NOTES ON UNCERTAINTY ANALYSIS
FOR MEL LABS
by
Matt Young

For more detail, see also
http://www.mines.edu/Academic/courses/physics/phgn471/uncertainty.pdf

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Matt Young’s Home Page

1. Definitions
   Error = deviation from True Value
   Uncertainty = estimate of probable error

2. Types of uncertainty
   • *Measured or statistical* uncertainty
     • for fluctuating or *random variables*
   • *All other* uncertainties
     • for variables whose statistics are not known
     • calculated from estimates
     • include but not limited to systematic errors
3. **Uncertainty due to measured or statistical errors**
   
   - Data are noisy, as due to voltage fluctuations
   - Temporarily ignore other sources of uncertainty
   - *Measurand* = quantity to be measured
     - Individual measured values are $m_i$
   - Mean = $\mu$, calculated from mean of data set
   - $N$ = number of data points
   - True value = $M = ???$
     - $\mu$ is called an estimator of $M$

Calculate the *standard deviation of the mean*, or *SDOM*:

$$
\sigma_m = \sqrt{\frac{\sum_{i=1}^{N} (m_i - \mu)^2}{N \cdot (N - 1)}}
$$

- $\sigma_m$ is the *standard uncertainty* due to random errors alone
- $\sigma_m$ decreases in proportion to $\sqrt{(N)}$
4. Significance of standard uncertainty $\sigma$

- $M$ falls between $\mu - \sigma$ and $\mu + \sigma$ with 68% probability
  - 68% confidence interval

- $M$ falls between $\mu - 2\sigma$ and $\mu + 2\sigma$ with 95% probability
  - 95% confidence interval

- $M$ falls between $\mu - 3\sigma$ and $\mu + 3\sigma$ with 99.7% probability
  - 99.7% confidence interval

Half the battle is learning the vocabulary!
Cheap and dirty way to estimate $\sigma$

- Calculate the maximum value of the dataset minus the minimum value
- Divide by $6\cdot\sqrt{(N)}$
- Result is a fair estimate of $\sigma$
- Use to check your calculation

¡Important note! $\sigma$ is the 68 % confidence interval; the $\Delta$'s below are your best estimates of 99.7 % confidence intervals. They are not directly comparable.
5. **Estimated errors** (all other errors)

- Usually errors whose *statistics are not known*

\[
\begin{pmatrix}
\text{Uncertainty} \\
\text{Analysis}
\end{pmatrix} = \begin{pmatrix}
\text{Partial} \\
\text{Differentiation}
\end{pmatrix} + (\text{Guessing})
\]

Suppose \( m = m(a,b,c) \) & we **calculate** \( m \) by measuring \( a, b, \) & \( c \)

Let \( \Delta a = \) extreme value of error of \( a \)

- \( \frac{1}{2} \) scale division, for example, or
- mechanical tolerance in a part, for example

\( \Delta a \) is

- an *estimate* or a *guess*
- of the 99.7 % confidence interval

Estimate error \( \Delta m \) of \( m \) due to error \( \Delta a \) of \( a \):

\[
\Delta m_a = \frac{\partial m}{\partial a} \cdot \Delta a
\]

- Assumes \( \Delta a \ll a \)
- \( \Delta a = \) half-width of 99.7 % confidence interval & is
  - always positive
  - *not* the standard uncertainty or standard deviation
6. **Relative error**

Often, \( m = K \cdot a \) (where \( K \) may be a function of \( b, c, ... \))

- Given \( \Delta a \), calculate confidence interval of \( m \):

\[
\Delta m_a = \frac{\partial}{\partial a} (K a) \cdot \Delta a = K \cdot \Delta a
\]

But \( K = \frac{m}{a} \)

\[
\text{so } \frac{\Delta m}{m} = \frac{\Delta a}{a}
\]

Best estimator of \( m \) is its mean \( \mu \), so we write

\[
\frac{\Delta m_a}{\mu} = \frac{\Delta a}{a}
\]

Similarly, if

- \( a \) is a *random variable*, and
- we measure \( \mu_a \) and \( \sigma_a \)
- then

\[
\frac{\sigma_m}{\mu} = \frac{\sigma_a}{\mu_a}
\]

where \( m \) means the measurand
7. **Painless uncertainty analysis**

- When $m$ is not a strong function (such as an exponential) of $a$, $b$, $c$

- Estimate $\Delta a$, $\Delta b$, $\Delta c$, ...
- Calculate $\Delta a/a$, $\Delta b/b$, $\Delta c/c$, ...
- Include only the *largest* in your analysis

- We will use the $\Delta$’s below to calculate standard uncertainties

- Note: If $m = K a^n$, then, similarly

\[
\frac{\Delta m_a}{\mu} = n \cdot \frac{\Delta a}{a}
\]

The relative error $\Delta m_a/\mu$ of $m$ due to $a$ is proportional to the relative error of $a$ itself
8. **Example 1** from MEL 2

Yield stress: \( Y = p \cdot \frac{S}{A} \)

- \( Y \) = yield stress
- \( p \) = measured pressure at yield
- \( S \) = area of piston
- \( A \) = area of wooden block

- Calculate component of uncertainty due to area of block
  - \( A = a^2 \), where \( a = 1.5 \text{ in} \)
  - Guess:
    \[
    \Delta a = \frac{1}{32} \text{ in} \approx 0.03 \text{ in}
    \]
    \[
    \frac{\Delta A}{A} = 2 \frac{\Delta a}{a} = 2 \cdot \frac{0.03 \text{ in}}{1.5 \text{ in}} = 0.04
    \]

- Similarly,
  \[
  S = \pi \frac{D^2}{4}, \quad D = 8.75 \text{ in}
  \]
  Guess: \( \Delta D = 0.015 \text{ in} \)
  \[
  \frac{\Delta S}{S} = 2 \frac{\Delta D}{D} = 0.003
  \]

- Ignore \( \Delta D \) in calculating \( \Delta Y \)
How to calculate $\sigma_p$ from the calibration curve

**Figure 1. Calibration curve.**

- Use the Excel spreadsheet to
  - Calculate line of best fit (solid line)
  - Find voltage $V_{\text{meas}}$, as at yield stress, calculate corresponding pressure $p_{\text{meas}}$
  - Use calculation of 68% confidence interval (dashed curves) to estimate the standard deviation $\sigma$
Now back to Example 1

\[ Y = p \cdot S / A = 2600 \text{ ksi}, \text{ from above} \]

By partial differentiation,

\[ \Delta Y_S = \Delta S \cdot p / A \]
\[ \Delta Y_A = \Delta A \cdot p S / A^2 \]

- \( S = 60 \text{ in}^2, \Delta S = 0.003 \cdot S, \text{ from above}; \)
  \( \Delta Y_S = \text{8 ksi}; \text{ small as predicted} \)

- \( \Delta A = 0.04 \cdot A, \text{ from above} \)
  \( \Delta Y_A = 108 \text{ ksi} \)

- We do not have to estimate \( \Delta p \), since we can
  *measure* \( \sigma_p = 2 \text{ ksi} \text{ from the calibration curve}. \)
  Thus,

\[ u_p = \sigma_p \cdot S / A, \]

where \( u_p \) is the *component of uncertainty* of \( Y \) due to the uncertainty \( \sigma_p \) of \( p \), and

\( u_p = 50 \text{ ksi} \)
9. **Uncertainties at last!**

(a) *Type A* or measured uncertainties

- *Standard uncertainty* \( u_r = \) the SDOM, that is,
  - \( u_r = \sigma \)
  - \( r \) stands for “random”

(b) *Type B* uncertainties, or all other uncertainties

- Estimate 99.7 % confidence interval \( \Delta m_a, \Delta m_b, \Delta m_c, \ldots \), as above
- Assume *uniform* distribution of errors (*not* Gaussian)
  - Standard deviation of uniform distribution with half-width \( \Delta m_a \) is \( \Delta m_a / \sqrt{3} \)
- Define *standard uncertainties* as
  \( u_a = \Delta m_a / \sqrt{3} \), \( u_b = \Delta m_b / \sqrt{3} \), ...
Back to Example 1 yet again

- Type A (measured) uncertainty
  \[ u_p = 50 \text{ ksi} \]

- Type B (all other) uncertainties
  \[ \Delta Y_s = 8 \text{ ksi} \text{ and} \]
  \[ \Delta Y_A = 108 \text{ ksi}, \text{ so} \]

  \[ u_s = \frac{8 \text{ ksi}}{\sqrt{3}} \text{ and} \]
  \[ u_A = \frac{108 \text{ ksi}}{\sqrt{3}}, \text{ so} \]

  \[ u_s = 5 \text{ ksi} \text{ and} \]
  \[ u_A = 62 \text{ ksi} \]
10. **Combined standard uncertainty**

- Uncertainties are added *in quadrature* (sum of squares)

\[ u_c = \sqrt{u_1^2 + u_2^2 + u_3^2 + \cdots} \]

- \( u_c \) is the *combined standard uncertainty*
- Note that there may be more than 1 source of random uncertainty

11. **Expanded uncertainty**

- \( u_c \) is multiplied by a *coverage factor*, usually 2
- Express experimental results as

\[ \mu \pm 2 \, u_c \]

- \( 2 \, u_c \) is the *expanded uncertainty*
- The interval \( 2 \, u_c \) defines the 95 % *confidence interval*
  - The True Value is presumably within the interval \( \mu \pm 2 \, u_c \), with 95 % probability
**Example 1**, one last time!

Table 1. Compilation of standard uncertainties

<table>
<thead>
<tr>
<th>Source of uncertainty</th>
<th>Type of uncertainty</th>
<th>Standard uncertainty, ksi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measurement of pressure</td>
<td>Measured (Type A)</td>
<td>50</td>
</tr>
<tr>
<td>Area $S$ of piston</td>
<td>Other (Type B)</td>
<td>5</td>
</tr>
<tr>
<td>Area $A$ of block</td>
<td>Other (Type B)</td>
<td>62</td>
</tr>
<tr>
<td></td>
<td>Combined standard uncertainty</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>Expanded uncertainty</td>
<td>160</td>
</tr>
</tbody>
</table>

Thus,

- $Y = 2600 \pm 160$ ksi

Note the use of round numbers for uncertainties and the correct number of significant digits in the expression for $Y$.
12. Example 2. Uncertainty of Young’s modulus

- Slope of curve of $\sigma$ vs. $\epsilon$

Standard deviation of the slope of a line

- Fitted equation is $y = A + Bx$

- Standard uncertainty $\sigma_B$ of the slope $B$ is

$$\sigma_B^2 = \frac{N \sigma_y^2}{\Delta}, \text{ where}$$

$$\sigma_y^2 = \frac{1}{N - 2} \sum_{i=1}^{N} (y_i - A - Bx_i)^2,$$

$$\Delta = N \sum_{i=1}^{N} x_i^2 - (\sum_{i=1}^{N} x_i)^2$$


- Use *slope uncertainty* when interested in slope as such (as when measuring Young’s modulus)
- Use *confidence interval* of line when interested in uncertainty of quantity on y-axis (such as displacement or pressure)
Estimated (Type B) uncertainties

- Formulas estimate uncertainty due to *random errors only*, not *calibration uncertainties* of axes
  \[ E = \sigma / \varepsilon \]
  \[ \Delta E_{\sigma} / E = \Delta \sigma / \sigma \] (Here \( \sigma \) means stress not SDOM)
  \[ \Delta E_{\varepsilon} / E = \Delta \varepsilon / \varepsilon \]

- Assume that electronics introduce negligible error
  - *Write down* a number for \( \Delta L \), where \( L \) is any length measurement. (Hint: what is the least count of the dial indicator?)
  - *Calculate* \( \Delta \varepsilon \) for a representative value of \( \varepsilon \)
  - Similarly, *write down* \( \Delta F \), where \( F \) is a force measurement, and *calculate* \( \Delta \sigma \)
  - *Calculate* the appropriate \( \Delta E \)'s, *divide* by \( \sqrt{3} \), and *combine* in quadrature

Details are left as a proverbial exercise for the student
13. **Example 3** from MEL 2

Use *displacement sensor* to measure $E$ of steel specimen

- Measured displacement = elongation of specimen + elongation of shafts holding specimen, or

- $d_m = d_{sp} + d_{sh}$

- **Measure** $d_m$ [Here $d$ is displacement; $L$ is original length]

- **Calculate**
  
  
  $$d_{sh} = L_{sh} \times \varepsilon_{sh}, \text{ where}$$
  
  $$\varepsilon_{sh} = \frac{F}{A_{sh} \times E_{sh}}$$

- **Assume** $E_{sh} = 3 \times 10^7 \pm 3 \times 10^6$ psi (that is, \pm 10 \%)
  
  $F = 6000$ lb, $A_{sh} = 2$ in$^2$, $L_{sh} = 10$ in

- Then $d_{sh} = 1$ mil [1 mil = $10^{-3}$ in]

- **Subtract** systematic error, or bias, $d_{sh}$ from measured value $d_m$
Example 3, continued

- *Calculate* confidence interval of correction, assuming that $E_{sp} = 10^7$ psi, $F = 6000$ lb, $L_{sp} = 10$ in

- 99.7 % confidence interval $\Delta d_m$ of bias:
  - Suppose $\varepsilon_{sp} = 3 \times 10^{-3}$ (measured)
  - $\Delta d_{sh} = 0.1 \times d_{sh}$ (Why?)
  - $\Delta \varepsilon_{sp} = \Delta d_{sh} / L_{sp}$ ... (Why?) ... = $10^{-5}$
  - $\Delta E_{sp} / E_{sp} = \Delta \varepsilon_{sp} / \varepsilon_{sp}$; $\Delta E_{sp} = \frac{1}{3} \times 10^5$

- Standard uncertainty of $E$ due to $\varepsilon$:

  $$ u_{\varepsilon} = \frac{1}{3} \times 10^5 / \sqrt{3} \approx 19000 \text{ psi} $$

- Around 0.2 %, for the made-up numbers I have chosen