1. Consider a hollow waveguide with a rectangular cross section of dimensions \( a \times b \). It is oriented along the z-axis with the x-axis along the \( a \) side and y-axis along the \( b \) side. By z-translational symmetry, we may assume the electric and magnetic fields take the form:

\[
\vec{E} = \vec{E}(x, y)e^{i(k_zz - \omega t)} \\
\vec{B} = \vec{B}(x, y)e^{i(k_zz - \omega t)}
\]

(Note the difference between bold-faced and normal-faced vectors.)

a. Take the \( \hat{x} \)-component of Maxwell’s third equation and the \( \hat{y} \)-component of Maxwell’s fourth equation, and, using the forms above, solve for \( E_y \) and \( B_x \) in terms of \( \{ E_z, B_z \} \) and the plane wave factors, \( \{ k_z, \omega \} \). Show all work.

Solution:

MIII: \( \hat{x} \cdot \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \cdot \hat{x} \)

\[
\frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_x}{\partial t}
\]

Using the plane wave form for the z- and t-dependence gives

\[
\frac{\partial}{\partial z} e^{i(k_zz - \omega t)} = ik_z e^{i(k_zz - \omega t)} \quad \text{and} \quad \frac{\partial}{\partial t} e^{i(k_zz - \omega t)} = -i\omega e^{i(k_zz - \omega t)}
\]

\[
\frac{\partial E_z}{\partial y} - ik_z E_y = i\omega B_x \\
\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} = -\frac{\omega}{c^2} E_y
\]

This is two equations for the two unknowns, \( \{ E_y, B_x \} \). Solve by substitution to find:

\[
B_x = \frac{i}{(\omega^2/c^2 - k_z^2)} \left( k_z \frac{\partial B_z}{\partial x} - \omega \frac{\partial E_z}{\partial y} \right)
\]

\[
E_y = \frac{i}{(\omega^2/c^2 - k_z^2)} \left( k_z \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right)
\]

b. Consider the special case of transverse magnetic modes (\( B_z = 0 \)). Assume the general solution of the z-component of the electric field is:

\[
E_z = E_{z0} \left[ \alpha \sin(k_zx) + \beta \cos(k_zx) \right] \left[ \gamma \sin(k_yy) + \delta \cos(k_yy) \right]
\]

(1)

where \( E_{z0} \) is the amplitude of \( E_z(x, y) \). Apply the boundary condition that \( E_{||} \) is continuous across a boundary to find \( \{ \alpha, \beta, k_x \} \). Show all work.

Solution:

Since \( E = 0 \) inside the conductor, for \( E_{||} \) to be continuous across the surface of the conductor, \( E_y(x = 0, y) = 0 \) and \( E_y(x = a, y) = 0 \). Using the expression above, and setting \( B_z = 0 \) for transverse magnetic modes, we have

\[
E_y \propto \frac{\partial E_z}{\partial y} \propto [\alpha \sin(k_x x) + \beta \cos(k_x x)].
\]

(2)

(The y-derivative doesn’t change the x-dependence.) Thus, applying the boundary conditions gives \( \beta = 0 \) and \( k_x = \frac{\pi}{a} \).

Finally, since \( E_{z0} \) is the stated amplitude of the wave, \( \alpha = 1 \).