PHGN 422: Nuclear Physics
Lecture 10: The Nuclear Shell Model II - Application

Prof. Kyle Leach
Last Class...

- Introduction of the nuclear shell model
- Discussion of how we cannot solve the S.E. exactly for this problem
- First look at the Woods-Saxon potential as our mean-field....
- But this still could not reproduce our observed shell effects
The Schrödinger Equation for $A$ Nucleons

Recall from last class, that we can write down the Schrödinger equation for the nucleons...

The Schrödinger Equation

$$\hat{H}\psi(\vec{r}_1 \ldots \vec{r}_A) = E\psi(\vec{r}_1 \ldots \vec{r}_A)$$

where:

$$\hat{H} = \sum_{i=1}^{A} \left[ -\frac{\hbar^2}{2m_i} \nabla_i^2 \right]$$

Kinetic term

$$+ \sum_{i>j}^{A} \sum_{j=1}^{A} v_{ij}(\vec{r}_i, \vec{r}_j)$$

Two-body potential interactions

$$+ \sum_{i>j}^{A} \sum_{j>k}^{A} \sum_{k=1}^{A} v_{ijk}(\vec{r}_i, \vec{r}_j, \vec{r}_k)$$

Three-body potential interactions

$$+ \ldots$$

But we can’t solve it exactly...
The Woods-Saxon Potential

Our most successful approach so far was to approximate the nuclear potential by a *Woods-Saxon* form:

\[ V(r) = \frac{-V_0}{1 + e^{(r-R)/d}} \]
The Woods-Saxon Potential

\[ V(r) = \frac{-V_0}{1 + e^{(r-R)/d}} \]

• By solving the single-particle Schrödinger equation using this potential we lifted some of the degeneracies for the different \( \ell \) values from the H.O. solution

• We still couldn’t reproduce the observed shell closures...we need to go further in our model of the interaction

Source: Krane, Fig. 5.6
Atomic Spin-Orbit Coupling

We know that spin-orbit coupling exists as a relativistic correction in atomic physics, and is responsible for the fine structure of atoms....perhaps we should consider what something like this would look like for nuclei....

Nuclear Spin-Orbit Coupling

Although it takes a similar form (below), the nuclear spin-orbit coupling is not a relativistic correction, but a first order term in the bare nucleon-nucleon interaction:

\[ V_{\ell s}^{\text{atomic}}(r) \propto \frac{1}{r} \frac{d}{dr} V_{\text{Coulomb}}(r) \vec{\ell} \cdot \vec{s} \]  

(1)

\[ V_{\ell s}^{\text{nuclear}}(r) \propto -\frac{1}{r} \frac{d}{dr} V_{\text{WS}}(r) \vec{\ell} \cdot \vec{s} \]  

(2)

So we must consider the effect that this interaction has on our potential....
Expanding our Potential $V(r)$

Recall that we are trying to approximate the complicated form for the potential in our nuclear Hamiltonian:

$$
\Rightarrow \quad V(r) \approx \sum_{i>j}^{\text{A}} \sum_{j=1}^{\text{A}} v_{ij}(\vec{r}_i, \vec{r}_j) + \sum_{i>j}^{\text{A}} \sum_{j>k}^{\text{A}} \sum_{k=1}^{\text{A}} v_{ijk}(\vec{r}_i, \vec{r}_j, \vec{r}_k) + \ldots
$$

We should now include all of the interactions that we think go into our potential:

$$
V(r) = V_{\text{WS}}(r) + V_{\ell s}(r)
$$
Expanding our Potential $V(r)$

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$$\implies V(r) \approx \sum_{i>j}^{A} \sum_{j=1}^{A} v_{ij}(\vec{r}_i, \vec{r}_j) + \sum_{i>j}^{A} \sum_{j>k}^{A} \sum_{k=1}^{A} v_{ijk}(\vec{r}_i, \vec{r}_j, \vec{r}_k) + \ldots$$

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$$V(r) = V_{\text{WS}}(r) + V_{\ell s}(r)$$

- The Woods-Saxon central potential
Expanding our Potential $V(r)$

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$$

We should now include all of the interactions that we think go into our potential:

$$
V(r) = V_{WS}(r) + V_{\ell s}(r)
$$

- The Woods-Saxon central potential
- The potential that exists from the nucleon-nucleon spin-orbit interaction
Effect of the Spin-Orbit Interaction

Quantum energy states of potential well including angular momentum effects.

Further splitting from spin-orbit effect

Multiplicity of states

1g 1g\textsubscript{\(7/2\)}

1g\textsubscript{\(9/2\)} 10

2p 2p\textsubscript{\(3/2\)}

1f 1f\textsubscript{\(5/2\)}

2s 2s\textsubscript{\(1/2\)} 2s\textsubscript{\(3/2\)}

1d 1d\textsubscript{\(3/2\)} 1d\textsubscript{\(5/2\)}

1p 1p\textsubscript{\(1/2\)} 1p\textsubscript{\(3/2\)}

1s 1s

Closed shells indicated by "magic numbers" of nucleons.

Source: http://hyperphysics.phy-astr.gsu.edu
Reproducing the Observed Magic Numbers

Intermediate form

4s 3d 2g 1i 3p 2f 1h 3s 2d 1g 2p 1f 2s 1d 1p 1s
2 10 18 26 14 22 20 18 14 6 10 20 18 6 2 2

Intermediate form with spin orbit

1j_{15/2} 2d_{3/2} 4s_{1/2} 2g_{7/2} 1i_{11/2} 3d_{5/2} 2g_{9/2}
1i_{13/2} 3p_{1/2} 3p_{3/2} 2f_{5/2} 2f_{1/2} 1h_{9/2}
1h_{11/2} 3s_{1/2} 2d_{3/2} 2d_{5/2} 1g_{7/2} 1g_{9/2} 2p_{1/2} 1f_{5/2} 2p_{3/2} 1f_{7/2} 1d_{3/2} 1d_{5/2} 1s_{1/2}

Source: Krane, Fig. 5.6
What Else Can It Tell Us?

After several attempts at trying to model the nuclear potential, we are now able to reproduce our observed shell gaps! But let’s go a step further...what else can we learn from our new model?

First let’s go back to our spin/parity rules for nuclei that we presented last week:

1. **ALL** even-even nuclei have $J^\pi = 0^+$ ground states

2. For odd-$A$ nuclei, the ground-state spin and parity is determined by the only unpaired nucleon, ie. $J^\pi = j_{\text{odd}}^\pi$

3. For odd-odd nuclear ground states, $J^\pi = j_p^\pi \otimes j_n^\pi$

Can our model predict this without any further inputs?
Application of the Shell Model:

Predicting the Ground State $J^\pi$

Example: $A = 15$ and 17 Isotopes of Oxygen

First, let’s start with $^{16}\text{O}_8$:

External Orbital States

Completely Closed Core

Slide 12 — Prof. Kyle Leach — PHGN 422: Nuclear Physics
Application of the Shell Model:

The $^{16}$O Ground State

OK, even-even nuclei with $J^\pi = 0^+$ ground states...fine, that’s easy. What about odd-$A$ nuclei?
Application of the Shell Model:

The $^{15}\text{O}$ and $^{17}\text{O}$ Ground States

Source: Krane, Fig. 5.7
Application of the Shell Model:

The $^{15}$O and $^{17}$O Ground States

Using this schematic from the shell model picture:

*For odd-$A$ nuclei, the $J^{π}$ of the ground-state is determined by the last unpaired nucleon....in this case, a neutron.*

1. $^{15}_{8}$O$_{7}$: The last neutron $\rightarrow 1p_{1/2}$
   - $j = \frac{1}{2}$
   - $\ell = 1$ (from $p$ orbital)
   - $\implies$ parity: $\pi = (-1)^{\ell} = -$  
   - $\therefore J^{π} gs({^{15}\text{O}}) = \frac{1}{2}^{-}$

2. $^{17}_{8}$O$_{9}$: The last neutron $\rightarrow 1d_{5/2}$
   - $j = \frac{5}{2}$
   - $\ell = 2$ (from $d$ orbital)
   - $\implies$ parity: $\pi = (-1)^{\ell} = +$  
   - $\therefore J^{π} gs({^{17}\text{O}}) = \frac{5}{2}^{+}$
Experimental Data for $^{15,17}\text{O}$ G.S.

<table>
<thead>
<tr>
<th>Level</th>
<th>$^{15}\text{O}$</th>
<th>$^{17}\text{O}$</th>
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<td>4.15</td>
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Excited Nuclear States in the Shell Model

Source: Krane, Fig. 5.11
Excited States in $^{17}_{8}\text{O}$ and $^{17}_{9}\text{F}$

Referring to this picture, we can make some statements about how excited states nuclear states are formed within our shell model:

- A low-lying (low-energy relative to the ground-state) results when the last particle (the valence nucleon) is excited to the next shell.
Excited States in $^{17}_8\text{O}_{9}$ and $^{17}_9\text{F}_{8}$

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- Higher energy states can result when a pair is broken at a lower configuration in the shell structure.
Excited States in $^{17}_{8}O_9$ and $^{17}_{9}F_8$

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- Higher energy states can result when a pair is broken at a lower configuration in the shell structure.
- This can result in *coexistence* of the unpaired particles, which leads to much more complicated couplings between the single-particle states.

*More on this can be read on Pg. 131-134 in Krane, the section on valence nucleons*
Conclusions on the Shell Model

We have finally developed a model that can describe the nucleus from a microscopic approach. Let’s recap what we’ve done, and what we’ve achieved:

1. **The exact S.E. for nucleons could not be solved**
   
   *So..we approximated our nucleon interaction by a Woods-Saxon potential based on our observations of the nuclear density*  
   
   *This potential got close to reproducing the magic numbers, but not all the way*

2. **We therefore needed to consider the \( \ell \cdot s \) (spin-orbit) interaction**
   
   *By lifting the shell degeneracies, we got back shell closures at 2,8,20,28,50,82...*

3. **We applied it to the *doubly-magic* nucleus \(^{16}\text{O}_8\)**
   
   *We reproduced the ground state \( J^\pi \) values of nearby nuclei*  
   
   *We could even investigate \( J^\pi \) values for excited states*
Supplemental Material for the Shell Model

There is so much more that the nuclear shell model can give us, but sadly we must move on. I have included the following supplemental materials on our course website to satisfy any further thirst you may have for reading on:

1. A brief recap (and a selection of more advanced topics) that I wrote on the nuclear shell model.

2. A very advanced look at new techniques with the nuclear shell model from one of the world’s experts, Dr. Jason Holt (TRIUMF).

   There are 5 lectures from a graduate-level summer school that are uploaded

I also encourage you to read the relevant chapters in the texts that I have listed in the syllabus on the shell-model for further information....
Next Week...

Reading Before Next Class

- **Assignment #1!** due before Monday. Email to Patrick (or hard copy in mailbox before Friday)
- **Assignment #2!** is now posted online. Due October 10.
- Radioactive Decay Supplement and Chapter 6 in Krane

Next Class Topics

- Guest Lecturer: Patrick Hunt!
- The start of the largest “section” of our course... *Radioactive Decay*, its modes, and properties
- Decay statistics