|}.$$  

where $\vec{r}' = \vec{r} - \vec{r}'(t_R)$ with the retarded time given by $t_R = t - |\vec{r}' - \vec{r}|/c$.

a. Through what angle will the loop have rotated in time, $t$?

Solution: Integrating the angular acceleration twice gives, $\beta(t) = \frac{1}{2} \alpha t^2$ (just like constant linear acceleration).

b. What is the retarded time for a point on the z-axis?

Solution:  
$t_R = t - |\vec{r}' - \vec{r}|/c = t - \sqrt{R^2 + z^2}/c$, independent of time.

c. For this case, we can write $d^3r'\vec{J}(\vec{r}', t_R) \rightarrow R\omega(t_R)\lambda(\theta, t_R)d\theta$, where $\lambda(\theta, t_R)$ is the linear charge density at $t = t_R$ similar to the $t = 0$ case shown above. Find $\lambda(\theta, t_R)$.

Solution: Using the result from part b, the linear charge density at arbitrary time, $t$, is

$$\lambda(\theta, t) = \begin{cases} 
+\lambda_0, & \frac{\alpha t^2}{2} \leq \theta < \pi + \frac{\alpha t^2}{2} \\
-\lambda_0, & \pi + \frac{\alpha t^2}{2} < \theta \leq 2\pi + \frac{\alpha t^2}{2}.
\end{cases}$$

d. Find the vector potential, $\vec{A}(z, t)$, for a point on the z-axis.

Solution: Using the result from part c and setting $\omega(t_R) = \alpha t_R$, the vector potential is

$$\vec{A}(z, t) = \frac{\mu_0}{4\pi} \int_0^{2\pi} d\theta \frac{R(\alpha t_R)\lambda(\theta, t_R)}{\sqrt{R^2 + z^2}} (\cos \theta \hat{y} - \sin \theta \hat{x})$$

$$= \frac{\mu_0 R\alpha t_R\lambda_0}{4\pi\sqrt{R^2 + z^2}} \left( \int_{\pi + \frac{\alpha t^2}{2}}^{\pi + \frac{\alpha t^2}{2}} d\theta - \int_{\pi + \frac{\alpha t^2}{2}}^{2\pi + \alpha t^2} d\theta \right) (\cos \theta \hat{y} - \sin \theta \hat{x})$$

$$= -\frac{\mu_0 R\alpha t_R\lambda_0}{\pi\sqrt{R^2 + z^2}} (\cos(\frac{\alpha t^2 R}{2})\hat{x} + \sin(\frac{\alpha t^2 R}{2})\hat{y})$$

where $t_R = t - \sqrt{R^2 + z^2}/c$ and $\hat{\phi} = -\sin \theta \hat{x} + \cos \theta \hat{y}$ was used.