MATH225, Fall 2011 **Name:**
Worksheet 12 (10.1, 10.2, 10.3, 10.4) **Section:**

For full credit, you must show all work and box answers.

1. Find all the critical (equilibrium) points of the given systems.

   (a) $$\frac{dx}{dt} = x - y$$
   $$\frac{dy}{dt} = 5x + 2y$$

   (b) $$x' = x^2 + xy$$
   $$y' = x^3 + 3y + 2x$$

   (c) $$\frac{dx}{dt} = (x+1)e^y$$
   $$\frac{dy}{dt} = 5y(e^x - 1)$$

2. Given the system

$$\frac{d\mathbf{X}}{dt} = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix} \mathbf{X}$$

(a) Find the general solution.

(b) Classify the critical (equilibrium) point $(0, 0)$ of the system.

(c) Discuss the nature of solutions in a neighborhood of $(0, 0)$.

(d) Find the solution that satisfies the initial condition $\mathbf{X}(0) = (-2, 2)$. Report your solution as one real vector.

3. Classify the critical (equilibrium) point $(0, 0)$ of the linear system by computing the trace τ and the determinant Δ. Use this information to match the given linear systems to their phase portraits.

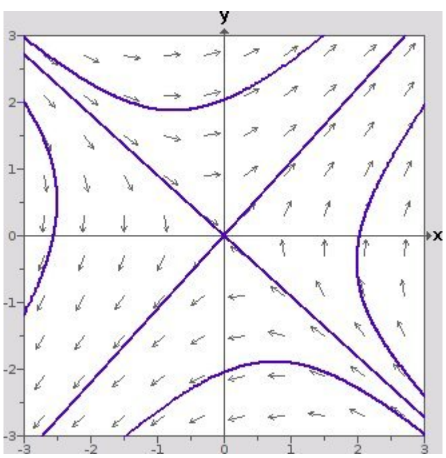
(a)
$$\frac{dx}{dt} = 4x + y \qquad \frac{dy}{dt} = x + 2y$$

(b)
$$\frac{dx}{dt} = -x \qquad \frac{dy}{dt} = 2x - y$$

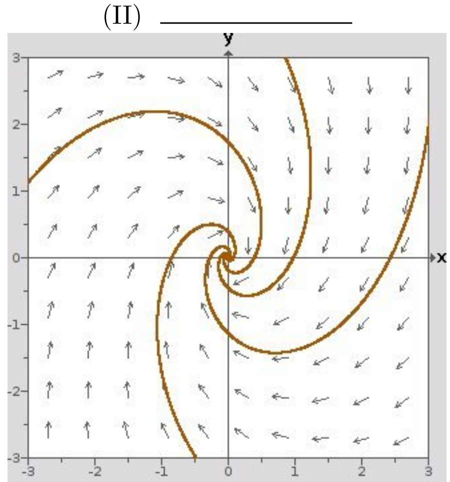
(c)
$$\frac{dx}{dt} = -x + y \qquad \frac{dy}{dt} = -2x - y$$

(d)
$$\frac{dx}{dt} = x + 5y \qquad \frac{dy}{dt} = 5x + 2y$$

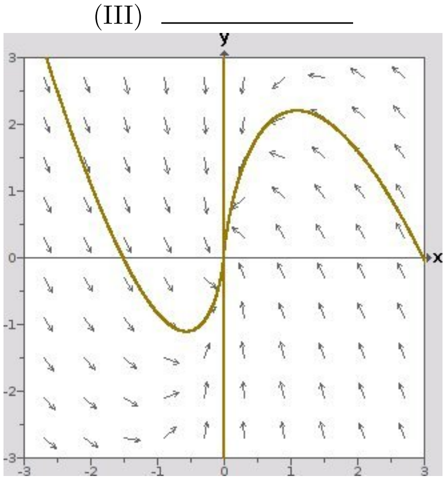
(I) ______________



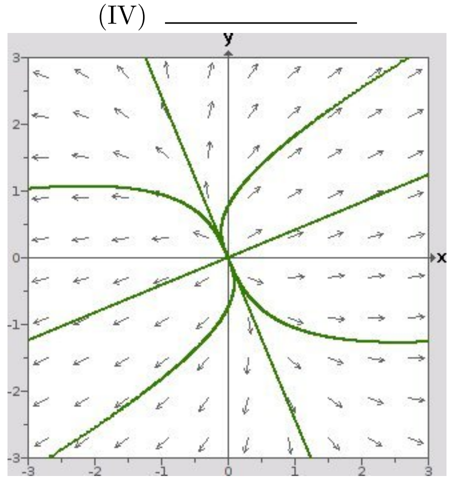
(II) ______________



(III) ______________



(IV) ______________



4. Consider the system:

$$\begin{aligned}\frac{dx}{dt} &= x(1-3x+y)\\ \frac{dy}{dt} &= y(4-y-2x)\end{aligned}$$

(a) Are these species predator-prey or competing?

(b) What type of growth does species y exhibit in absence of species x?

(c) Find all critical (equilibrium) points.

(d) Using the Jacobian matrix, classify (if possible) each critical (equilibrium) point as a stable node, a stable spiral point, an unstable node, an unstable spiral point, or a saddle point.

5. Consider a pendulum made of a rigid rod with a ball at one end. The second-order differential equation which models the damped pendulum is:

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = 0$$

where $\theta(t)$ is the angle at time, t, measured in a counterclockwise direction from the downward vertical. The parameter g is gravity, l is the length of the pendulum, b is the damping coefficient, and m is the mass of the ball (we neglect the mass of the rod). For simplicity, let $l = m = 1$, $b = 2$, and $g \approx 9.8$, thus the equation becomes:

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + 9.8\sin(\theta) = 0$$

(a) Let $v = \dfrac{d\theta}{dt}$ and convert the second-order differential equation to a first-order system.

(b) Find the equilibrium points of the system for $0 \leq \theta < 2\pi$.

(c) Using the Jacobian matrix, classify (if possible) each critical (equilibrium) point for $0 \leq \theta < 2\pi$ as a stable node, a stable spiral point, an unstable node, an unstable spiral point, or a saddle point.