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3, 4, 12, 13

3. $\tau = 0$, $\Delta \neq 0$

$\tau^2 - 4\Delta = -4\Delta$

$\boxed{\begin{array}{l}\Delta > 0 : \text{Center} \\ \Delta < 0 : \text{Saddle point}\end{array}}$

4. stable spiral point

$\boxed{\lambda = a \pm bi, \; a < 0}$

12. $\vec{X}' = A\vec{X}$

(a) $\vec{X}(t) = c_1 \binom{1}{1} e^{-t} + c_2 \binom{1}{-2} e^{-2t}$ $\quad$ (0,0): $\boxed{\text{Stable Node}}$

$\lambda_1 = -1, \; \lambda_2 = -2 < 0$

straight-lines:

$\vec{V}_1 = \binom{1}{1}$ $\qquad$ $\vec{V}_2 = \binom{1}{-2}$

$v_1 = 1, \; v_2 = 1 = v_1$ $\qquad$ $v_1 = 1, \; v_2 = -2 = -2v_1$

$y = x$ $\qquad$ $y = -2x$

$$\vec{X}(t) = \underbrace{c_1 e^{-t} \binom{1}{1}}_{\text{Dominates as } t \to \infty} + c_2 e^{-2t} \binom{1}{-2}$$

If $c_1 = 0, c_2 \neq 0$: Initial value on $y = -2x$.
$\vec{X}(t) \to \vec{0}$ as $t \to \infty$ along line $y = -2x$

If $c_1 \neq 0, c_2 = 0$: Initial value on $y = x$
$\vec{X}(t) \to \vec{0}$ as $t \to \infty$ on $y = x$

If $c_1 \neq 0, c_2 \neq 0$: $\vec{X}(t) \to 0$ as $t \to \infty$ in direction of $y = x$

12. (b) $\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$

$\lambda_1 = -1 < 0$, $\lambda_2 = 2 > 0$

$\boxed{(0,0) : \text{Saddle Point}}$

<u>straight-lines</u>

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$v_1 = 1$, $v_2 = -1 = -v_1$

$y = -x$

$\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$v_1 = 1$, $v_2 = 2 = 2v_1$

$y = 2x$

$$\vec{X}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \underbrace{c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}}_{\text{Dominates as } t \to \infty}$$

If $c_1 = 0$, $c_2 \neq 0$: Initial value on $y = 2x$. $\vec{X}(t)$ becomes unstable as $t \to \infty$ on $y = 2x$.

If $c_1 \neq 0$, $c_2 = 0$: Initial value on $y = -x$. $\vec{X}(t) \to \vec{0}$ as $t \to \infty$ on $y = -x$

If $c_1 \neq 0$, $c_2 \neq 0$: $\vec{X}(t)$ becomes unstable as $t \to \infty$ in direction of line $y = 2x$.
$y = 2x$ is an asymptote for solutions.

13. (a) $x' = -3x + 4y$

$y' = -5x + 3y$

$$A = \begin{pmatrix} -3 & 4 \\ -5 & 3 \end{pmatrix}$$

$\tau = 0$, $\Delta = (-3)(3) - 4(-5) = 11 > 0$

$\tau^2 - 4\Delta = -44 < 0$

$\boxed{(0,0) \text{ is center}}$

(b) $x' = -3x + 2y$

$y' = -2x + y$

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$$

$\tau = -3+1 = -2 < 0$, $\Delta = (-3)(1) - (2)(-2) = 1 > 0$

$\tau^2 - 4\Delta = 4 - 4 = 0$

$\boxed{(0,0) \text{ is a degenerate stable node}}$

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6, 7, 14, 20 (a)

6. $A = g'(\vec{X}_1) = J(\vec{X}_1)$

$\tau > 0$, $\Delta > 0$

$\tau^2 - 4\Delta = ?$

$\vec{X}_1$ is an unstable node, or an unstable spiral point, or an unstable degenerate node

$\boxed{\text{True}}$

7. $\boxed{\text{False}}$

Periodic solutions would mean the equilibrium point is a center & it is not possible to determine centers from linearization.

14. $x' = x + xy - 3x^2$

$y' = 4y - 2xy - y^2$

EP: $x(1+y-3x) = 0$ and $y(4-2x-y) = 0$

$x = 0$ $\quad 1 + y - 3x = 0$

$x = \frac{1+y}{3}$

$x = 0$: $y(4-y) = 0$

$y = 0$ $\quad 4 - y = 0$

$y = 4$

$x = \frac{1}{3} + \frac{y}{3}$: $y(4 - \frac{2}{3} - \frac{2y}{3} - y) = 0$

$y = 0$ $\quad \frac{10}{3} - \frac{5y}{3} = 0$

$\downarrow$ $\quad y = 2$

$x = \frac{1}{3}$ $\quad \downarrow$

$x = 1$

$\boxed{\text{EP}: (0,0), (0,4), (\tfrac{1}{3},0), (1,2)}$

14. (cont.)

$$g'(x,y) = J(x,y) = \begin{pmatrix} 1+y-6x & x \\ -2y & 4-2x-2y \end{pmatrix}$$

$(0,0)$: $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = A$

$\tau = 1+4 = 5 > 0$

$\Delta = 4 - 0 > 0$

$\tau^2 - 4\Delta = 25 - 16 > 0$

$\boxed{\text{Unstable node}}$

$(0,4)$: $J(0,4) = \begin{pmatrix} 5 & 0 \\ -8 & -4 \end{pmatrix}$

$\tau = 5 - 4 = 1 > 0$

$\Delta = -20 - 0 < 0$

$(0,4)$ is a $\boxed{\text{saddle point}}$

$(\frac{1}{3},0)$: $J(\frac{1}{3},0) = \begin{pmatrix} -1 & \frac{1}{3} \\ 0 & \frac{10}{3} \end{pmatrix}$

$\tau = -1 + \frac{10}{3} = \frac{7}{3} > 0$

$\Delta = -\frac{10}{3} - 0 < 0$

$(\frac{1}{3},0)$ is a $\boxed{\text{saddle point}}$

$(1,2)$: $J(1,2) = \begin{pmatrix} -3 & 1 \\ -4 & -2 \end{pmatrix}$

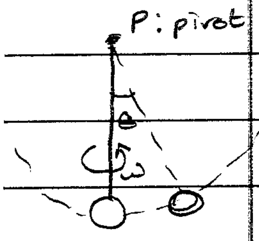
$\tau = -3-2 = -5 < 0$

$\Delta = 6 + 4 = 10 > 0$

$\tau^2 - 4\Delta = 25 - 40 < 0$

$(1,2)$ is a $\boxed{\text{stable spiral point}}$

20. $m\ell \frac{d^2\theta}{dt^2} = \omega^2 m\ell \sin\theta\cos\theta - mg\sin\theta - \beta\frac{d\theta}{dt}$



(a) If $\omega^2 < \frac{g}{\ell}$: $(0,0)$ is stable critical point & only critical point in $-\pi < \theta < \pi$.

Convert to 1st-order system

Let $y = \frac{d\theta}{dt}$

$$\frac{dy}{dt} = \frac{d^2\theta}{dt^2} = \omega^2\sin\theta\cos\theta - \frac{g}{\ell}\sin\theta - \frac{\beta}{m\ell}y$$

$$\frac{d\theta}{dt} = y$$
$$\frac{dy}{dt} = \omega^2\sin\theta\cos\theta - \frac{g}{\ell}\sin\theta - \frac{\beta}{m\ell}y$$

Find EP:

$\underline{y=0}$ and $\omega^2\sin\theta\cos\theta - \frac{g}{\ell}\sin\theta - \frac{\beta}{m\ell}y = 0$

$\underline{y=0}$: $\omega^2\sin\theta\cos\theta - \frac{g}{\ell}\sin\theta = 0$

$\sin\theta(\omega^2\cos\theta - \frac{g}{\ell}) = 0$

$\sin\theta = 0$ $\quad$ $\omega^2\cos\theta - \frac{g}{\ell} = 0$

$\theta = 0, \cancel{\pm\pi}, \cancel{\pm 2\pi}, \ldots$ not in domain $\quad$ $\cos\theta = \frac{g}{\ell\omega^2}$

If $\frac{g}{\ell} > \omega^2$, $\frac{g/\ell}{\omega^2} > 1$,

$\cos\theta \neq \frac{g/\ell}{\omega^2}$

No EP.

$\boxed{\text{EP: } (0,0) \text{ in } -\pi < \theta < \pi}$

Jacobian: $g'(\theta,y) = J(\theta,y) = \begin{pmatrix} 0 & 1 \\ \omega^2\cos^2\theta - \omega^2\sin^2\theta - \frac{g}{\ell}\cos\theta & -\frac{\beta}{m\ell} \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ \omega^2 - \frac{g}{\ell} & -\frac{\beta}{m\ell} \end{pmatrix}$

$\tau = -\frac{\beta}{m\ell} < 0$, $\beta, m, \ell > 0$

$\Delta = 0 - (\omega^2 - \frac{g}{\ell}) = -\omega^2 + \frac{g}{\ell} > 0$, $\frac{g}{\ell} > \omega^2$

(0,0) is a stable node, stable degenerate node, or stable spiral point

20. (a) (cont.)

What occurs physically when $\theta(0)=\theta_0$, $\theta'(0)=0$, + θ_0 is small?

Since $(0,0)$ is a stable EP, if θ_0 is small, thus the bob is close to the EP, $\vec{x}(t) = \begin{pmatrix} \theta(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \theta(t) \\ \theta'(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$ so the bob approaches the rest position or $\theta=0$, $\theta'=0$.

Optional (b) If $\omega^2 > \frac{g}{\ell}$ show that $(0,0)$ is unstable + there are 2 additional EP $(\pm\hat{\theta}, 0)$ in $-\pi < \theta < \pi$.

Find EP:

$y=0$ and $\omega^2 \sin\theta\cos\theta - \frac{g}{\ell}\sin\theta - \frac{\beta}{m\ell}y = 0$

$y=0$: $\sin\theta(\omega^2\cos\theta - \frac{g}{\ell}) = 0$

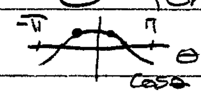
$\sin\theta = 0$ $\qquad$ $\omega^2\cos\theta = \frac{g}{\ell}$

$\theta = 0, \cancel{\pm\pi}, \cancel{\pm 2\pi}, \ldots$ Not in domain $\qquad$ $\cos\theta = \frac{g}{\ell\omega^2}$

$\theta = \cos^{-1}\left(\frac{g}{\ell\omega^2}\right)$

$0 < \frac{\frac{g}{\ell}}{\omega^2} < 1$ if $\frac{g}{\ell} < \omega^2$

$\theta = \pm\hat{\theta}$ for $-\pi < \theta < \pi$



$$\boxed{\text{EP}: (0,0), (\pm\hat{\theta}, 0)}$$

Jacobian: $J(\theta, y) = \begin{pmatrix} 0 & 1 \\ \omega^2\cos^2\theta - \omega^2\sin^2\theta - \frac{g}{\ell}\cos\theta & -\frac{\beta}{m\ell} \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ \omega^2 - \frac{g}{\ell} & -\frac{\beta}{m\ell} \end{pmatrix}$

$\tau = -\frac{\beta}{m\ell} < 0$, $\Delta = -(\omega^2 - \frac{g}{\ell}) < 0$, $\frac{g}{\ell} < \omega^2$

20. (b) (cont.)

$\tau < 0$, $\Delta < 0$

$\boxed{(0,0) \text{ is a saddle point}}$ + so unstable

$$J(\pm\hat{\theta}, 0) = \begin{pmatrix} 0 & 1 \\ \omega^2\cos^2\hat{\theta} - \omega^2\sin^2\hat{\theta} - \frac{g}{\ell}\cos(\hat{\theta}) & -\frac{\beta}{m\ell} \end{pmatrix}$$

$\cos(-\hat{\theta}) = \cos(\hat{\theta})$

$\sin(-\hat{\theta}) = -\sin(\hat{\theta})$

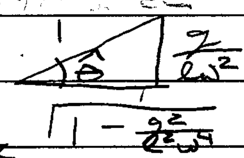
$$\tau = 0 - \frac{\beta}{m\ell} = -\frac{\beta}{m\ell} < 0$$

$$\Delta = -(1)\left(\omega^2\cos^2\hat{\theta} - \omega^2\sin^2\hat{\theta} - \frac{g}{\ell}\cos(\hat{\theta})\right)$$

$$= \frac{g}{\ell}\cos(\hat{\theta}) + \omega^2\sin^2\hat{\theta} - \omega^2\cos^2\hat{\theta}$$

$$= \frac{g}{\ell}\left(\frac{g}{\ell\omega^2}\right) + \omega^2\left(1 - \frac{g^2}{\ell^2\omega^4}\right) - \omega^2\left(\frac{g}{\ell\omega^2}\right)^2$$

* $\cos(\hat{\theta}) = \frac{g}{\ell\omega^2}$ from EP calculations



$$\sin\hat{\theta} = \sqrt{1 - \frac{g^2}{\ell^2\omega^4}} = \sqrt{\frac{\ell^2\omega^4 - g^2}{\ell^2\omega^4}}$$

$$\Delta = \frac{g^2}{\ell^2\omega^2} + \omega^2 - \frac{2g^2}{\ell^2\omega^2} = \omega^2 - \frac{g^2}{\ell^2\omega^2} > 0, \quad \frac{g}{\ell} < \omega^2$$

$\boxed{(\pm\hat{\theta}, 0) \text{ are stable.}}$

What occurs physically when $\theta(0) = \theta_0$, $\theta'(0) = 0$, + θ_0 is small?

Since (0,0) is a saddle point and $(\pm\hat{\theta}, 0)$ are both stable EP, we expect solutions to move away from (0,0) + towards one of the two stable EP. Thus, the pendulum will reach one of the 2 stable equilibrium position, $\pm\hat{\theta}$.