Linear Open Loop Systems

1st Order Systems

Output modeled with a 1\textsuperscript{st} order ODE:

\[ a_1 \frac{dy}{dt} + a_0 y = b \cdot f(t). \]

If \( a_0 \neq 0 \), then:

\[ \frac{a_1}{a_0} \frac{dy}{dt} + y = \frac{b}{a_0} f(t) \Rightarrow \tau_p \frac{dy}{dt} + y = K_p \cdot f(t) \]

where: \( \tau_p \) is the \textit{time constant}.

\( K_p \) is the \textit{steady state gain}, \textit{static gain}, or \textit{gain}.

For deviation variables, where \( y(0) = f(0) = 0 \), the Laplace transform will be:

\[ (\tau_p s + 1) \bar{y}(s) = K_p \bar{f}(s) \Rightarrow G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K_p}{\tau_p s + 1} \]

This transfer function is referred to as \textit{1st order lag}, \textit{linear lag}, or \textit{exponential transfer lag}.

But what if \( a_0 = 0 \)? Then:

\[ a_1 \frac{dy}{dt} = bf(t) \Rightarrow \frac{dy}{dt} = \frac{b}{a_1} f(t) \Rightarrow \frac{dy}{dt} = K'_p \cdot f(t). \]

For deviation variables, the Laplace transform will be:
$$\bar{y}(s) = K'_p \cdot \bar{f}(s) \Rightarrow G(s) = \frac{\bar{y}(s)}{\bar{f}(s)} = \frac{K'_p}{s}.$$  

This transfer function is referred to as purely capacitive or pure integrator.

$\bar{f}(s) \xrightarrow{1^st \text{Order lag}} \frac{K_p}{\tau_p s + 1}, \quad \bar{y}(s) \quad \bar{f}(s) \xrightarrow{\text{Pure Integrator}} \frac{K_p}{s}$

**Example 1st Order Systems — Mercury Thermometer**

Last time we developed the following equation for the reading from a mercury thermometer:

$$\hat{m} \hat{C}_p \frac{dT}{dt} = T_a - T \quad \Rightarrow \quad \hat{m} \hat{C}_p \frac{dT}{dt} + T = T_a$$

So this is a $1^{st}$ order lag system with:

$$\tau_p = \frac{m \hat{C}_p}{hA}, \quad K_p = 1$$

**Example 1st Order Systems — Mass Storage in Tank**

Mass storage in a tank is a $1^{st}$ order system, but we don’t know which type until we say something about how the flow out of the tank is controlled.

For constant density & constant cross-sectional area:

$$\frac{d(\rho_1 Ah)}{dt} = \rho_0 F_0 - \rho_1 F_1 \quad \Rightarrow \quad A \frac{dh}{dt} = F_0 - F_1.$$
For flow through a valve where we can linearize to \( F_i \approx C_i h = h/R \), then:

\[
A \frac{dh}{dt} = F_0 - \frac{h}{R} \quad \Rightarrow \quad AR \frac{dh}{dt} + h = RF_0. 
\]

So this is a 1\(^{st}\) order lag system with:

\[
\tau_p = AR \\
K_p = R
\]

However, if the flowrate out is controlled separately from the level in the tank, e.g., with a pump, then:

\[
A \frac{dh}{dt} = F_0 - F_1 \quad \Rightarrow \quad \frac{dh}{dt} = \frac{F_0 - F_1}{A}.
\]

So this is pure integrator system with:

\[
f(t) = F_0 - F_1 \\
K'_p = \frac{1}{A}.
\]

**Response of 1st Order Systems**

Look at response to 4 typical driving functions.

**Impulse disturbance.** \( f(t) = \alpha \cdot \delta(0) \) \( \Rightarrow \) \( \tilde{f}(s) = \alpha \). So, if 1\(^{st}\) order lag:

\[
\bar{y}(s) = G(s) \cdot \tilde{f}(s) = \frac{\alpha K_p}{\tau_p s + 1} \quad \Rightarrow \quad y(t) = \alpha K_p \frac{e^{-\tau_p/\tau_p}}{\tau_p}
\]

If pure integrator:

\[
\bar{y}(s) = G(s) \cdot \tilde{f}(s) = \frac{K_p}{s} \cdot \alpha \quad \Rightarrow \quad y(t) = \alpha K_p
\]
Unit step change. \( f(t) = \alpha \cdot H(t) \Rightarrow \bar{f}(s) = \alpha / s \). So, if 1\textsuperscript{st} order lag:

\[
\bar{y}(s) = G(s) \cdot \bar{f}(s) = \frac{K_p}{\tau_p s + 1} \frac{\alpha}{s} = \alpha K_p \left( \frac{1}{s} - \frac{\tau_p}{\tau_p s + 1} \right)
\]

\[
= \alpha K_p \left( \frac{1}{s} - \frac{1}{\tau_p} \right)
\]

\[
\Rightarrow y(t) = \alpha K_p \left( 1 - e^{-t/\tau_p} \right).
\]

Notice that \( K_p \) is the fraction of the value of the input disturbance that will show up on the output signal. Also notice that the slope is:

\[
\frac{dy}{dt} = \frac{\alpha K_p}{\tau_p} e^{-t/\tau_p} \Rightarrow \frac{dy}{dt} \bigg|_{t=0} = \frac{\alpha K_p}{\tau_p}
\]

If the system would maintain its initial rate of change, then it would achieve its ultimate value in one time constant, i.e., when \( t = \tau_p \). In reality, the final value is reached in an exponential decay manner — in reality, it takes about 4\( \tau_p \) to reach the ultimate value (when \( y \approx 0.98 \alpha K_p \)).
If pure integrator:

$$\bar{y}(s) = G(s) \cdot \bar{f}(s) = \frac{K_p}{s} \frac{\alpha}{s} = \frac{\alpha K_p}{s^2} \Rightarrow y(t) = \alpha K_p t$$

This shows the integrating nature of this type of 1st order process.

Ramp. \( f(t) = mt \) \( \Rightarrow \bar{f}(s) = m/s^2 \). So, if 1st order lag:

$$\bar{y}(s) = G(s) \cdot \bar{f}(s) = \frac{K_p}{\tau_p s + 1} \frac{m}{s^2} = mK_p \left( \frac{1}{s^2} - \frac{\tau_p}{s} + \frac{\tau_p^2}{\tau_p s + 1} \right)$$

$$\therefore \ y(t) = mK_p \left( t - \tau_p + \tau_p e^{-t/\tau_p} \right).$$

At large times, then:

$$y(t) \approx mK_p \left( t - \tau_p \right)$$

Again, \( K_p \) is the fraction of the value of the input disturbance that will show up on the output signal. Now \( \tau_p \) represents a time offset — how far behind the output signal lags behind the input signal.
Sinusoidal Response. \( f(t) = \alpha \sin \omega t \) \( \Rightarrow \) \( \tilde{f}(s) = \frac{\alpha \omega}{s^2 + \omega^2} \). So, if 1st order lag:

\[
\tilde{y}(s) = G(s) \cdot \tilde{f}(s) = \frac{K_p}{\tau_p s + 1} \frac{\alpha \omega}{s^2 + \omega^2} = \frac{\alpha \omega K_p}{1 + \tau_p^2 \omega^2} \left( \frac{\tau_p^2}{\tau_p s + 1} + \frac{1 - \tau_p s}{s^2 + \omega^2} \right)
\]

\[
y(t) = \frac{\alpha \omega K_p}{1 + \tau_p^2 \omega^2} \left( \tau_p e^{-t/\tau_p} + \frac{1}{\omega} \sin \omega t - \tau_p \cos \omega t \right)
\]

\[
y(t) = \frac{\alpha \omega \tau_p K_p}{1 + \tau_p^2 \omega^2} e^{-t/\tau_p} + \frac{\alpha K_p}{1 + \tau_p^2 \omega^2} \sin \omega t - \frac{\alpha \omega \tau_p K_p}{1 + \tau_p^2 \omega^2} \cos \omega t
\]

There is a trigonometric identity:

\[p \cos \theta + q \sin \theta = r \sin (\theta + \phi)\]

where: \( r = \sqrt{p^2 + q^2} \)
\[
\tan \phi = \frac{p}{q}
\]

So for this problem:

\[
r = \sqrt{\left( \frac{\alpha \omega \tau_p K_p}{1 + \tau_p^2 \omega^2} \right)^2 + \left( \frac{\alpha K_p}{1 + \tau_p^2 \omega^2} \right)^2} = \frac{\alpha K_p \sqrt{1 + \tau_p^2 \omega^2}}{1 + \tau_p^2 \omega^2} = \frac{\alpha K_p}{1 + \tau_p^2 \omega^2}
\]

\[
\tan \phi = \frac{\alpha \omega \tau_p K_p}{1 + \tau_p^2 \omega^2} = -\omega \tau_p \Rightarrow \phi = \tan^{-1}(\omega \tau_p)
\]

and:

\[
y(t) = \frac{\alpha \omega \tau_p K_p}{1 + \tau_p^2 \omega^2} e^{-t/\tau_p} + \frac{\alpha K_p}{1 + \tau_p^2 \omega^2} \sin \omega t - \tan^{-1}(\omega \tau_p)
\]

At large times, then:

\[
y(t) \approx \frac{\alpha K_p}{\sqrt{1 + \tau_p^2 \omega^2}} \sin (\omega t - \tan^{-1}(\omega \tau_p))
\]
Again, $K_p$ represents part of the fraction of the value of the input disturbance that will show up on the output signal. $\tau_p$ also plays a part in the gain, but, more importantly, represents a time offset that manifests itself as a phase angle lag (lag because the angel is subtracted from the input signal). At most, even with very large $\tau_p$ values, the output can lag by no more than 90°.

**Determination of Coefficients from Data**

How could we determine the parameters in the transfer function for a 1st order process? We can perturb the process in a controlled manner & look at the transient results.

**Impulse disturbance.** $f(t) = \alpha \cdot \delta(0)$ gives:

$$y(t) = \alpha \frac{K_p}{\tau_p} e^{-t/\tau_p}$$

Can put into “linear form” by taking the logarithm:

$$\ln(y) = \ln \left( \frac{\alpha K_p}{\tau_p} \right) - \left( \frac{1}{\tau_p} \right) t$$

Using linear regression of $\ln(y)$ vs. $t$ data, the slope will be $-1/\tau_p$ and the intercept will be $\ln \left( \frac{\alpha K_p}{\tau_p} \right)$.

**Unit step change.** $f(t) = \alpha \cdot H(t)$ gives:

$$y(t) = \alpha K_p \left( 1 - e^{-t/\tau_p} \right)$$

with the initial slope of:

$$\left. \frac{dy}{dt} \right|_{t=0} = \frac{\alpha K_p}{\tau_p}$$

One would get the gain from the ultimate value, $K_p = y_\infty / \alpha$; however, this requires that you take data long enough to be confident of the ultimate value, $y_\infty$. You could then get the initial slope & determine the time constant:
However, then numerical determination of the derivative from data is inherently difficult. You could also put this equation into "linear form:"

$$\ln \left(1 - \frac{y}{\alpha K_p}\right) = -\left(\frac{1}{\tau_p}\right)t$$

and linear regression can again be used. The difficulty now is that the form of the independent variable, \(\ln \left(1 - \frac{y}{\alpha K_p}\right)\), includes one of the variables to be determined, \(K_p\). There are two ways this could be handled:

1. Adjust the value of \(K_p\) until the intercept from the linear regression is zero.
2. Do the regression by forcing the intercept to be zero & adjust \(K_p\) to maximize the regression coefficient, \(r^2\).

The following table shows the response of a tank when the flow rate is increased from 2 ft\(^3\)/min to 3 ft\(^3\)/min. Using deviation variables, the value of \(\alpha\) will be \(\alpha = 3 - 2 = 1\) ft\(^3\)/min.

<table>
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<th>ft</th>
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<th>ft</th>
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<td>1.8</td>
</tr>
<tr>
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</tr>
<tr>
<td>9</td>
<td>1.6</td>
<td></td>
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</table>

Doing linear regression with the linear form and allowing an intercept, a value of \(K_p = 7.92\) min/ft\(^2\) gives an intercept of zero; the corresponding slope gives \(\tau_p = 39.0\) min. Doing linear regression with the linear form and not allowing an intercept, a value of \(K_p = 8.05\) min/ft\(^2\) gives an intercept of zero; the corresponding slope gives \(\tau_p = 41.0\) min. Note that the values were generated with random perturbations added to a linear model with \(K_p = 8\) min/ft\(^2\) and \(\tau_p = 40\) min. The following chart shows the
Linearization

Not all dynamic systems are described by linear ODEs — in truth, probably none of the systems are really linear. To use transfer functions & Laplace transforms, however, we must linearize the system of equations. We have already looked at an example of this with flow through a valve. If the flow through a valve is considered to be proportional to $\sqrt{h}$, then the material balance around a tank containing a fluid of constant density is described by:

$$\frac{dV}{dt} = F_0 - F_1 = F_0 - C_v \sqrt{h}$$

and for a constant cross-sectional area in the tank, $A$, then:

$$A \frac{dh}{dt} = F_0 - C_v \sqrt{h}.$$  

We've discussed ways to linearize the square root term in this ODE. We can effectively do a Taylor series expansion by first taking the total differential of the right-hand-side:

$$d\left(F_0 - C_v \sqrt{h}\right) = dF_0 - \left(\frac{C_v}{2 \sqrt{h}}\right) dh$$
Now, replace the differentials with deviation variables & evaluate any coefficients at the initial steady state. So:

\[
\left( F_0 - C_v \sqrt{h} \right)' \approx F_0' - \left( \frac{C_v}{2\sqrt{h}} \right) h'
\]

and the linearized ODE will be:

\[
A \frac{dh'}{dt} = F_0' - \left( \frac{C_v}{2\sqrt{h}} \right) h'
\]

\[
A \frac{dh'}{dt} + \left( \frac{C_v}{2\sqrt{h}} \right) h' = F_0'
\]

\[
\left( \frac{2A\sqrt{h}}{C_v} \right) \frac{dh'}{dt} + h' = \left( \frac{2\sqrt{h}}{C_v} \right) F_0'.
\]

From this, we see that the linearized ODE is simply a 1st order ODE with the following gain and time constant:

\[
K_p = \frac{2\sqrt{h}}{C_v}
\]

\[
\tau_p = \frac{2A\sqrt{h}}{C_v}
\]

Effectively, we have found that the gain and time constant are not constant, but rather functions of the liquid level. Whether or not the linearized equation is a good representation depends upon how far we perturb the system from its steady state value. For example, given a tank with a cross-sectional area of 5 m² which maintains a level of 16 m when the flow in is 2 m³/min, what happens when we shut off the flow? In the full non-linear solution:

\[
\frac{dh}{dt} = F_0 - \frac{C_v}{A} \sqrt{h}
\]

where:

\[
h(0) = h^* = 16 \text{ m}
\]

\[
F_0^* = 2 \text{ m}^3/\text{min}
\]

\[
F_0(t) = 0
\]
then:
\[
\frac{dh}{\sqrt{h}} = -\frac{C_v}{A} \, dt \quad \Rightarrow \quad \int_{h_0}^{h} \frac{dh}{\sqrt{h}} = -\frac{C_v}{A} \int_{0}^{t} dt \quad \Rightarrow \quad h = \left( \sqrt{h_0} - \frac{C_v \cdot t}{2A} \right)^2
\]

The linearized equation is:
\[
A \frac{d^2h}{dt^2} + \frac{C_v (h-h^*)}{2\sqrt{h}} = -F_0^* \quad \Rightarrow \quad A \frac{d^2h}{dt^2} + \frac{C_v}{2\sqrt{h}} \cdot h' = -F_0^*
\]
\[
\left( As + \frac{C_v}{2\sqrt{h^*}} \right) \cdot \bar{R} = -\frac{F_0^*}{s}
\]
\[
\bar{R} = -K_p \frac{F_0^*}{\tau_p s + 1} \quad \text{where} \quad K_p = \frac{2\sqrt{h}}{C_v} \quad \text{and} \quad \tau_p = \frac{2A\sqrt{h}}{C_v}
\]

So:
\[
h' = -F_0^* K_p \left( 1 - e^{-t/\tau_p} \right) \quad \Rightarrow \quad h = h^* - \frac{2F_0^* \sqrt{h}}{C_v} \left( 1 - \exp \left( -\frac{tC_v}{2A\sqrt{h}} \right) \right)
\]

The chart below shows the difference in the drainage curve for the two equations. Notice that the linearized equation stays fairly close to the exact solution for the 1st 8 ft change of level. Also note that even though the ODE is linearized, it does not predict a straight line answer — there is still curvature to the final \(h(t)\) result.