MATH225, Fall 2011
Worksheet 9 (7.3, 7.4, 7.5)

Name:
Section: Solutions

For full credit, you must show all work and box answers.

1. Find the Laplace transform of the following functions. (Use the table.)

(a) $g(t) = e^{2t}\cos(3t)$

$\boxed{G(s) = \mathcal{L}\{g(t)\} = \frac{s-2}{(s-2)^2+9}}$ #13, a=2, b=3

or $g(t) = e^{2t}\cos(3t)$, #9, a=2, $f(t) = \cos(3t)$

$F(s) = \frac{s}{s^2+9}$, #6, b=3

$\boxed{G(s) = F(s-a) = \frac{s-2}{(s-2)^2+9}}$

(b) $g(t) = 4t^2e^t$

$\boxed{G(s) = \mathcal{L}\{g(t)\} = \frac{8}{(s-1)^3}}$ #11, a=1, n=2

or $g(t) = 4e^t t^2$, #9, a=1, $f(t) = t^2$

$F(s) = \frac{2}{s^3}$ #4, n=2

$\boxed{G(s) = 4F(s-a) = \frac{8}{(s-1)^3}}$

(c) $h(t) = te^{6t}\sin(3t)$
~~Hint~~ Use Formula 10, #10, n=1, $f(t) = e^{6t}\sin(3t)$

$F(s) = \mathcal{L}\{f(t)\} = \frac{3}{(s-6)^2+9} = 3[(s-6)^2+9]^{-1}$
#12, a=6, b=3

$\frac{dF}{ds} = -3[(s-6)^2+9]^{-2}(2)(s-6) = \frac{-6(s-6)}{[(s-6)^2+9]^2}$

$\boxed{H(s) = \mathcal{L}\{h(t)\} = (-1)^1\frac{dF}{ds} = \frac{6(s-6)}{[(s-6)^2+9]^2}}$

or without Formula 10: $h(t) = e^{6t}t\sin(3t)$, #9, a=6, $f(t) = t\sin(3t)$

$F(s) = \frac{6s}{(s^2+9)^2}$

$\boxed{H(s) = F(s-a) = \frac{6(s-6)}{[(s-6)^2+9]^2}}$

(d) $f(t) = 7e^tU(t-2)$, #19, a=2, $g(t) = 7e^t$, $g(t+2) = 7e^{t+2} = 7e^2e^t$

$F(s) = \mathcal{L}\{f(t)\} = e^{-2s}\mathcal{L}\{7e^2e^t\} = e^{-2s}(7e^2)\mathcal{L}\{e^t\}$, #3, a=

$\boxed{F(s) = 7e^2e^{-2s}\left(\frac{1}{s-1}\right)}$

(e) $f(t) = t^2U(t-3)$, #19, a=3, $g(t) = t^2$, $g(t+3) = (t+3)^2 = t^2+6t+9$

$F(s) = \mathcal{L}\{f(t)\} = e^{-3s}\mathcal{L}\{t^2+6t+9\} = \boxed{e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)}$
#4, n=2, n=1 #2

(f) $h(t) = \begin{cases} e^{2t}, & 0 \le t < 1 \\ 6t, & t \ge 1 \end{cases}$
Hint: Write $h(t)$ in terms of Heaviside functions first.

$h(t) = e^{2t} + U(t-1)(6t - e^{2t})$

#3, a=2 # 19, a=1, $g(t) = 6t - e^{2t}$

$g(t+1) = 6(t+1) - e^{2(t+1)}$
$= 6t + 6 - e^2e^{2t}$

$H(s) = \mathcal{L}\{h(t)\} = \frac{1}{s-2} + e^{-s}\mathcal{L}\{6t + 6 - e^2e^{2t}\}$
#4 n=1 #2 #3, a=2

$\boxed{H(s) = \frac{1}{s-2} + e^{-s}\left[\frac{6}{s^2} + \frac{6}{s} - e^2\left(\frac{1}{s-2}\right)\right]}$

2. Find the inverse Laplace transform of the following.

(a) $F(s) = \frac{s+4}{s^2+2s+10}$

$b^2 - 4ac = 4 - 40 < 0$
No real factors

CS: $s^2+2s+10 = \underbrace{s^2+2s+\left(\frac{2}{2}\right)^2}+10-\left(\frac{2}{2}\right)^2$
$= (s+1)^2+9$

$F(s) = \frac{s+4}{(s+1)^2+9}$

$= \frac{s+1}{(s+1)^2+9} + \frac{3}{(s+1)^2+9}$

13, a=-1, b=3 # 12, a=-1, b=3

$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t}\cos(3t) + e^{-t}\sin(3t)}$

(b) $G(s) = \frac{e^{-3s}+1}{(s+4)(s+1)} = \frac{1}{(s+4)(s+1)} + e^{-3s}\left(\frac{1}{(s+4)(s+1)}\right)$

PF: $\frac{1}{(s+4)(s+1)} = \frac{A}{s+4} + \frac{B}{s+1}$

$A(s+1) + B(s+4) = 1$

$s=-1$: $3B = 1$, $B = \frac{1}{3}$

$s=-4$: $-3A = 1$, $A = -\frac{1}{3}$

$G(s) = -\frac{1}{3}\left(\frac{1}{s+4}\right) + \frac{1}{3}\left(\frac{1}{s+1}\right) + e^{-3s}\left[-\frac{1}{3}\left(\frac{1}{s+4}\right) + \frac{1}{3}\left(\frac{1}{s+1}\right)\right]$

$\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$, $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$, #3, a=-4, -

18, a=3

$\boxed{g(t) = \mathcal{L}^{-1}\{G(s)\} = -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} + U(t-3)\left[-\frac{1}{3}e^{-4(t-3)} + \frac{1}{3}e^{-(t-3)}\right]}$

(c) $G(s) = \frac{e^{-2s}}{s^2-4s+9}$

$b^2 - 4ac = 16 - 36 < 0$
No real factors

CS: $s^2-4s+9 = \underbrace{s^2-4s+\left(-\frac{4}{2}\right)^2}+9-\left(-\frac{4}{2}\right)^2$
$= (s-2)^2+5$

$G(s) = e^{-2s}\frac{1}{\sqrt{5}}\left(\frac{1(\sqrt{5})}{(s-2)^2+5}\right)$

$\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{(s-2)^2+5}\right\} = e^{2t}\sin(\sqrt{5}t)$

12, a=-2, b=$\sqrt{5}$

18, a=2

$\boxed{g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{\sqrt{5}}U(t-2)e^{2(t-2)}\sin(\sqrt{5}(t-2))}$

3. Find the Laplace transform of the following functions. (Use the table.)

(a) $h(t) = e^{2t} * \cosh(3t)$

Do not evaluate the integral before transforming.

$\#16$, $f(t) = e^{2t}$, $g(t) = \cosh(3t)$

$F(s) = \frac{1}{s-2}$, $G(s) = \frac{s}{s^2-9}$

$\#3, a=2$ $\quad \#8, b=3$

$$H(s) = \mathcal{L}\{h(t)\} = F(s)G(s) = \left(\frac{1}{s-2}\right)\left(\frac{s}{s^2-9}\right) = \boxed{\frac{s}{(s-2)(s^2-9)}}$$

(b) $h(t) = \int_0^t \tau \sin(\tau) d\tau$

Do not evaluate the integral before transforming.

$$f*g = \int_0^t f(\tau) g(t-\tau) d\tau$$

$h(t) = \int_0^t \tau \sin(\tau) d\tau = f*g$, $f(\tau) = \tau \sin \tau$, $g(t-\tau) = 1$,

$\#14, b=1$ $f(t) = t \sin t$, $g(t) = 1$

$\#16$ $\quad \#2$

$$H(s) = \mathcal{L}\{h(t)\} = F(s)G(s) = \left(\frac{2s}{(s^2+1)^2}\right)\left(\frac{1}{s}\right) = \boxed{\frac{2}{(s^2+1)^2}}$$

4. Find the inverse Laplace transform of $H(s) = \frac{\quad}{(s+2)(s+1)}$ using:

(a) Partial Fractions

$$H(s) = \frac{4}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} = \frac{-4}{s+2} + \frac{4}{s+1}$$

$\#3, a=-2, a=-1$

$A(s+1) + B(s+2) = 4$

$\boxed{h(t) = \mathcal{L}^{-1}\{H(s)\} = -4e^{-2t} + 4e^{-t}}$

$s=-1$: $B = 4$

$s=-2$: $-A = 4$, $A = -4$

(b) The Convolution Integral. Simplify your solution. Your final answer should not contain integrals or convolutions.

$$H(s) = \frac{4}{(s+2)(s+1)} = \left(\frac{4}{s+2}\right)\left(\frac{1}{s+1}\right)$$

$\#16$, $F(s) = \frac{4}{s+2}$, $G(s) = \frac{1}{s+1}$

$f(t) = 4e^{-2t}$ $\quad g(t) = e^{-t}$

$\#3, a=-2$ $\quad \#3, a=-1$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = f(t) * g(t) = 4e^{-2t} * e^{-t}$$

$$= \int_0^t 4e^{-2\tau} e^{-(t-\tau)} d\tau$$

$$= \int_0^t 4e^{-t} e^{-\tau} d\tau$$

$$= -4e^{-t}e^{-\tau} \Big|_0^t$$

$$= -4e^{-t}e^{-t} - (-4e^{-t}e^{0})$$

$$\boxed{h(t) = -4e^{-2t} + 4e^{-t}}$$

5. Given $y'' + 9y = \cos(3t)$, $y(0) = y'(0) = 0$. Solve using

(a) The Method of Undetermined Coefficients.

y_h: $y_h'' + 9y_h = 0$, $y_h = e^{rt}$

$r^2 + 9 = 0$, $r^2 = -9$, $r = \pm 3i$

$e^{rt} = e^{3it} = \cos(3t) + i\sin(3t)$

$y_h = C_1\cos(3t) + C_2\sin(3t)$

y_p: $y_p'' + 9y_p = \cos(3t)$

$y_p = \cancel{A\cos(3t) + B\sin(3t)}$ * Appears in y_h

$y_p = At\cos(3t) + Bt\sin(3t)$

$y_p' = A\cos(3t) - 3At\sin(3t) + B\sin(3t) + 3Bt\cos(3t)$

$y_p'' = -3A\sin(3t) - 3A\sin(3t) - 9At\cos(3t) + 3B\cos(3t) + 3B\cos(3t) - 9Bt\sin(3t)$

$[-6A\sin(3t) - \cancel{9At\cos(3t)} + 6B\cos(3t) - \cancel{9Bt\sin(3t)}] + 9[\cancel{At\cos(3t) + Bt\sin(3t)}] = \cos(3t)$

$-6A = 0$, $A = 0$ $\quad$ $6B = 1$, $B = \frac{1}{6}$, $y_p = \frac{t}{6}\sin(3t)$

$y = C_1\cos(3t) + C_2\sin(3t) + \frac{t}{6}\sin(3t)$ $\quad$ $y(0) = C_1 = 0$

$y' = -3C_1\sin(3t) + 3C_2\cos(3t) + \frac{1}{6}\sin(3t) + \frac{t}{2}\cos(3t)$ $\quad$ $y'(0) = 3C_2 = 0$

$\boxed{y = \frac{t}{6}\sin(3t)}$

(b) Laplace Transforms.

$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\cos(3t)\}$

#1(b) $\quad$ # 6, b=3

$s^2Y(s) - 0 - 0 + 9Y(s) = \frac{s}{s^2+9}$

$(s^2+9)Y(s) = \frac{s}{s^2+9}$

$Y(s) = \frac{s(2)(3)}{(s^2+9)^2}\frac{1}{(2)(3)}$, # 14, b=3

$\boxed{y(t) = \frac{1}{6}t\sin(3t)}$

(c) Assuming this initial-value problem models a spring-mass system, what phenomenon is occurring?

$\boxed{\text{Resonance}}$

6. Use the Laplace transform to solve the following equation. Simplify your solution. Your final answer should not contain integrals or convolutions.

$y'(t) + \int_0^t y(t-\tau)d\tau = t, \quad y(0) = 0$, $f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau$

$\int_0^t (1) y(t-\tau) d\tau = 1 * y(t)$

$f(\tau) = 1, \quad g(t-\tau) = y(t-\tau)$

$f(t) = 1, \quad g(t) = y(t)$

$\mathcal{L}\{y'\} + \mathcal{L}\{1 * y\} = \mathcal{L}\{t\}$

#1(a) #16 #4, n=1

$sY(s) - 0 + \left(\frac{1}{s}\right) Y(s) = \frac{1}{s^2}$

$sY(s) + \frac{1}{s} Y(s) = \frac{1}{s^2}$

$Y(s)\left(s + \frac{1}{s}\right) = \frac{1}{s^2}$

$Y(s)\left(\frac{s^2+1}{s}\right) = \frac{1}{s^2}$

$Y(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

P.F.:

$A(s^2+1) + s(Bs+C) = 1$

$(A+B)s^2 + Cs + A = 1$

$A+B=0 \quad C=0 \quad A=1$

$B=-1$

$Y(s) = \frac{1}{s} - \frac{s}{s^2+1}$

#2 #6, b=1

$\boxed{y(t) = 1 - \cos t}$

7. Solve the following initial-value problem. Simplify your solution. Your final answer should not contain integrals or convolutions.

$y' + 4y = \delta(t-\pi) + \delta(t-2\pi), \quad y(0) = 1.$

$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\} + \mathcal{L}\{\delta(t-2\pi)\}$

#1(a) # 20, a=π #20, a=2π

$sY(s) - 1 + 4Y(s) = e^{-\pi s} + e^{-2\pi s}$

$(s+4)Y(s) = 1 + e^{-\pi s} + e^{-2\pi s}$

$Y(s) = \frac{1}{s+4} + e^{-\pi s}\left(\frac{1}{s+4}\right) + e^{-2\pi s}\left(\frac{1}{s+4}\right)$

$\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$, # 3, a=-4

18, a=π, a=2π

$\boxed{y(t) = e^{-4t} + u(t-\pi)e^{-4(t-\pi)} + u(t-2\pi)e^{-4(t-2\pi)}}$

8. Consider the LRC series circuit with $R = 40\,\Omega$, $L = 1$ h, $C = 0.001$ f. A battery supplying 90 V is attached to the circuit. The switch to the battery is initially closed and left closed for 1 second. At time $t = 1$ it is left open. Initially, $q(0) = i(0) = 0$. The differential equation that describes this situation is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad E(t) = \begin{cases} 90, & 0 \le t < 1 \\ 0, & t > 1 \end{cases}$$

Solve the initial-value problem. Simplify your solution. Your final answer should not contain integrals or convolutions.

$$q'' + 40q' + \frac{1}{\frac{1}{1000}} q = 90 + U(t-1)(-90)$$

$$q'' + 40q' + 1000q = 90 - 90\,U(t-1)$$

$$\underset{\#1(b)}{\mathcal{L}\{q''\}} + 40\underset{\#1(a)}{\mathcal{L}\{q'\}} + 1000\,\mathcal{L}\{q\} = \underset{\#2}{\mathcal{L}\{90\}} - 90\underset{\#17,\ a=1}{\mathcal{L}\{U(t-1)\}}$$

$$s^2Q(s) - 0 - 0 + 40(sQ(s) - 0) + 1000\,Q(s) = \frac{90}{s} - \frac{90e^{-s}}{s}$$

$$(s^2 + 40s + 1000)\,Q(s) = \frac{90}{s} - e^{-s}\left(\frac{90}{s}\right)$$

$$Q(s) = \frac{90}{s(s^2 + 40s + 1000)} - e^{-s}\left(\frac{90}{s(s^2+40s+1000)}\right)$$

$s^2 + 40s + 1000$

$b^2 - 4ac = 1600 - 4000 < 0$

No real Factors

CS: $s^2 + 40s + 1000 = \underbrace{s^2 + 40s + \left(\frac{40}{2}\right)^2} + 1000 - \left(\frac{40}{2}\right)^2$

$= (s+20)^2 + 600$

PF: $\dfrac{90}{s(s^2+40s+1000)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+40s+1000}$

$A(s^2 + 40s + 1000) + s(Bs + C) = 90$

$(A+B)s^2 + (40A + C)s + 1000A = 90$

$A + B = 0$	$40A + C = 0$	$1000A = 90$
$B = -A$	$C = -40A$	$A = \frac{9}{100}$
$B = -\frac{9}{100}$	$C = \frac{-360}{100}$	

$$Q(s) = \frac{9}{100}\left(\frac{1}{s}\right) - \frac{9}{100}\left(\frac{s+40}{(s+20)^2+600}\right) - e^{-s}\left[\frac{9}{100}\left(\frac{1}{s}\right) - \frac{9}{100}\left(\frac{s+40}{(s+20)^2+600}\right)\right]$$

$$Q(s) = \underset{\#2}{\frac{9}{100}\left(\frac{1}{s}\right)} - \frac{9}{100}\left[\frac{s+20}{(s+20)^2+600} + \frac{20}{10\sqrt{6}}\left(\frac{10\sqrt{6}}{(s+20)^2+600}\right)\right] - e^{-s}\left[\frac{9}{100}\left(\frac{1}{s}\right) - \frac{9}{100}\left[\frac{s+20}{(s+20)^2+600} + \frac{20}{10\sqrt{6}}\left(\frac{10\sqrt{6}}{(s+20)^2+ \ldots}\right.\right.\right.$$

8. (cont.)

$$g(t) = \frac{9}{100} - \frac{9}{100} e^{-20t} \cos(10\sqrt{6}\, t) - \frac{9}{50\sqrt{6}} e^{-20t} \sin(10\sqrt{6}\, t)$$
$$+ u(t-1) \left[\frac{9}{100} - \frac{9}{100} e^{-20(t-1)} \cos(10\sqrt{6}(t-1)) - \frac{9}{50\sqrt{6}} e^{-20(t-1)} \sin(10\sqrt{6}(t-1)) \right]$$

<u>or</u>

$$g(t) = 0.09 - 0.09 e^{-20t} \cos(\sqrt{600}\, t) - \frac{1.8}{\sqrt{600}} e^{-20t} \sin(\sqrt{600}\, t)$$
$$- u(t-1) \left[0.09 - 0.09 e^{-20(t-1)} \cos(\sqrt{600}(t-1)) - \frac{1.8}{\sqrt{600}} e^{-20(t-1)} \sin(\sqrt{600}(t-1)) \right]$$

Table of Common Taylor/Maclaurin Series

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\end{aligned}
$$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1. (a) $f'(t)$ (b) $f''(t)$	$sF(s) - f(0)$ $s^2F(s) - sf(0) - f'(0)$	11. $e^{at}t^n,\ n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
2. 1	$\frac{1}{s}$	12. $e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
3. e^{at}	$\frac{1}{s-a}$	13. $e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
4. $t^n,\ n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$	14. $t\sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
5. $\sin(bt)$	$\frac{b}{s^2+b^2}$	15. $t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
6. $\cos(bt)$	$\frac{s}{s^2+b^2}$	16. $f * g$	$F(s)G(s)$
7. $\sinh(bt)$	$\frac{b}{s^2-b^2}$	17. $U(t-a) = u_a(t),\ a \geq 0$	$\frac{e^{-as}}{s}$
8. $\cosh(bt)$	$\frac{s}{s^2-b^2}$	18. $f(t-a)U(t-a)$ $= f(t-a)u_a(t),\ a \geq 0$	$e^{-as}F(s)$
9. $e^{at}f(t)$	$F(s-a)$	19. $g(t)U(t-a)$ $= g(t)u_a(t),\ a \geq 0$	$e^{-as}\mathcal{L}\{g(t+a)\}$
10. $t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$	20. $\delta_a(t) = \delta(t-a),\ a \geq 0$	e^{-as}