PHGN 422: Nuclear Physics
Lecture 10: The Nuclear Shell Model II - Application

Prof. Kyle Leach
Last Class...

- Introduction of the nuclear shell model
- Discussion of how we cannot solve the S.E. exactly for this problem
- First look at the Woods-Saxon potential as our mean-field....
- But this *still* could not reproduce our observed shell effects
The Schrödinger Equation for $A$ Nucleons

Recall from last class, that we can write down the Schrödinger equation for the nucleons...

The Schrödinger Equation

$$\hat{H}\psi(\vec{r}_1 \ldots \vec{r}_A) = E\psi(\vec{r}_1 \ldots \vec{r}_A)$$

where:

$$\hat{H} = \sum_{i=1}^{A} \left[-\frac{\hbar^2}{2m_i} \nabla_i^2\right]$$

Kinetic term

$$+ \sum_{i>j}^{A} \sum_{j=1}^{A} v_{ij}(\vec{r}_i, \vec{r}_j)$$

Two – body potential interactions

$$+ \sum_{i>j}^{A} \sum_{j>k}^{A} \sum_{k=1}^{A} v_{ijk}(\vec{r}_i, \vec{r}_j, \vec{r}_k)$$

Three – body potential interactions

$$+ \ldots$$

But we can’t solve it exactly...
The Woods-Saxon Potential

Our most successful approach so far was to approximate the nuclear potential by a *Woods-Saxon* form:

\[ V(r) = \frac{-V_0}{1 + e^{(r-R)/d}} \]
The Woods-Saxon Potential

\[ V(r) = \frac{-V_0}{1 + e^{(r-R)/d}} \]

- By solving the single-particle Schrödinger equation using this potential we lifted some of the degeneracies for the different \( \ell \) values from the H.O. solution.

- We still couldn’t reproduce the observed shell closures...we need to go further in our model of the interaction.

Source: Krane, Fig. 5.6
Atomic Spin-Orbit Coupling

We know that spin-orbit coupling exists as a relativistic correction in atomic physics, and is responsible for the fine structure of atoms....perhaps we should consider what something like this would look like for nuclei....

Nuclear Spin-Orbit Coupling

Although it takes a similar form (below), the nuclear spin-orbit coupling is *not* a relativistic correction, but a first order term in the bare nucleon-nucleon interaction:

$$V_{\ell s}^{\text{atomic}}(r) \propto \frac{1}{r} \frac{d}{dr} V_{\text{Coulomb}}(r) \vec{\ell} \cdot \vec{s}$$ (1)

$$V_{\ell s}^{\text{nuclear}}(r) \propto -\frac{1}{r} \frac{d}{dr} V_{\text{WS}}(r) \vec{\ell} \cdot \vec{s}$$ (2)

So we must consider the effect that this interaction has on our potential....
Expanding our Potential $V(r)$

Recall that we are trying to approximate the complicated form for the potential in our nuclear Hamiltonian:

$$
\Rightarrow \quad V(r) \approx \sum_{i>j}^{A} \sum_{j=1}^{A} v_{ij}(\vec{r}_i, \vec{r}_j) + \sum_{i>j}^{A} \sum_{j>k}^{A} \sum_{k=1}^{A} v_{ijk}(\vec{r}_i, \vec{r}_j, \vec{r}_k) + \ldots
$$

We should now include all of the interactions that we think go into our potential:

$$
V(r) = V_{WS}(r) + V_{\ell s}(r)
$$
Expanding our Potential $V(r)$

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- The Woods-Saxon central potential
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$$

- The Woods-Saxon central potential
- The potential that exists from the nucleon-nucleon spin-orbit interaction
Effect of the Spin-Orbit Interaction

Quantum energy states of potential well including angular momentum effects.

Further splitting from spin-orbit effect

Multiplicity of states

1g7/2

1g9/2

2p3/2

1f5/2

2p1/2

1f7/2

1d3/2

2s1/2

1d5/2

1p3/2

1p1/2

1s

Closed shells indicated by "magic numbers" of nucleons.

Source: http://hyperphysics.phy-astr.gsu.edu
Reproducing the Observed Magic Numbers

Source: Krane, Fig. 5.6
What Else Can It Tell Us?

After several attempts at trying to model the nuclear potential, we are now able to reproduce our observed shell gaps! But let’s go a step further...what else can we learn from our new model?

First let’s go back to our spin/parity rules for nuclei that we presented last week:

1. *ALL* even-even nuclei have $J^\pi = 0^+$ ground states

2. For odd-$A$ nuclei, the ground-state spin and parity is determined by the only unpaired nucleon, ie. $J^\pi = j_{\text{odd}}^\pi$

3. For odd-odd nuclear ground states, $J^\pi = j_p^\pi \otimes j_n^\pi$

Can our model predict this without any further inputs?
Application of the Shell Model:

Predicting the Ground State $J^\pi$

Example: $A = 15$ and 17 Isotopes of Oxygen

First, let’s start with $^{16}\text{O}_8$:
Application of the Shell Model:

The $^{16}$O Ground State

<table>
<thead>
<tr>
<th>N</th>
<th>E(level)</th>
<th>T$_{1/2}$</th>
<th>Decay Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>16O</td>
<td>E(0.0)</td>
<td>0+STABLE</td>
<td></td>
</tr>
</tbody>
</table>


OK, even-even nuclei with $J^\pi = 0^+$ ground states...fine, that’s easy. What about odd-$A$ nuclei?
Application of the Shell Model:

The $^{15}$O and $^{17}$O Ground States

- Source: Krane, Fig. 5.7
Application of the Shell Model:

The $^{15}$O and $^{17}$O Ground States

Using this schematic from the shell model picture:

*For odd-$A$ nuclei, the $J^\pi$ of the ground-state is determined by the last unpaired nucleon....in this case, a neutron.*

1. $^{15}$O$_7$: The last neutron $\rightarrow 1p_{1/2}$
   
   - $j = \frac{1}{2}$
   - $\ell = 1$ (from $p$ orbital)
     
     $\Rightarrow$ parity: $\pi = (-1)^\ell = -$  

   - $\therefore J^\pi_{gs}(^{15}\text{O}) = \frac{1}{2}^-$

2. $^{17}$O$_9$: The last neutron $\rightarrow 1d_{5/2}$
   
   - $j = \frac{5}{2}$
   - $\ell = 2$ (from $d$ orbital)
     
     $\Rightarrow$ parity: $\pi = (-1)^\ell = +$  

   - $\therefore J^\pi_{gs}(^{17}\text{O}) = \frac{5}{2}^+$
# Experimental Data for $^{15,17}\text{O}$ G.S.

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>$^{15}\text{O}$</th>
<th>$^{17}\text{O}$</th>
<th>$E_{\text{level}}$</th>
<th>$J^\pi_\text{J}$</th>
<th>$T_1/2$</th>
<th>Decay Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.80</td>
<td>125.0</td>
<td>125.6</td>
<td>0.0</td>
<td>1/2</td>
<td>122.24</td>
<td>$\alpha: 100%$</td>
</tr>
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</tr>
<tr>
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<td>125.0</td>
<td>125.6</td>
<td>0.0</td>
<td>1/2</td>
<td>122.24</td>
<td>$\alpha: 100%$</td>
</tr>
</tbody>
</table>

Excited Nuclear States in the Shell Model

Source: Krane, Fig. 5.11
Excited States in $^{17}\text{O}_9$ and $^{17}\text{F}_8$

Referring to this picture, we can make some statements about how excited states nuclear states are formed within our shell model:

- A low-lying (low-energy relative to the ground-state) results when the last particle (the valence nucleon) is excited to the next shell.
Excited States in $^{17}_{8}\text{O}_9$ and $^{17}_{9}\text{F}_8$

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- Higher energy states can result when a pair is broken at a lower configuration in the shell structure
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- Higher energy states can result when a pair is broken at a lower configuration in the shell structure.

- This can result in *coexistence* of the unpaired particles, which leads to much more complicated couplings between the single-particle states.

*More on this can be read on Pg. 131-134 in Krane, the section on valence nucleons*
Conclusions on the Shell Model

We have finally developed a model that can describe the nucleus from a microscopic approach. Let’s recap what we’ve done, and what we’ve achieved:

1. The exact S.E. for nucleons could not be solved
   So.. we approximated our nucleon interaction by a Woods-Saxon potential based on our observations of the nuclear density
   This potential got close to reproducing the magic numbers, but not all the way

2. We therefore needed to consider the $\ell \cdot s$ (spin-orbit) interaction
   By lifting the shell degeneracies, we got back shell closures at 2, 8, 20, 28, 50, 82...

3. We applied it to the doubly-magic nucleus $^{16}\text{O}_8$
   We reproduced the ground state $J^\pi$ values of nearby nuclei
   We could even investigate $J^\pi$ values for excited states
Supplemental Material for the Shell Model

There is so much more that the nuclear shell model can give us, but sadly we must move on. I have included the following supplemental materials on our course website to satisfy any further thirst you may have for reading on:

1. A brief recap (and a selection of more advanced topics) that I wrote on the nuclear shell model.

2. A very advanced look at new techniques with the nuclear shell model from one of the world’s experts, Dr. Jason Holt (TRIUMF). 

   There are 5 lectures from a graduate-level summer school that are uploaded

I also encourage you to read the relevant chapters in the texts that I have listed in the syllabus on the shell-model for further information....
Next Week...

Reading Before Next Class

- Radioactive Decay Supplement and Chapter 6 in Krane

Next Class Topics

- The start of the largest “section” of our course... *Radioactive Decay*, its modes, and properties
- Decay statistics
- *Assignment #2!* on Thursday