MATH225, Fall 2009
Worksheet 7 (3.4, 3.5)

Name: Solutions
Section:

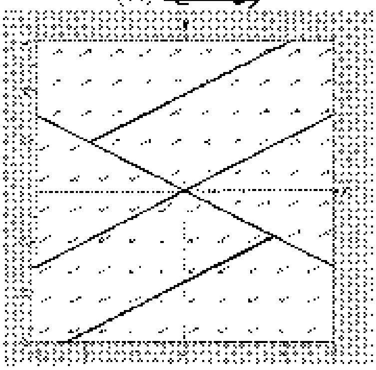
For full credit, you must show all work and box answers.

1. Match each of the following matrices with its possible phase portrait. Hint: Find the eigenvalues.

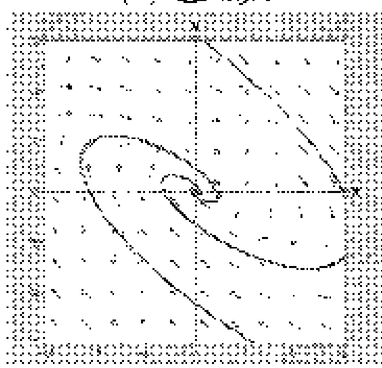
(i) $A = \begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ (iv) $A = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$

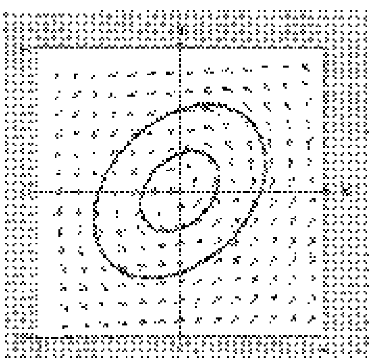
(A) (iii)



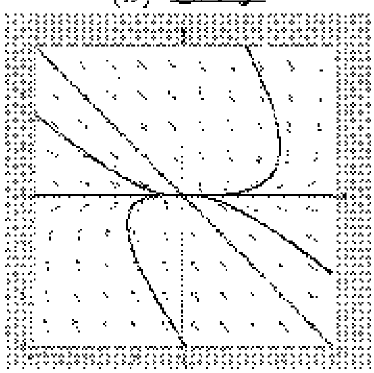
(B) (ii)



(C) (i)



(D) (iv)



(i) $\det(A - \lambda I) = 0$

$\det\begin{pmatrix} 1-\lambda & -3 \\ 3 & -1-\lambda \end{pmatrix} = 0$

$(1-\lambda)(-1-\lambda) - (-3)(3) =$

$\lambda^2 + 8 = 0$

$\lambda^2 = -8$

$\lambda = \pm\sqrt{8}\, i$

center

(ii) $\det\begin{pmatrix} 1-\lambda & 4 \\ -2 & -3-\lambda \end{pmatrix} = 0$

$(1-\lambda)(-3-\lambda) - (4)(-2) = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$

$\lambda = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

Spiral sink

(iii) $\det\begin{pmatrix} 2-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix} = 0$

$(2-\lambda)(2-\lambda) - (4)(1) = 0$

$\lambda^2 - 4\lambda = 0$

$\lambda(\lambda - 4) = 0$

$\lambda_1 = 0$, $\lambda_2 = 4$

zero eigenvalue

(iv) $\det\begin{pmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{pmatrix} = 0$

$(-1-\lambda)(-3-\lambda) - (2)(0) = 0$

$(-1-\lambda)(-3-\lambda) = 0$

$\lambda_1 = -1$, $\lambda_2 = -3$

sink

2. Given the system

$$\frac{dx}{dt} = 8y \qquad A = \begin{pmatrix} 0 & 8 \\ -3 & 0 \end{pmatrix}$$

$$\frac{dy}{dt} = -3x \qquad A - \lambda I = \begin{pmatrix} -\lambda & 8 \\ -3 & -\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\det \begin{pmatrix} -\lambda & 8 \\ -3 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (8)(-3) = 0$$

$$\lambda^2 + 24 = 0$$

$$\lambda^2 = -24$$

$$\lambda = \pm\sqrt{24}\, i$$

$$\lambda = \pm 2\sqrt{6}\, i$$

$\lambda_1 = \sqrt{24}\, i$: $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} -\sqrt{24}\, i & 8 \\ -3 & -\sqrt{24}\, i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-\sqrt{24}\, i\, x_1 + 8y_1 = 0$$

$$y_1 = \frac{\sqrt{24}\, i\, x_1}{8}, \quad x_1 = \alpha$$

$$y_1 = \frac{\sqrt{24}\, i\alpha}{8}$$

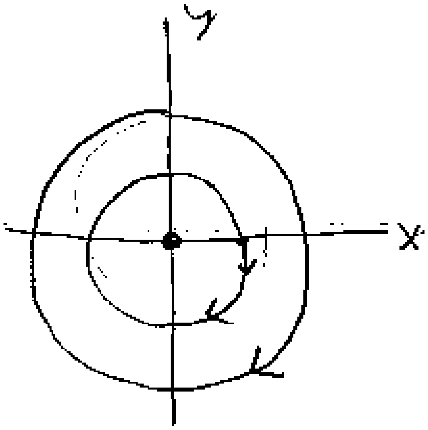
$$\begin{pmatrix} \alpha \\ \frac{\sqrt{24}\, i\alpha}{8} \end{pmatrix}, \quad \alpha = 8, \quad \vec{v}_1 = \begin{pmatrix} 8 \\ \sqrt{24}\, i \end{pmatrix}$$

$$e^{\lambda_1 t}\vec{v}_1 = e^{\sqrt{24}\, i t}\begin{pmatrix} 8 \\ \sqrt{24}\, i \end{pmatrix} = \left(\cos(\sqrt{24}\, t) + i\sin(\sqrt{24}\, t)\right)\begin{pmatrix} 8 \\ \sqrt{24}\, i \end{pmatrix} = \begin{pmatrix} 8\cos(\sqrt{24}\, t) + 8i\sin(\sqrt{24}\, t) \\ \sqrt{24}\, i\cos(\sqrt{24}\, t) - \sqrt{24}\sin(\sqrt{24}\, t) \end{pmatrix}$$

$$= \begin{pmatrix} 8\cos(\sqrt{24}\, t) \\ -\sqrt{24}\sin(\sqrt{24}\, t) \end{pmatrix} + i\begin{pmatrix} 8\sin(\sqrt{24}\, t) \\ \sqrt{24}\cos(\sqrt{24}\, t) \end{pmatrix},$$

$$\boxed{\vec{Y}(t) = K_1\begin{pmatrix} 8\cos(\sqrt{24}\, t) \\ -\sqrt{24}\sin(\sqrt{24}\, t) \end{pmatrix} + K_2\begin{pmatrix} 8\sin(\sqrt{24}\, t) \\ \sqrt{24}\cos(\sqrt{24}\, t) \end{pmatrix}}$$

(b) Sketch the phase portrait. Classify the origin.



$\lambda = \pm\sqrt{24}\, i$

origin is a $\boxed{\text{center}}$

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 8 \\ -3 & 0 \end{pmatrix}\vec{Y}, \quad \text{Let } \vec{Y} = (1, 0)$$

$$= \begin{pmatrix} 0 & 8 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad \downarrow \text{ : vector from vector field} \Rightarrow \text{CW}$$

(c) Find the particular solution that satisfies the initial condition $(x(0), y(0)) = (6, 0)$. Report your solution as one real vector.

$$\vec{Y}_1(0) = K_1\begin{pmatrix} 8 \\ 0 \end{pmatrix} + K_2\begin{pmatrix} 0 \\ \sqrt{24} \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 8K_1 \\ \sqrt{24}\, K_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}, \quad 8K_1 = 6, \quad K_1 = \frac{6}{8} = \frac{3}{4} \qquad \sqrt{24}\, K_2 = 0, \quad K_2 = 0$$

$$\vec{Y}(t) = \frac{3}{4}\begin{pmatrix} 8\cos(\sqrt{24}\, t) \\ -\sqrt{24}\sin(\sqrt{24}\, t) \end{pmatrix} = \begin{pmatrix} 6\cos(\sqrt{24}\, t) \\ -\frac{3\sqrt{24}}{4}\sin(\sqrt{24}\, t) \end{pmatrix} = \begin{pmatrix} 6\cos(2\sqrt{6}\, t) \\ -\frac{3\sqrt{6}}{2}\sin(2\sqrt{6}\, t) \end{pmatrix}$$

3. Given the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix}\mathbf{Y}$$

$A = \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 7-\lambda & -5 \\ 4 & 3-\lambda \end{pmatrix}$

(a) Find the general solution.

$\det\begin{pmatrix} 7-\lambda & -5 \\ 4 & 3-\lambda \end{pmatrix} = 0$

$(7-\lambda)(3-\lambda) - (-5)(4) = 0$

$\lambda^2 - 10\lambda + 41 = 0$

$\lambda = \dfrac{10 \pm \sqrt{100-164}}{2}$

$\begin{array}{r} 41 \\ \times 4 \\ \hline 164 \end{array}$

$\lambda = \dfrac{10 \pm \sqrt{-64}}{2} = \dfrac{10 \pm 8i}{2} = 5 \pm 4i$

$\lambda_1 = 5+4i : (A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$\begin{pmatrix} 2-4i & -5 \\ 4 & -2-4i \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(2-4i)x_1 - 5y_1 = 0$

$y_1 = \dfrac{(2-4i)x_1}{5}, \quad x_1 = \alpha$

$y_1 = \dfrac{(2-4i)\alpha}{5}$

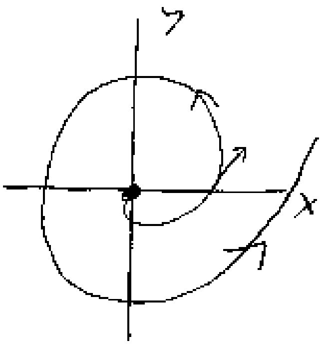
$\begin{pmatrix} \alpha \\ \frac{(2-4i)\alpha}{5} \end{pmatrix}, \ \alpha = 5, \ \vec{v}_1 = \begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$

$e^{\lambda_1 t}\vec{v}_1 = e^{(5+4i)t}\begin{pmatrix} 5 \\ 2-4i \end{pmatrix} = e^{5t}(\cos(4t) + i\sin(4t))\begin{pmatrix} 5 \\ 2-4i \end{pmatrix}$

$= e^{5t}\begin{pmatrix} 5\cos(4t) + 5i\sin(4t) \\ 2\cos(4t) + 2i\sin(4t) - 4i\cos(4t) + 4\sin(4t) \end{pmatrix} = e^{5t}\begin{pmatrix} 5\cos(4t) \\ 2\cos(4t) + 4\sin(4t) \end{pmatrix} + ie^{5t}\begin{pmatrix} 5\sin(4t) \\ 2\sin(4t) - 4\cos(4t) \end{pmatrix}$

$\boxed{\vec{Y}(t) = K_1 e^{5t}\begin{pmatrix} 5\cos(4t) \\ 2\cos(4t) + 4\sin(4t) \end{pmatrix} + K_2 e^{5t}\begin{pmatrix} 5\sin(4t) \\ 2\sin(4t) - 4\cos(4t) \end{pmatrix}}$

(b) Sketch the phase portrait. Classify the origin.



$\lambda = 5 \pm 4i$, origin is a $\boxed{\text{spiral source}}$

$\dfrac{d\vec{Y}}{dt} = \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix}\vec{Y}$, Let $\vec{Y} = (1, 0)$

$= \begin{pmatrix} 7 & -5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} \rightarrow$: vector from vector field $\Rightarrow$ ccw

(c) Find the particular solution that satisfies the initial condition $\mathbf{Y}(0) = (5, 1)$. Report your solution as one real vector.

$\vec{Y}(0) = K_1\begin{pmatrix} 5 \\ 2 \end{pmatrix} + K_2\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 5K_1 \\ 2K_1 - 4K_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$5K_1 = 5$

$K_1 = 1$

$2K_1 - 4K_2 = 1$

$-4K_2 = 1 - 2$

$-4K_2 = -1, \ K_2 = \frac{1}{4}$

$\vec{Y}(t) = \begin{pmatrix} 5e^{5t}\cos(4t) + \frac{5}{4}e^{5t}\sin(4t) \\ e^{5t}\cos(4t) + \frac{9}{2}e^{5t}\sin(4t) \end{pmatrix} = e^{5t}\begin{pmatrix} 5\cos(4t) + \frac{5}{4}\sin(4t) \\ \cos(4t) + \frac{9}{2}\sin(4t) \end{pmatrix}$

4. Given the second-order differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2y = 0 \quad , \; y'' = -2y - y'$$

(a) Find the corresponding first-order system by letting $v = \frac{dy}{dt}$.

$$v = \frac{dy}{dt}$$

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -2y - y' = -2y - v$$

$$\boxed{\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2y - v \end{aligned}}$$

$$\boxed{\begin{aligned} &\text{or } \frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \vec{Y}, \quad \vec{Y} = \begin{pmatrix} y \\ v \end{pmatrix} \\ &\text{or } \frac{d\vec{Y}}{dt} = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix} \vec{Y}, \quad \vec{Y} = \begin{pmatrix} v \\ y \end{pmatrix} \end{aligned}}$$

(b) Find the eigenvalues. Do not find the general solution.

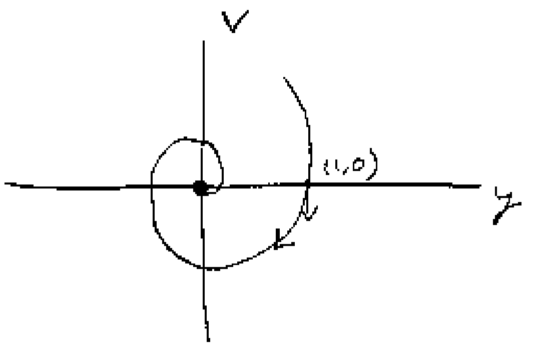
$$\det \begin{pmatrix} -\lambda & 1 \\ -2 & -1-\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-1-\lambda) - (1)(-2) = 0$$

$$\lambda^2 + \lambda + 2 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-8}}{2} = \boxed{\frac{-1 \pm \sqrt{7}\, i}{2} = \lambda}$$

(c) Sketch the phase portrait. Classify the origin.



$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

origin is a $\boxed{\text{spiral sink}}$

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \vec{Y}, \text{ let } \vec{Y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

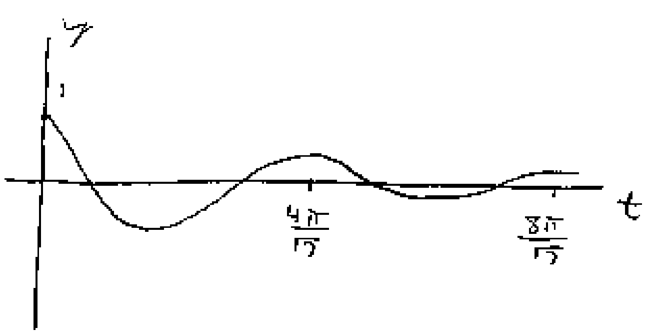
$$= \begin{pmatrix} 0 \\ -2 \end{pmatrix} \downarrow \Rightarrow$$

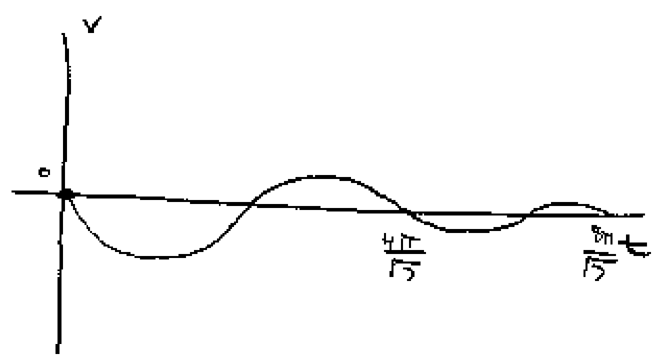
cw

(d) What is the natural period of the system? $\lambda = a \pm bi$, $a = -\frac{1}{2}$, $b = \frac{\sqrt{7}}{2}$

$$\text{period} = \frac{2\pi}{b} = \frac{2\pi}{\frac{\sqrt{7}}{2}} = \boxed{\frac{4\pi}{\sqrt{7}}}$$

(e) Sketch the $y(t)$- and $v(t)$-graphs for the particular solution that satisfies the initial condition $(y(0), v(0)) = (1, 0)$.





5. Given the system

$$Y' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} Y \qquad A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix}$$

(a) Find the particular solution with the initial condition $\mathbf{Y}(0) = (1, 0)$. Report your solution as one real vector.

$$\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - (1)(-1) = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda-3)(\lambda-3) = 0$$

$$\lambda = 3$$

Repeated

$$\vec{Y}(t) = e^{\lambda t}\vec{V_0} + te^{\lambda t}\vec{V_1}$$

$$\vec{V_0} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \vec{Y}(0), \quad x_0 = 1, \ y_0 = 0$$

$$\vec{V_1} = (A - \lambda I)\vec{V_0}$$

$$= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}$$

$$\vec{Y}(t) = e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{3t}\begin{pmatrix} -x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}$$

$$\vec{Y}(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} e^{3t} - te^{3t} \\ -te^{3t} \end{pmatrix}}$$

(b) Graph the phase portrait using HPGSystemSolver (from the software associated with your book). Graph at least three separate trajectories in the phase portrait. Print and include your results.

See attached.

6. Given the system

$$\frac{dx}{dt} = 2x - 4y$$
$$\frac{dy}{dt} = -3x + 6y$$

$$A = \begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 2-\lambda & -4 \\ -3 & 6-\lambda \end{pmatrix}$$

(a) Find the general solution.

$$\det\begin{pmatrix} 2-\lambda & -4 \\ -3 & 6-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(6-\lambda) - (-4)(-3) = 0$$

$$\lambda^2 - 8\lambda = 0$$

$$\lambda(\lambda - 8) = 0$$

$$\lambda_1 = 0, \ \lambda_2 = 8$$

$\lambda_1 = 0$: $(A - \lambda_1 I)\vec{V_1} = \vec{0}$

$$\begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 - 4y_1 = 0$$

$$y_1 = \frac{x_1}{2}, \quad x_1 = \alpha, \quad y_1 = \frac{\alpha}{2}$$

$$\begin{pmatrix} \alpha \\ \frac{\alpha}{2} \end{pmatrix}, \ \alpha = 2, \ \vec{V_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda_2 = 8$: $(A - \lambda_2 I)\vec{V_2} = \vec{0}$

$$\begin{pmatrix} -6 & -4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-6x_2 - 4y_2 = 0$$

$$y_2 = \frac{-6x_2}{4} = \frac{-3x_2}{2}$$

$$x_2 = \alpha, \ y_2 = \frac{-3\alpha}{2}$$

$$\begin{pmatrix} \alpha \\ -\frac{3\alpha}{2} \end{pmatrix}, \ \alpha = 2, \ \vec{V_2} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\vec{Y}(t) = K_1 e^{0t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + K_2 e^{8t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\boxed{\vec{Y}(t) = K_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + K_2 e^{8t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}}$$

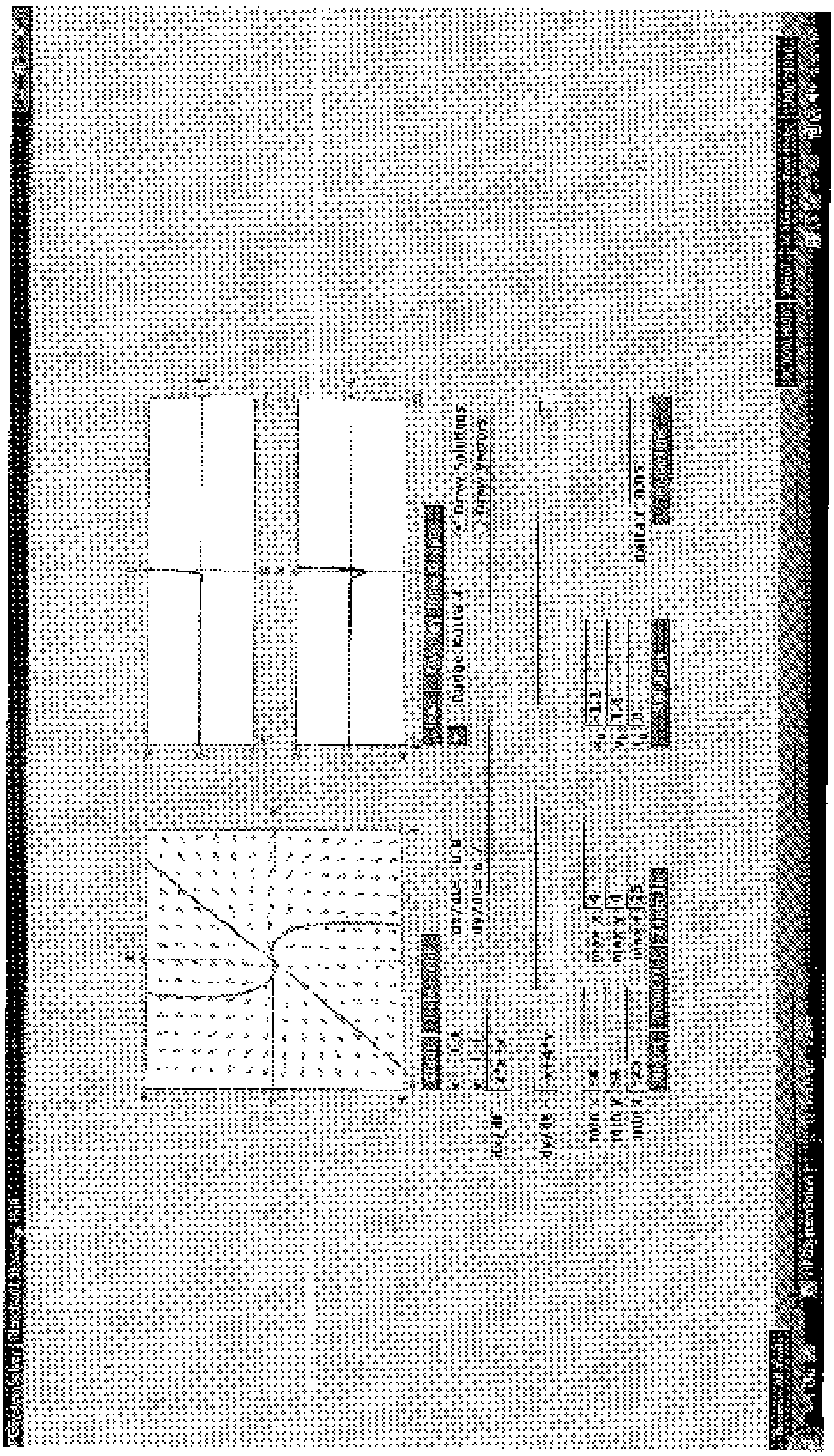
(b) Graph the phase portrait using HPGSystemSolver (from the software associated with your book). Graph at least three separate trajectories in the phase portrait. Print and include your results. See attached.

(c) For the solution that satisfies the initial condition $(x(0), y(0)) = (1, 1)$, find

$\lim_{t\to\infty} x(t) = -\infty$

$\lim_{t\to\infty} y(t) = \infty$

5(b)



#6(b)