MATH225, Fall 2009
Worksheet 6 (3.1, 3.2, 3.3)

Name: Solutions
Section:

For full credit, you must show all work and box answers.

1. Given

$$\frac{dx}{dt} = 2x + 7y$$
$$\frac{dy}{dt} = -x - 6y$$

(a) Is this system linear?

Yes

(b) Rewrite the system in matrix-vector form.

$$\boxed{\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\vec{Y},\ \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix} \ \underline{\text{or}}\ \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$$

$$\boxed{\underline{\text{or}}\ \frac{d\vec{Y}}{dt} = A\vec{Y},\ \vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}}$$

(c) Show $\mathbf{Y}_1(t) = \begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix}$ and $\mathbf{Y}_2(t) = \begin{pmatrix} -7e^{t} \\ e^{t} \end{pmatrix}$ are solutions to the system.

$$\vec{Y}_1 = \begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix},\quad \vec{Y}_2 = \begin{pmatrix} -7e^{t} \\ e^{t} \end{pmatrix}$$

$$\frac{d\vec{Y}_1}{dt} = \begin{pmatrix} -5e^{-5t} \\ 5e^{-5t} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\vec{Y}_1 = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix} = \begin{pmatrix} 2e^{-5t} - 7e^{-5t} \\ -e^{-5t} + 6e^{-5t} \end{pmatrix} = \begin{pmatrix} -5e^{-5t} \\ 5e^{-5t} \end{pmatrix} = \frac{d\vec{Y}_1}{dt}$$

$$\frac{d\vec{Y}_2}{dt} = \begin{pmatrix} -7e^{t} \\ e^{t} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\vec{Y}_2 = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} -7e^{t} \\ e^{t} \end{pmatrix} = \begin{pmatrix} -14e^{t} + 7e^{t} \\ 7e^{t} - 6e^{t} \end{pmatrix} = \begin{pmatrix} -7e^{t} \\ e^{t} \end{pmatrix} = \frac{d\vec{Y}_2}{dt}$$

(d) Show $\mathbf{Y}_3(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$ is not a solution to this system.

$$\frac{d\vec{Y}_3}{dt} = \begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\vec{Y}_3 = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} 2e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 4e^{2t} + 14e^{2t} \\ -2e^{2t} - 12e^{2t} \end{pmatrix} = \begin{pmatrix} 18e^{2t} \\ -14e^{2t} \end{pmatrix} \neq \frac{d\vec{Y}_3}{dt}$$

(e) Show $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent.

$\vec{Y}_1 \neq k\vec{Y}_2$, k: constant

$$\begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix} \neq k\begin{pmatrix} -7e^{t} \\ e^{t} \end{pmatrix}$$

$\vec{Y}_1$ + $\vec{Y}_2$ are Linearly independent

$\underline{\text{or}}$ $\vec{Y}_1(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \neq k\vec{Y}_2(0) = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$

$\vec{Y}_1(0) + \vec{Y}_2(0)$ Linearly independent

Thus so are $\vec{Y}_1(t) + \vec{Y}_2(t)$

(f) Find the general solution to the system. What principle are you using to do this?

$$\boxed{\vec{Y}(t) = k_1\begin{pmatrix} e^{-5t} \\ -e^{-5t} \end{pmatrix} + k_2\begin{pmatrix} -7e^{t} \\ e^{t} \end{pmatrix}}$$

Linearity Principle or Principle of Superposition

2. Given the system

$$\frac{dx}{dt} = 2x + 7y$$
$$\frac{dy}{dt} = -x - 6y$$

$A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix}$

(a) Is this system linear?

Yes

(b) Find the general solution.

$\det \begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix} = 0$

$(2-\lambda)(-6-\lambda) - (-7) = 0$

$\lambda^2 + 4\lambda - 5 = 0$

$(\lambda+5)(\lambda-1) = 0$

$\lambda_1 = -5, \ \lambda_2 = 1$

$\lambda_1 = -5: (A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$7x_1 + 7y_1 = 0$

$y_1 = -x_1, \ x_1 = \alpha$

$\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}, \ \alpha = 1, \ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 1: (A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$\begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_2 + 7y_2 = 0$

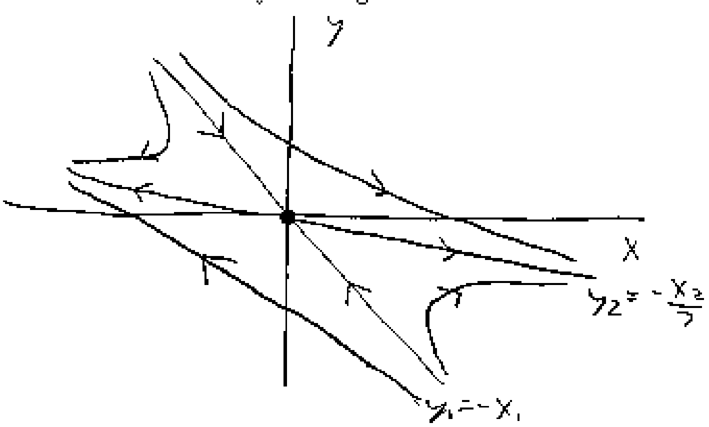
$y_2 = \frac{-x_2}{7}, \ x_2 = \alpha$

$\begin{pmatrix} \alpha \\ -\frac{\alpha}{7} \end{pmatrix}, \ \alpha = 7$

$\vec{v}_2 = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$

$$\boxed{\vec{Y}(t) = K_1 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + K_2 e^{t}\begin{pmatrix} 7 \\ -1 \end{pmatrix}}$$

(c) Sketch the phase portrait. Make sure you include the straight-line solutions and at least two other trajectories. Classify the origin.



$\boxed{\text{origin is a saddle}}$

straight-lines

$\lambda_1 = -5 < 0: \ y_1 = -x_1$

$\lambda_2 = 1 > 0: \ y_2 = -\frac{x_2}{7}$

$$\vec{Y}(t) = \underbrace{K_1 e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\substack{\text{Dominates as} \\ t \to -\infty \\ (e^t \to 0)}} + \underbrace{K_2 e^{t}\begin{pmatrix} 7 \\ -1 \end{pmatrix}}_{\substack{\text{Dominates as } t \to \infty \\ (e^{-5t} \to 0)}}$$

(d) Find the particular solution that satisfies the initial condition $(x(0), y(0)) = (6, 0)$. Report your solution as one real vector.

$\vec{Y}(0) = K_1 e^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + K_2 e^0 \begin{pmatrix} 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$\begin{pmatrix} K_1 + 7K_2 \\ -K_1 - K_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$K_1 + 7K_2 = 6$

$-K_2 + 7K_2 = 6$

$6K_2 = 6, \ K_2 = 1$

$-K_1 - K_2 = 0$

$K_1 = -K_2$

$K_1 = -1$

$\vec{Y}(t) = -e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{t}\begin{pmatrix} 7 \\ -1 \end{pmatrix}$

$$\boxed{\vec{Y}(t) = \begin{pmatrix} -e^{-5t} + 7e^{t} \\ e^{-5t} - e^{t} \end{pmatrix}}$$

3. Given the system

$$Y' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} Y \qquad A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \; A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\det\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-3-\lambda) - (-2) = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0$$

$$\lambda_1 = -2, \; \lambda_2 = -1$$

$\lambda_1 = -2$: $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 + y_1 = 0$$

$$y_1 = -2x_1, \; x_1 = \alpha$$

$$y_1 = -2\alpha$$

$$\begin{pmatrix} \alpha \\ -2\alpha \end{pmatrix}, \; \alpha = 1, \; \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\lambda_2 = -1$: $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2 + y_2 = 0$$

$$y_2 = -x_2, \; x_2 = \alpha$$

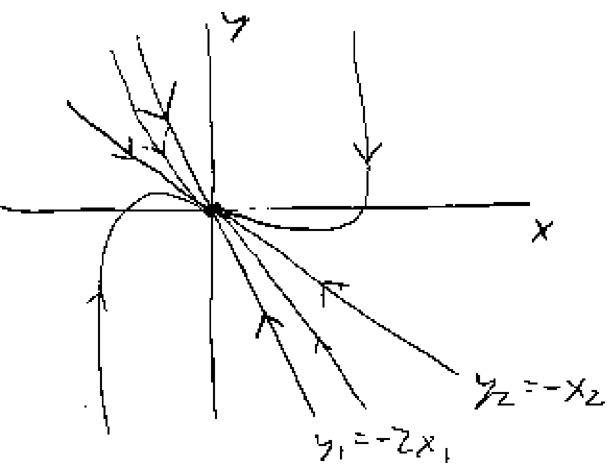
$$y_2 = -\alpha$$

$$\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}, \; \alpha = 1,$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boxed{\vec{Y}(t) = K_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + K_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

(b) Sketch the phase portrait. Make sure you include the straight-line solutions and at least two other trajectories. Classify the origin.



Straight-Lines

$\lambda_1 = -2 < 0$: $y_1 = -2x_1$

$\lambda_2 = -1 < 0$: $y_2 = -x_2$

$$\vec{Y}(t) = \underbrace{K_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{\text{Dominates as } t \to -\infty} + \underbrace{K_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\text{Dominates as } t \to \infty}$$

$(e^{-t}, e^{-2t} \to 0)$

$\boxed{\text{Origin is a sink}}$

(c) Find the particular solution that satisfies the initial condition $Y(0) = (1, 5)$. Report your solution as one real vector.

$$\vec{Y}(0) = K_1 e^0 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + K_2 e^0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} K_1 + K_2 \\ -2K_1 - K_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$K_1 + K_2 = 1$ $\qquad$ $-2K_1 - K_2 = 5$

$K_1 - 2K_1 - 5 = 1$ $\qquad$ $K_2 = -2K_1 - 5$

$-K_1 = 6, \; K_1 = -6$ $\qquad$ $K_2 = 12 - 5 = 7$

$$\vec{Y}(t) = -6e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 7e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boxed{\vec{Y}(t) = \begin{pmatrix} -6e^{-2t} + 7e^{-t} \\ 12e^{-2t} - 7e^{-t} \end{pmatrix}}$$

4. Given the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}\mathbf{Y} \qquad A=\begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}, \quad A-\lambda I = \begin{pmatrix} 3-\lambda & 0 \\ -4 & 5-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\det\begin{pmatrix} 3-\lambda & 0 \\ -4 & 5-\lambda \end{pmatrix}=0$$

$$(3-\lambda)(5-\lambda)-0=0$$

$$(3-\lambda)(5-\lambda)=0$$

$$\lambda_1=3,\ \lambda_2=5$$

$\lambda_1=3$: $(A-\lambda_1 I)\vec{v}_1=\vec{0}$

$$\begin{pmatrix} 0 & 0 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4x_1+2y_1=0$$

$$y_1=2x_1,\ x_1=\alpha$$

$$y_1=2\alpha$$

$$\begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix},\ \alpha=1,\ \vec{v}_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\lambda_2=5$: $(A-\lambda_2 I)\vec{v}_2=\vec{0}$

$$\begin{pmatrix} -2 & 0 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_2=0$$

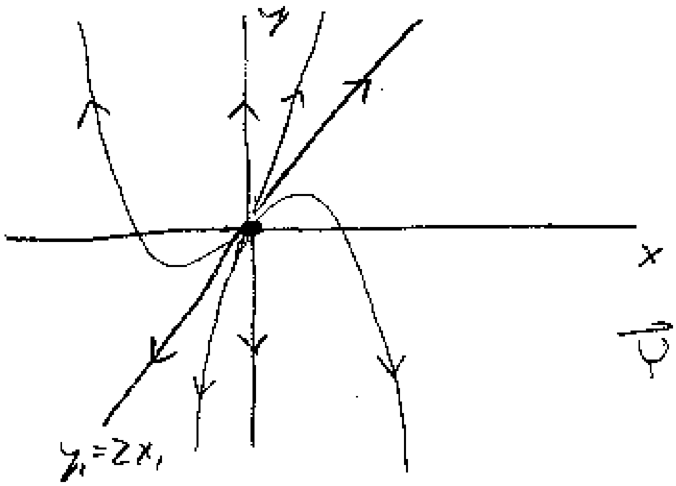
$$x_2=0,\ y_2=\alpha$$

$$\begin{pmatrix} 0 \\ \alpha \end{pmatrix},\ \alpha=1$$

$$\vec{v}_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{Y}(t)=K_1e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}+K_2e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

(b) Sketch the phase portrait. Make sure you include the straight-line solutions and at least two other trajectories. Classify the origin.



$\boxed{\text{origin is a source}}$

Straight-lines

$\lambda_1=3>0$: $y_1=2x_1$

$\lambda_2=5>0$: $x_2=0$

$$\vec{Y}(t)=\underbrace{K_1e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\text{Dominates as } t\to-\infty}+\underbrace{K_2e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{Dominates as } t\to\infty}$$

$(e^{5t}, e^{3t}\to\infty)$

(c) Find the particular solution that satisfies the initial condition $\mathbf{Y}(0)=(-3,1)$. Report your solution as one real vector.

$$\vec{Y}(0)=K_1e^0\begin{pmatrix} 1 \\ 2 \end{pmatrix}+K_2e^0\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} K_1 \\ 2K_1+K_2 \end{pmatrix}=\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$K_1=-3 \qquad 2K_1+K_2=1$$

$$-6+K_2=1$$

$$K_2=7$$

$$\vec{Y}(t)=-3e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}+7e^{5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{Y}(t)=\begin{pmatrix} -3e^{3t} \\ -6e^{3t}+7e^{5t} \end{pmatrix}}$$