Kinematics & Momentum
CEEN 598D: Fluid Mechanics for Hydro Systems

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Momentum

Newton’s second law of motion says:

\[
\begin{align*}
\text{time rate of change of the} & \quad \text{Sum of external forces} \\
\text{linear momentum of the} & \quad \text{acting on a system} \\
\text{system} & \quad \end{align*}
\]
Momentum = mass \bullet velocity

How does this relate to Newton’s second law that we have discussed previously in class?
Momentum = mass * velocity!

If we sum the momentum of all of the “particles” in our system:

\[ \sum V \times m = \sum V \rho d \forall = \sum F_{\text{sys}} \]

At a moment in time, we can define a CV so that:

\[ \sum F_{CV} = \sum F_{\text{sys}} \]
Reynolds Transport Theorem (RTT) relates Lagrangian and Eulerian approaches

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<th>Control Volume Approach</th>
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<td>Mass cannot cross the boundaries.</td>
<td>Mass is allowed to cross the boundaries.</td>
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<td>The mass of the closed system must stay constant with time; always the same number of kilograms.</td>
<td>The mass of the materials inside the CV can stay constant or can change with time.</td>
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<td>Mass (identity)</td>
<td>Always contains the same matter.</td>
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<td>Solid mechanics, fluid mechanics, thermodynamics, and other thermal sciences.</td>
<td>Fluid mechanics, thermodynamics, and other thermal sciences.</td>
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Reynolds Transport Theorem (RTT)

\[
\frac{DB_{SYS}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \sum \rho_{OUT} A_{OUT} V_{OUT} b_{OUT} - \sum \rho_{IN} A_{IN} V_{IN} b_{IN}
\]

Fixed control surface and system boundary at time \( t \)

---

System boundary at time \( t + \delta t \)
Recall: Intensive and Extensive Properties

- **Intensive property:** property that is *independent* of the amount of mass

- **Extensive property:** property that *depends* on amount of matter present (i.e., mass)
RTT and Conservation of Momentum

where $B = \text{momentum}$ and $b = \text{velocity}$ in RTT

\[
\text{RTT: } \frac{DB_{\text{SYS}}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \sum \rho_{\text{OUT}} A_{\text{OUT}} V_{\text{OUT}} b_{\text{OUT}} - \sum \rho_{\text{IN}} A_{\text{IN}} V_{\text{IN}} b_{\text{IN}}
\]

For momentum $(M)$:

\[
\frac{DM_{\text{SYS}}}{Dt} = \frac{\partial M_{CV}}{\partial t} + \sum \rho_{\text{OUT}} A_{\text{OUT}} V_{\text{OUT}} V_{\text{OUT}} - \sum \rho_{\text{IN}} A_{\text{IN}} V_{\text{IN}} V_{\text{IN}}
\]
Momentum

This means, for such a system, the Reynolds Transport Theorem allows us to say:

\[
\frac{D}{Dt} \int_{\text{SYS}} V \rho dV = \frac{\partial}{\partial t} \int_{\text{CV}} V \rho dV + \sum V_{\text{OUT}} \rho_{\text{OUT}} A_{\text{OUT}} V_{\text{OUT}} - \sum V_{\text{IN}} \rho_{\text{IN}} A_{\text{IN}} V_{\text{IN}}
\]

\[
\begin{align*}
\text{time rate of change} & \quad \text{time rate of change} & \quad \text{net rate of flow} \\
\text{of the linear} & \quad \text{of the linear} & \quad \text{of linear momentum} \\
\text{momentum of the} & \quad \text{momentum of the} & \quad \text{through the control} \\
\text{system} & \quad \text{contents of the} & \quad \text{surface} \\
\end{align*}
\]
In other words:

As a particle of mass moves in or out of the control volume they carry linear momentum in or out just like temperature, mass, etc!

Just like conservation of mass principals used in simple watershed models (e.g. the linear reservoir model)
Momentum = mass * velocity!

If we sum the momentum of all of the “particles” in our system:

\[ \sum F_{sys} = \frac{D}{Dt} \int V \rho d\mathcal{V} \]

At a moment in time, we can define a CV so that:

\[ \sum F_{CV} = \sum F_{sys} \]

\[
\frac{\partial}{\partial t} \int_{CV} V \rho d\mathcal{V} + \sum V_{OUT} \rho_{OUT} A_{OUT} V_{OUT} - \sum V_{IN} \rho_{IN} A_{IN} V_{IN} = \sum F_{CV}
\]
Linear Momentum Equation

For steady flow, where the momentum is also constant in time:

\[ \sum V_{OUT} \rho_{OUT} A_{OUT} V_{OUT} - \sum V_{IN} \rho_{IN} A_{IN} V_{IN} = \sum F_{CV} \]
Momentum: Interpretation

• Application of momentum equation to fluid flow is just like free-body approach to solid mechanics.

• Forces summed and equated to momentum changes of the flow.
Example

GIVEN: A “balloon rocket” is a balloon suspended from a taut wire by a hollow tube and string. The nozzle is formed of a 1 cm-diameter tube, and an air jet exits the nozzle with a speed of 40 m/s and a density of 1.2 kg/m^3.

FIND: the force $F$ needed to hold the balloon stationary neglecting friction.
GIVEN: As shown in the figure, a horizontal jet of water exits a nozzle with a uniform speed of \( V_1 = 10 \text{ ft/s} \), strikes a vane and is turned through an angle of theta.

FIND: Determine the anchoring force needed to hold the vane stationary if gravity and viscous effects are neglected.
Example

GIVEN: A sluice gate across a channel of width $b$ is shown in the closed and open position.

FIND: Is the anchoring force required to hold the gate in place larger when closed or open?
Navier-Stokes Equations

• You cannot leave a fluids class without hearing the term Navier-Stokes equations!

• Applies to viscous flow.
Claude-Louis Navier  
(1785-1836)

- Formulated the general theory of elasticity in a mathematically usable form (1821) and made it available to the field of construction with sufficient accuracy for the first time.

- He succeeded in determining the zero line of mechanical stress, finally correcting Galileo Galilei's incorrect results (1819).

- Established the elastic modulus as a property of materials independent of the second moment of area (1826).

- Navier is therefore often considered to be the founder of modern structural analysis.
Sir George Gabriel Stokes
(1819-1903)

- Fluid Mechanics (in addition to the N-S eqns), Stokes flow/Creeping Flow, steady motion of incompressible fluids and some cases of fluid motion (1842)

- Light, polarization and diffraction (1845) dynamical theory of diffraction, in which he showed that the plane of polarization must be perpendicular to the direction of propagation.
  - Florescence (1852)
  - Polarization (1852)
  - Chemical Analysis
Start with a control volume:
Write momentum in $x$:

$$
\sum F_x = \frac{d}{dt} \int_{CV} \rho v_x dV + \sum_{CS} \dot{m}_o v_{ox} - \sum_{CS} \dot{m}_i v_{ix}
$$
Add pressure, stress and body force to the control volume:
Write the forces as

Pressure: \[ F_{x,p} = \left( p \bigg|_{x-\frac{\Delta x}{2}} - p \bigg|_{x+\frac{\Delta x}{2}} \right) \Delta y \]

Gravity: \[ F_{x,g} = g_x \rho \Delta x \Delta y \]
Shear forces a bit more complicated:

\[
F_{x,\tau} = \left( \begin{array}{c} \tau_{yx} & y+\frac{\Delta y}{2} - \tau_{yx} & y-\frac{\Delta y}{2} \\ \end{array} \right) \Delta x + \left( \begin{array}{c} \tau_{xx} & x+\frac{\Delta x}{2} - \tau_{xx} & x-\frac{\Delta x}{2} \\ \end{array} \right) \Delta y
\]

There are TWO sets of shear forces:

1) Shear stress in x-direction on the “north” face
2) Normal stress (not pressure) that acts on east/west faces proportional to strain-rate of fluid
We apply Liebnitz Theorem to momentum:

$$\frac{d}{dt} \int_{CV} \rho u dV = \int_{CV} \frac{\partial}{\partial t} (\rho u) dV + \sum_{CS} \rho u (V_c \cdot A)$$

Velocity of the CV

Assume that the velocity of the control volume, $V_c=0$ (i.e., non-moving control volume).
Our rate of change of momentum now becomes:

$$\frac{d}{dt} \int_{CV} \rho u dV = \int_{CV} \frac{\partial}{\partial t} (\rho u) dx dy = \frac{\partial}{\partial t} (\rho u) \Delta x \Delta y$$

We can use our control volume to sum momentum flow over all faces:

$$\sum_{CS} m_o v_o x - \sum_{CS} m_i v_i x = \left( \rho uu \bigg|_{x+\frac{\Delta x}{2}} - \rho uu \bigg|_{x-\frac{\Delta x}{2}} \right) \Delta y$$

$$+ \left( \rho vu \bigg|_{y+\frac{\Delta y}{2}} - \rho vu \bigg|_{y-\frac{\Delta y}{2}} \right) \Delta x$$
This is the FLUX of momentum across all faces in the control volume:
Collect terms and divide through by $\Delta x \Delta y$:

$$
\frac{1}{\Delta x \Delta y} \int_{CV} \frac{\partial}{\partial t} (\rho u) dx dy + \frac{\rho uu}{\Delta x} \bigg|_{x+\frac{\Delta x}{2}}^{x-\frac{\Delta x}{2}} - \frac{\rho uu}{\Delta x} \bigg|_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} + \frac{\rho vu}{\Delta y} \bigg|_{y+\frac{\Delta y}{2}}^{y-\frac{\Delta y}{2}} - \frac{\rho vu}{\Delta y} \bigg|_{y-\frac{\Delta y}{2}}^{y+\frac{\Delta y}{2}} = 
$$

$$
\begin{align*}
\frac{p}{\Delta x} & \bigg|_{x+\frac{\Delta x}{2}}^{x-\frac{\Delta x}{2}} - \frac{p}{\Delta x} \\
\frac{\tau_{yx}}{\Delta y} & \bigg|_{y+\frac{\Delta y}{2}}^{y-\frac{\Delta y}{2}} - \frac{\tau_{yx}}{\Delta y} \\
\frac{\tau_{xx}}{\Delta x} & \bigg|_{x+\frac{\Delta x}{2}}^{x-\frac{\Delta x}{2}} - \frac{\tau_{xx}}{\Delta x} \\
+ \rho g_x
\end{align*}
$$

and take the limit as $\Delta x$ and $\Delta y$ approach zero:

$$
\lim_{\Delta x, \Delta y \to 0} \frac{1}{\Delta x \Delta y} \int_{CV} \frac{\partial}{\partial t} (\rho u) dx dy = \frac{\partial}{\partial t} (\rho u)
$$
Differential Form of Momentum

This gives the differential form of momentum:

\[
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho uu) + \frac{\partial}{\partial y} (\rho uv) = \\
- \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho g_x
\]
Differential Form of Momentum

If we apply the continuity equation (not shown) we can differentiate by parts and simplify:

\[ \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial v}{\partial y} = \]

\[ - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \rho g_x \]
Differential Form of Momentum

The left hand side of the below equation is called the Material Derivative and can be shortened to the form on the right hand side:

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho u \frac{\partial v}{\partial y} = \rho \frac{Du}{Dt}$$

This whole process can be repeated for the $y$-direction.
Stress and Strain

We can include relationships for the shear of an element and relate shear strain to the constant of proportionality between shear stress and shear strain:

\[ \tau_{yx} = \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]

And normal stress:

\[ \tau_{xx} = 2\mu \frac{\partial u}{\partial x} \]
Differential Form of Momentum

Substituting stress relationships back into the momentum:

\[
\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ 2\mu \frac{\partial u}{\partial x} \right] + \rho g_x
\]
Differential Form of Momentum

If we assume viscosity is constant:

\[
\frac{\partial}{\partial x} \left[ \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ 2\mu \frac{\partial u}{\partial x} \right] = \\
\mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \mu \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right]
\]
Navier-Stokes Equations

Now, we can write the N-S Equations for planar flow for an incompressible fluid:

\[
\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x
\]

\[
\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_x
\]
Differential Form of Momentum

- All the equations we work with in this class are simplifications of the N-S.
- Some analytical solutions exist
  - Poiseuille flow – slow, viscous, incompressible flow through a tube
  - Couette flow – laminar flow of a viscous fluid between two moving plates
  - Stokes boundary layer – boundary layer (close to a wall) in oscillatory, viscous flow
  - Taylor Green vortex – two-dimensional, unsteady, decaying vortex
- Mostly numerical solutions exist, though these can be very problematic at high Reynolds Numbers.