Several empirical models have been formulated over the last 70 years in an attempt to describe observed nuclear-structure characteristics. There are two basic types of models used:

1. Those which describe the nucleus as individual nucleons that interact with each other, and give rise to the observed structure (microscopic models), and

2. Those that attempt to describe nuclear structure by considering the motion of many nucleons simultaneously (collective models).

The nuclear shell model is an example of the former. The shell model has been among the most successful, and widely used, microscopic models of the nucleus. The following material attempts to lay the groundwork necessary for an understanding of the primary motivation for this study, as well as describing the theoretical framework behind the experimental reaction mechanisms used to probe the specific shell-model methods.

1 INTRODUCTION

Analogous to ionization energy in atomic systems, evidence of discrete energy jumps in two-nucleon separation energies in nuclear-structure experiments were observed over several decades (Fig. 1.1). Like electrons, nucleons are fermions which must obey the Pauli exclusion principle,
and therefore, in the same way that the structure of an atomic element is determined by filling discrete-energy shells with electrons, the properties of any nucleus should depend on the combined number of protons and neutrons within their respective shells.

Unlike the central Coulomb potential in atomic systems, however, the nuclear shell model includes an attractive potential arising from the short-range interaction between neighbouring nucleons which is directly dependent on the shape of the nuclear distribution [3]. Therefore, the shell model is able to treat each nucleon as an independent particle that acts within a mean field of all the rest, thus allowing nucleons to occupy the various orbitals within the shells. These fundamental ideas form the basis for the description of nuclear structure within the shell model, as well as characterizing the formalism for single-nucleon transfer reactions. More formally, the shell model is predicated on the notion that the full nuclear Hamiltonian can be expressed as

\[ \hat{H} = \hat{H}_0 + \hat{V}, \]  

where \( \hat{V} \) is a residual interaction and \( \hat{H}_0 \) is the independent-particle Hamiltonian, that is a sum of the single particle Hamiltonians [3],

\[ \hat{H}_0 = \sum_{i=1}^{A} \hat{h}_i. \]  

This concept relies on being able to model the entire system by describing the motion of individual nucleons acting in a single-particle potential, \( U(r) \). Since empirical evidence suggests that this potential should be proportional to the nuclear density distribution [3], a potential of Woods-Saxon form is used. The Woods-Saxon potential is defined as

\[ U(r) = \frac{-V_0}{1 + \exp \left( \frac{r-R_a}{a} \right)}, \]  

where \( V_0 \) is the potential depth (generally in MeV), \( r \) is the magnitude of the position vector, \( R = r_0 A^{1/3} \) is the nuclear radius, and \( a \) is a length representing the surface thickness of the nucleus. An example of the Woods-Saxon potential is shown in Fig. 1.2.

Although the Woods-Saxon potential provides a more realistic description of the nucleon mean-field interaction, it is still unable to directly reproduce the observed shell closures seen in Fig. 1.1. These closures are determined by the cumulative occupation of the fermions within the shells, where the total occupation number for a given level is determined via the \( m \) quantum number. Since each \( \ell \) value can span \((2\ell + 1)\) \( m \) states, it would stand to reason that each orbital could be occupied with \( N_0 = 2\ell + 1 \) fermions, however this is not the case. Due to the spin quantum number \( s \), each \( m \) state is effectively split, and can contain a proton or neutron with \( s = \pm 1/2 \). Therefore, the total occupation number for any shell in the Woods-Saxon potential is

\[ N_0 = 2(2\ell + 1). \]  

Although this breaks the degeneracy which arises from a simple harmonic oscillator picture, this is not the entire story. In atomic systems, an electromagnetic spin-orbit interaction is present
Figure 1.1: (upper) Two proton and (lower) two neutron separation energies as a function of the nucleon number for sequences of isotones and isotopes, respectively. The plot is adapted from Ref. [1], and the measured values are from Ref. [2].
which lifts the level degeneracies further to reproduce the atomic shell closures. Since the energy scales for nuclear systems are much higher, a nucleon-nucleon spin-orbit force must be considered in an analogous way. This spin interaction assumes the form

$$V_{so}(r)[\vec{\ell} \cdot \vec{s}],$$

(1.5)

where the $\vec{\ell} \cdot \vec{s}$ term results from the nucleon-nucleon spin-orbit force. It is at this point that $m$ and $s_z$ are no longer good quantum numbers, since the spin and orbital angular momentum are now coupled. Therefore, the total angular momentum $\vec{j} = \vec{\ell} + \vec{s}$, and its projection $m_j$ become good quantum numbers in the system. For each $j$, there are $(2j + 1)$ $m_j$ values, which implies that each $\pi$ and $\nu$ orbit can contain up to $N_0 = 2j + 1$ particles. With the inclusion of the spin-orbit coupling, the experimentally observed shell closures at 2, 8, 20, 28, 50, 82, and 126 are predicted exactly by the shell model, shown in Fig. 1.3. The model also predicts a nucleon shell-closure at 184 as well, however this is yet to be observed experimentally. In general, the ordering of the levels within the major shells has a heavy dependence on the radial part of the spin-orbit potential $V_{so}(r)$, which is peaked at the nuclear surface and is often chosen as the derivative of the average Woods-Saxon potential.

**2 Closed-Shell Cores**

The basis of the nuclear shell model centres on the fact that the individual nucleon-nucleon collisions which occur within the nucleus in its ground-state, do not supply enough energy to
Figure 1.3: The shell-structure energy levels for a Woods-Saxon nucleon potential given by Eqn. 1.3 for the cases of (left) no spin-orbit splitting, and (right) energy splitting caused by a spin-orbit interaction. The values at the far left of the figure denote the \( n\ell \) quantum numbers of each state (using \( spdf \) spectroscopic notation for \( \ell \) values). The numbers directly to the right of each level display the total number of \( \pi \) or \( \nu \) which each state is allowed to contain, and further right shows the total cumulative number of nucleons in the entire system. The values within the level gaps are the calculated magic numbers for the two cases. It should be noted that once the spin-orbit interaction is included, the experimentally observed magic numbers are exactly reproduced. Figure is adapted from Ref. [1].
Figure 2.1: The neutron and proton particle configurations for the doubly-magic nucleus $^{16}$O. When the nucleus is treated as a core, the Pauli principle prevents particles from moving within the closed system, and the $1d_{5/2}$ state is closed to interaction since it is external to the $^{16}$O-core model space.

move nucleons from one major shell to another. The theory can therefore function by building individual nucleons on the cumulative interaction of all of the others which occupy the levels up to the closest shell-closure below. These cumulative states which are magic for protons and neutrons, are referred to as cores. As previously shown in Fig. 1.1, the energy required to remove a nucleon from a closed shell is on the order of a few MeV or more. Thus, the core can be approximated as a closed system which is not open to interactions with external nucleons. As an example, the case of $^{16}$O, with 8 neutrons and 8 protons, represents the lightest\(^1\) doubly-magic shell-model core (Fig. 2.1) Although this is a seemingly gross approximation of what occurs within a realistic nuclear system, there is good evidence to support the theory of nuclear cores. If an additional neutron is introduced into the $^{16}$O system, the structure of the resulting nucleus should thus be determined by the one particle outside of the doubly-magic core. Since the additional neutron is located in the $d_{5/2}$ orbital, the model predicts a ground state for $^{17}$O of $5/2^+$ (since parity is determined by $(-1)^{\ell}$), and indeed this is what is observed experimentally. The states which occur as a result of adding a nucleon above a closed shell are referred to as particle states, and similarly, states below the closed core which are introduced are known as hole states. Using these two forms of interactions with the closed cores can describe nearly all structure related information within the nuclear-shell model. The introductory material outlined in

\(^1\)The $\alpha$ particle system is lighter, however using these closed-cores in calculations is usually referred to as cluster theory.
Section ?? explains how, by using various reaction mechanism, the particle and hole states can be selectively probed. To first order, these reactions can be approximated as one-body interactions that induce changes in the final-state nuclear structure.

3 INDEPENDENT-PARTICLE FORMALISM

One of the best ways to understand the fundamental nature of a quantum mechanical model is to test its observable matrix elements. For the nuclear shell model, the logical place to start is to construct the full Schrödinger equation for a particle in a central-potential. This is given as:

\[ H_0 |\Psi\rangle = E |\Psi\rangle. \tag{3.1} \]

The representation of this expression within the nuclear-shell model thus takes the form of

\[ H_0 |\Psi\rangle = \left\{ \sum_{i=1}^{A} h_i \right\} |\Psi\rangle = \sum_{i=1}^{A} \left\{ -\frac{\hbar^2}{2m} \Delta_i + V(i) \right\} |\Psi\rangle = E |\Psi\rangle, \tag{3.2} \]

where the \( |\Psi\rangle \) wave function solutions are anti-symmetric products of single-particle wave functions, which in-turn are eigenfunctions of the single-particle Hamiltonian;

\[ h_i \phi_k(i) = \epsilon_k \phi_k(i). \tag{3.3} \]

The \( \phi_k \) and \( \epsilon_k \) from Eqn. 3.3 are the single-particle wave functions and eigenvalues, respectively, for a given level \( k \), where the \( k \) quantum numbers represent the full set of single-particle quantum numbers \( n j \ell \). For each \( k \) state, corresponding creation \( (a_k^+) \) and annihilation \( (a_k) \) operators which act on the single-particle wave functions \( \phi_k \) can be included within the full shell-model Hamiltonian,

\[ H_0 = \sum \epsilon_k a_k^+ a_k. \tag{3.4} \]

Since the nuclear shell model attempts to describe microscopic interactions within the nucleus, these creation and annihilation operators must obey the Pauli exclusion principle, and as such, will only allow each \( k \) state to be occupied once. The eigenfunctions of the Hamiltonian in Eqn. 3.4 can also be represented in terms of the creation and annihilation operators,

\[ |a_{k_1} \ldots a_{k_A}\rangle = a_{k_1}^+ \ldots a_{k_A}^+ |0\rangle, \tag{3.5} \]

which have eigenvalues given by,

\[ E_{k_1} \ldots k_A = \epsilon_{k_1} + \ldots + \epsilon_{k_A}. \tag{3.6} \]

The \( |0\rangle \) state in Eqn. 3.5 represents the vacuum where the particles are created, and the \( k_i \) subscripts are the individual nucleons \( i \) occupying various discrete levels \( k \) in increasing order from \( 1 \rightarrow A \).
Figure 3.1: The energy potentials for (left) the single-particle model and (right) the nuclear shell model, displaying energy levels both above and below the Fermi surface. The lower dashed line for the proton potential shows where the nuclear contribution only would place the potential, whereas the upper dashed line displays the contribution from the Coulomb potential. It is the combination of these terms which creates the final proton potential, represented by the solid curve. Adapted from Ref. [4].

If the case of a nucleus in its ground-state is considered, the experimental results presented in the previous section provide strong evidence for sequential filling of shells with single particles. Using this formalism for an $A$-particle ground state, the wave function can be written using Eqn. 3.5 as:

$$|\alpha_0\rangle = a_1^\dagger \cdots a_A^\dagger |0\rangle,$$

which shows the successive filling of the ground-state level, from the vacuum, with $A$ particles. Using this idea, all $|n s^J m\rangle$ levels are filled particle-by-particle until all $A$ have been created. As successive particles are created, each level increases in energy since the Pauli principle prevents them from being in the same place. Once all $A$ particles have been generated, the highest occupied closed state is known as the Fermi surface. Thus, all levels below the Fermi surface are occupied levels, and all states above the Fermi surface are unoccupied. The independent-particle level scheme, including the Fermi surface is shown in Fig. 3.1.

Of course, this “independent-particle” treatment of the nucleus does not exactly translate since there are two types of particles within the nucleus, protons and neutrons. Due to the slightly different nature of the two particles, the average potential felt by protons differs from that felt by neutrons primarily due to charge-dependant forces. The Coulomb force results from the
repulsive charge of the protons, which adds a term to the nuclear potential of

\[ V_C(r) = \begin{cases} \frac{Ze^2}{r} \left( \frac{3r}{R} \right)^2 & r \leq R, \\ \frac{Ze^2}{r} & r > R. \end{cases} \]  

(3.8)

This Coulomb force is of the form for the classical potential of a uniformly charged sphere, and is included within \( V(i) \) in Eqn. 3.2 [4]. The energy difference between the proton and neutron potentials is shown in Fig. 3.1.

The lowest-energy excitations within the shell model are represented in second quantized form as:

\[ |\alpha_{mi}\rangle = a_{\alpha}^\dagger a_{\beta}^\dagger \ldots a_{A}^\dagger |0\rangle, \]  

(3.9)

where the \( |\alpha_{mi}\rangle \)'s represent a complete set of antisymmetric single-particle wave functions. If the core is considered as a vacuum state, nuclei near closed cores can be treated in the same way, using the above formalism.

### 4 Two-Body Interactions

The basic shell model outlined above is an independent-particle model, where no direct interactions between the particles exist, and each particle moves within a mean field of all the others. However, this above treatment can also be expanded to include an \( N \)-body interaction within the Hamiltonian. To a good approximation, only central-force and two-body interactions are necessary to properly reproduce observed nuclear-structure trends. If Eqn. 3.1 is considered again, the Schrödinger equation can be written as

\[ H |\Psi\rangle = \left( \sum_{i=1}^{A} \left\{ -\frac{\hbar^2}{2m} \Delta_i + V(i) \right\} + \sum_{i<j}^{A} V_{ij} \right) |\Psi\rangle = (H_0 + W) |\Psi\rangle, \]  

(4.1)

where \( H_0 \) is the sum of the single-particle Hamiltonians \( h_i \), and is represented in terms of creation and annihilation operators in Eqn. 3.4. The full effective Hamiltonian above can be rewritten in terms of the second quantized formalism as,

\[ \hat{H}_{\text{eff}} = \sum_{\alpha} \langle \alpha | U | \alpha \rangle a_{\alpha}^\dagger a_{\alpha} + \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha \beta | V_{\text{eff}} | \gamma \delta \rangle a_{\alpha}^\dagger a_{\beta}^\dagger a_{\gamma} a_{\delta}, \]  

(4.2)

where \( V_{\text{eff}} \) is the effective two-body nucleon-nucleon (\( NN \)) interaction [5]. These residual interactions, as mentioned previously, are included within the Hamiltonian to reproduce the observed pairing interaction between nucleons within the nuclear medium. Although there are several different residual interactions which are commonly used, the GXPF1 [6, 7], and GXPF1A [8] interactions (modified \( G \)-interactions [9]), and the surface-\( \delta \) interaction (SDI) are focused on here.
4.1 G-INTERACTION

The Brückner G-matrix [10, 11] is one of the most important effective \( NN \) interactions in nuclear physics [12], since it is analogous to the scattering matrix for two nucleons in free space. If the idea of nucleon scattering is therefore considered as part of the effective interaction, a scattering G-matrix can be defined to describe the interaction of nucleons below the Fermi surface. The scattering G-matrix for nuclear physics is generally referred to as the Bethe-Goldstone equation, and takes the operator form

\[
G = \bar{\mathcal{V}} + \mathcal{V} \frac{Q_F}{E - H_0} G, \tag{4.3}
\]

where \( H_0 \) is the shell-model Hamiltonian and \( \bar{\mathcal{V}} \) is the bare microscopic interaction. The Pauli operator,

\[
Q_F = \sum_{m<n} |mn\rangle \langle mn|, \tag{4.4}
\]

is a projection operator that excludes all of the occupied states within the closed-shell core [12]. The formal solution to the Bethe-Goldstone equation is:

\[
G = \frac{\bar{\mathcal{V}}}{1 - \frac{\mathcal{V} Q_F}{E - H_0}}. \tag{4.5}
\]

It has been shown, however, that using the G-matrix solution above as the effective two-body interaction does not reproduce the observed data with reasonable accuracy [9]. Therefore, minimal adjustments to the monopole two-body matrix elements must be made in order to properly use the G interaction within the various model spaces. For example, in the \( sd \) shell the universal \( sd \) (USD) interaction is derived from the G-matrix formalism through a fit to 447 energy-data in the mass range \( A = 17 - 40 \) [5, 13, 14]. In the \( fp \) shell, the final result of this parameter tuning is the so-called GXPF1 interaction [5], and is determined by a fit to 699 energy data in the mass range \( A = 47 - 66 \) [5, 6].

4.2 SURFACE-DELTAL INTERACTION

The surface-\( \delta \) interaction is an interaction between individual nucleons which exist near the Fermi surface. The primary principle behind the SDI is that only the nucleons on the surface interact with each other, while those within the nuclear interior are inert. The majority of the nuclear collisions occur at the surface, where the effect of the Pauli exclusion principle is reduced, and the nucleons feel the strong attractive interaction [4]. Outside the surface, the interaction is also not important since the probability of having a nucleon outside of the mean nuclear radius rapidly approaches zero. Thus, it is logical to restrict the residual interaction to the nuclear surface, and define the SDI as,

\[
V_{ij} = -V_0 \delta(|\vec{r}_i - \vec{r}_j| - R), \tag{4.6}
\]
where $V$ is dependent on the spin and isospin of the system, and $R = 1.25A^{1/3}$ is the mean nuclear radius. Since the SDI has fundamentally basic geometric properties, it is possible to expand the $\delta$-function in terms of Legendre polynomials,

$$\delta(\vec{r}_i - \vec{r}_j) = \frac{\delta(r_i - r_j)}{r_ir_j} \sum_\ell \frac{2\ell + 1}{4\pi} P_\ell(\cos \theta_{ij}),$$  \hspace{1cm} (4.7)

where $\theta_{ij}$ is the angle between $\vec{r}_i$ and $\vec{r}_j$. Finally, using this expansion, the SDI potential term can be written in terms of spherical harmonics as:

$$V_{ij} = -V_0 \sum_{\ell m} \frac{\delta(r_i - R)}{r_i} Y_{\ell m}(\theta_i \phi_i) \frac{\delta(r_j - R)}{r_j} Y_{\ell m}(\theta_j \phi_j),$$  \hspace{1cm} (4.8)

which is an infinite sum over separable terms between $i$ and $j$. Using a spherical basis, the matrix elements which are coupled to good angular momentum values are greatly simplified since only one term in the sum survives. Although the SDI effectively describes the residual interaction locally, it is not meaningful to extend the configuration space over more than two major shells, since there is the potential of having levels with the same angular momentum quantum numbers [4].

### 5 Configuration Mixing and Model Spaces

In most practical cases, nuclei with more than one nucleon outside of a closed core consist of several active orbitals, and can in principle have any number of allowed combinations of the available particles. To illustrate this further, the case of $^{18}$O, with two neutrons outside of the $^{16}$O closed core can be considered (Fig. 2.1). In the simplest case, the two additional neutrons outside of the closed shell will exist in the most energetically favourable level, $1d_{5/2}$, which can form $0^+$, $2^+$, and $4^+$ final states in $^{18}$O [12]. The residual interaction within the shell-model Hamiltonian can then be chosen such that the $0^+ \rightarrow 2^+ \rightarrow 4^+$ energy spacings reproduce what is observed experimentally.

There are of course several ways to form these $0^+$, $2^+$, and $4^+$ states within the model space outside of the $^{16}$O core. For example, in the $sd$ model space, the $0^+$ final state in $^{18}$O can be formed by putting both neutrons in either the $1d_{5/2}$, $2s_{1/2}$, or $1d_{3/2}$ configurations. Because all three of these situations can occur, the wave functions will be linear combinations of the possible basis functions [12]. This means that for the $0^+$ state, two eigenfunctions exist,

$$\Psi_{0^+:1} = \sum_{k=1}^n a_{k,1}|\psi_k;0^+\rangle,$$  \hspace{1cm} (5.1)

$$\Psi_{0^+:2} = \sum_{k=1}^n a_{k,2}|\psi_k;0^+\rangle,$$  \hspace{1cm} (5.2)

where, for the particular case of the two neutrons in either the $1d_{5/2}$ or $2s_{1/2}$ orbitals,

$$|\psi_1;0^+\rangle = |C,(1d_{5/2})^2;0^+\rangle = |C,aa\rangle,$$  \hspace{1cm} (5.3)

$$|\psi_2;0^+\rangle = |C,(2s_{1/2})^2;0^+\rangle = |C,bb\rangle,$$  \hspace{1cm} (5.4)
where $C$ indicates all possible orbits within the $^{16}\text{O}$ core [15]. The ground state is a linear combination of particles in these two orbitals, and therefore the individual contributions are determined by diagonalizing the two component Hamiltonian, where the energies for the respective states are their eigenvalues. Since these orbitals can form excited $0^+$ states as well, the similarities of the excited- and ground-state nuclear wave functions can be probed by a special class of $\beta$-decay.

In practice, the above is calculated by starting with a closed-shell core, and the full Hamiltonian is solved in a particular basis or model space. This model space is typically chosen as a subset of orbitals that are energetically separated by shell gaps from other sets of orbitals [5]. The Hamiltonian matrix dimensions grow exponentially with increasing number of orbitals in the model space, and with increasing number of valence nucleons that occupy those orbitals. Historically, these types of shell-model calculations have been limited by the available computing power since the full calculation spaces are often not tractable. In the 1960s, the $(1p_{3/2}, 1p_{1/2})$ space, with a dimension of $10^2$ could be treated, and the $(2s_{1/2}, 1d_{5/2}, 1d_{3/2})$ model space, with dimensions up to $10^5$ could be treated into the 1980s.

6 One-Particle Transfer Formalism

Quantum observables for the removal or addition of a single nucleon from some initial state to a specific final state are defined by the nuclear matrix elements of the creation and annihilation operators, respectively [15]. These matrix elements are used to define the spectroscopic factors for the respective reactions responsible for the transfers, as well as providing the basis for both one-body (ie. $\beta$-decay) and two-body (ie. two-nucleon transfer) transition amplitudes [15].

As defined previously, the creation operator $a_{km}^\dagger$ is in fact a tensor of rank $j$, since it creates the single-particle state $|k m\rangle$, where $k$ stands for the single-particle quantum numbers $n \ell j$.

The annihilation operator $a_{km}$, however, is not in itself a tensor of rank $j$, but rather requires a tensor identity to convert the Hermitian conjugate [15],

$$a_{km} = (-1)^{j-m} \tilde{a}_{k,(-m)}, \quad (6.1)$$

where $\tilde{a}_{k,(-m)}$ is the annihilation operator represented in tensor form. The reduced matrix elements generated by the two respective tensors can thus be related to each other,

$$\langle k^{n-1} \omega \prime J' | \tilde{a}_k | k^n \omega J \rangle = (-1)^{j'+J'-J} \langle k^n \omega J | a_k^\dagger | k^{n-1} \omega \prime J' \rangle, \quad (6.2)$$

where the corresponding sum rules for the addition and subtraction of one particle are,

$$\sum_{\omega'J'} |\langle k^n \omega J | a_k^\dagger | k^{n-1} \omega' J' \rangle|^2 = \sum_{\omega'J'} |\langle k^{n-1} \omega' J' | \tilde{a}_k | k^n \omega J \rangle|^2 = n(2J + 1). \quad (6.3)$$

All of the relevant structure and reaction information regarding the final-state nucleus lies in these reduced matrix elements. If the final states are summed over, and subsequently normalized to unity, the matrix elements are referred to as coefficients of fractional parentage (CFP), $\rangle$. 


Figure 5.1: The shell-model levels for proton or neutron occupation $< 50$. The circled values are the magic numbers and denote the energy gaps that generate the cores. The shells that are included in the boxed region are the ones which are probed by the one- and two-neutron pickup reactions on $^{64}$Zn ($\pi = 30$, $\nu = 34$).
and are defined as:

\[
\langle j^n \omega | j^{n-1} \omega' J' \rangle = \frac{\langle k^n \omega J | a_k^\dagger | k^{n-1} \omega' J' \rangle}{\sqrt{n(2J+1)}},
\]

(6.4)

where the notation \( j \) is used here instead of \( k \), to emphasize the fact that the CFP depends only on \( j \), and not on \( n \) or \( \ell \) [15]. The associated sum-rule for the CFP is therefore

\[
\sum |\langle j^n \omega | j^{n-1} \omega' J' \rangle|^2 = 1,
\]

(6.5)

which represents the probability of removing one particle from some configuration \( |k^n \omega J \rangle \), and leaving it in a \( |k^{n-1} \omega' J' \rangle \) configuration.

In reality, the nuclear environment is not as sterile as the above formalism may suggest, and in general, the full many-body wave functions are required in the nuclear matrix element. The reduced matrix elements from Eqn. 6.2 can thus be rewritten in terms of the full nuclear wave function as:

\[
\langle \Psi_{A-1}^F \omega_f J_f | a_k^\dagger | \Psi_{A}^I \omega_i J_i \rangle = (-1)^{j_f - j_i} \langle \Psi_{A}^I \omega_i J_i | a_k^\dagger | \Psi_{A-1}^F \omega_f J_f \rangle,
\]

(6.6)

where, analogous to Eqn. 6.3, the corresponding sum rules for particle removal and addition, respectively, are:

\[
\sum_{\omega' J'} |\langle \Psi_{A-1}^F \omega_f J_f | a_k^\dagger | \Psi_{A}^I \omega_i J_i \rangle|^2 = (2J_i + 1) \langle n_k \rangle_i,
\]

(6.7)

\[
\sum_{\omega' J'} |\langle \Psi_{A}^I \omega_i J_i | a_k^\dagger | \Psi_{A-1}^F \omega_f J_f \rangle|^2 = (2J_f + 1) \langle n_k \rangle_i,
\]

(6.8)

where \( \langle n_k \rangle_i \) is the number of particles in the \( k \) orbital, for some state \( \Psi_i \). If this state has a pure configuration, then \( \langle n_k \rangle_i \) is an integer, however for mixed configurations this value becomes the average number of particles in orbit \( k \) [15].

### 6.1 Spectroscopic Factors

The spectroscopic factor, \( S \), can be defined in terms of the full many-body reduced matrix elements for both nucleon creation and annihilation,

\[
S = \frac{|\langle \Psi_{A}^I \omega_i J_i | a_k^\dagger | \Psi_{A-1}^F \omega_f J_f \rangle|^2}{(2J + 1)} = \frac{|\langle \Psi_{A-1}^F \omega_f J_f | a_k^\dagger | \Psi_{A}^I \omega_i J_i \rangle|^2}{(2J + 1)},
\]

(6.9)

where by convention, the \( 2J + 1 \) factor is associated with the \( A \)-nucleon nucleus [15]. If the wave functions are restricted to one orbital \( k \) with some angular momentum \( j \), then the spectroscopic factor can be expressed as the square of the CFP in Eqn. 6.4,

\[
S = n |\langle j^n \omega J | j^{n-1} \omega' J' \rangle|^2.
\]

(6.10)
The spectroscopic factor can be approximated by the reaction cross-sections for both single-nucleon pickup and transfer,

\[
\sigma^- \approx \frac{|\langle \Phi^A_i \omega J_i \mid a_k \mid \Phi^A_f \omega J_f \rangle|^2}{(2J_i + 1)} = S, \quad (6.11)
\]

\[
\sigma^+ \approx \frac{|\langle \Phi^{A+1}_f \omega J_f \mid a_k^\dagger \mid \Phi^A_i \omega J_i \rangle|^2}{(2J_i + 1)} = \frac{(2J_f + 1)}{(2J_i + 1)} S, \quad (6.12)
\]

where \(\sigma^-\) and \(\sigma^+\) are the reaction cross-sections for nucleon removal and addition from and to an \(A\) nucleon system, respectively. The extra factors of \(J\) in Eqn. 6.12 take into account the different \(M\)-state averaging for the two different types of reactions [15]. The spectroscopic factors can therefore be extracted from experimental observations of these reactions if the ability to calculate the nuclear matrix element is possible. Typically, these calculations are done as pure single-particle predictions, assuming the \(k\) orbital where the nucleon is involved is of a purely single-particle nature (analogous to one particle outside of a closed shell as described in Section 2). If this predicted cross-section \(\sigma_{s.p.}\) is provided, Eqns. 6.11 and 6.12 can be rewritten in terms of the experimentally measured cross-sections,

\[
\sigma^-_{\text{exp.}} = S \sigma_{s.p.}, \quad (6.13)
\]

\[
\sigma^+_{\text{exp.}} = \frac{(2J_f + 1)}{(2J_i + 1)} S \sigma_{s.p.}, \quad (6.14)
\]

where, in principle, the experimentally observed quantities can be directly compared to the shell-model calculated values in Eqns. 6.11 and 6.12.

There exists some arguments both for and against the use of experimental spectroscopic factors to provide nuclear structure information, based on the fact that the spectroscopic factor is not strictly a quantum observable. Due to the requirement of \(\sigma_{s.p.}\) in the extraction of an experimental spectroscopic factor, they of course contain a heavy model dependency. Some of the critics against using these experimentally extracted quantities as comparisons for the theoretical calculations, suggest that these dependencies are prohibitive, and prevent any useful information from being extracted [16]. Despite these criticisms, experimental spectroscopic factors can give insight into the complexity of the nuclear states involved in the interaction, and are useful in comparison to shell model theories [17]. The information contained in the ejectile from a transfer reactions can therefore provide very detailed information about the nature of the final nuclear state. The single-nucleon transfer probe can thus be used to determine the similarity between the initial and final-state wave functions.

6.2 RADIAL OVERLAP FUNCTION AND ANCs

The spectroscopic factors described in the previous section are proportional to the similarity between the specific initial- \(\Psi^A_i\), and final- \(\Psi^{A+1}_f\), state nuclei. Assuming the case of a nucleon
pickup reaction, the initial state wave function can be written in terms of a sum over the complete set of states in the final $A-1$ nucleus,

$$\Psi_i^A = \sum_{f \ell j} \theta_{i f \ell j}(\vec{r}) \Psi_{f}^{A-1}, \quad (6.15)$$

where an explicit sum over all possible $\ell$ and $j$ values of the single-particle overlap function is made [15]. This overlap function is defined as:

$$\langle \Psi_f^{A-1} | \Psi_i^A \rangle = \sum_{f \ell j} \theta_{i f \ell j}(\vec{r}), \quad (6.16)$$

for the removal of one nucleon from some initial state $\Psi_i^A$. The spectroscopic amplitude, $A_{i f \ell j}$ is defined as the normalization of this overlap function, and is related to the spectroscopic factor via:

$$S_{i f \ell j} = |A_{i f \ell j}|^2. \quad (6.17)$$

The angular distribution for a given reaction depends explicitly on both $\ell$ and $j$ (hence the dependence shown in $\theta$). It is therefore possible to separate the individual components for a given transition, however the observed $j$ dependence of the angular distribution is very small, and the spectroscopic factors for different $j$ values of a given $\ell$ are often summed [15]. It has also been suggested [18, 19], that these direct reactions are primarily peripheral, and probe only the tail of the overlap function [17]. Therefore, the relevant information should be contained in its asymptotic behaviour,

$$\theta_{i f \ell j}(\vec{r}) \rightarrow N_{\ell j} f_{\ell}(r), \quad (6.18)$$

where $N_{\ell j}$ is an overall normalization, and $f_{\ell}(r)$ is an asymptotic form of the overlap function that is independent of the nuclear potential that depends on $\ell$, and is usually taken as the Whittaker function [15]. Using this asymptotic form, Eqn. 6.16 can be rewritten in terms of the asymptotic normalization coefficients (ANCs), $A_{i f \ell j} N_{\ell j}$, as:

$$\langle \Psi_f^{A-1} | \Psi_i^A \rangle \rightarrow \sum_{\ell j} A_{i f \ell j} N_{\ell j} f_{\ell}(r), \quad (6.19)$$

where, from Eqn. 6.17, the resulting spectroscopic factor is the square of these ANC. Therefore, if experimental spectroscopic factors can be measured, direct information about the nature of the individual state wave functions can be deduced.
REFERENCES


