PHGN 471
Analytical Description of Gap Solitons

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1 Introduction and Background

There is no known analytical description of a gap soliton. The purpose of this project is to
find one. The gap soliton is one possible waveform developed in a Bose-Einstein condensate
(BEC) in a Kronig-Penney (KP) potential. Gap solitons have been observed in photonic
crystals.\(^1\) A photonic crystal is a periodic lattice of photonic fields, and is the optical ana-
log of a periodic potential in a semiconductor. Many iridescent materials are comprised of
photonic crystals.\(^2\)

The BEC is a state of matter that was predicted in 1924, but was not realized experimentally
until 1995.\(^3\) The BEC is formed when bosonic atoms, typically rubidium, are confined within
a magnetic trap and then cooled to extremely low temperatures, on the order of nanokelvins.\(^4\)

The one-dimensional mean-field behavior of the BEC is governed by the nonlinear Schrödinger
equation (NLS),

\[
\left[-\frac{1}{2} \frac{\partial^2}{\partial x^2} + g |\Psi(x, t)|^2 + V(x)\right] \Psi(x, t) = \mu \Psi(x, t),
\]

where \(g\) is the nonlinearity or interaction strength, \(V(x)\) is the external potential (energy),
\(\mu\) is the energy eigenvalue or chemical potential, and \(\Psi(x, t)\) is the wave function. This
equation has been scaled, so that everything is dimensionless.

The Kronig-Penney potential is an infinite, one-dimensional periodic lattice of Dirac delta
functions,\(^5\) as shown in Figure 1. Mathematically,

\[
V(x) = V_0 \sum_{n=-\infty}^{\infty} \delta(x - nx_\ell), \ n \in \mathbb{Z},
\]

where \(x_\ell\) is the lattice spacing, also called the lattice constant, and \(\mathbb{Z}\) is the set of all integers.
One solution to the NLS (1) with a Kronig-Penney potential is the Bloch wave. The Bloch wave has periodicity equal to that of the potential lattice, so that the waveform repeats at every site. A ground-state Bloch wave is shown in Figure 2.

In general, a soliton is a wave pulse which does not disperse. The gap soliton is a Bloch wave which is modulated by an envelope, as shown in Figure 3.
Gap solitons have been observed in nonlinear optical experiments, as well as BECs. For example, when a laser beam is shined through a nonlinear optical material, gap solitons may result. The laser beam need not be coherent in order to observe this effect.

2 Analysis

The mathematical form of the Bloch wave is

$$\psi_{\text{Bloch}}(x) = e^{iqx} \cdot f_q(x),$$

where $q$ is the wavenumber and $f_q(x)$ is a function having the same period as the potential,

$$f_q(x + n x_\ell) = f_q(x), \quad n \in \mathbb{Z}.$$  

In the regions between the Dirac delta spikes, that is, for $nx_\ell < x < (n + 1)x_\ell$, $n \in \mathbb{Z}$, the potential is constant (in fact, it is 0). Thus, we choose $f_q(x)$ to be consistent with a constant-potential solution. Stationary-state solutions of the NLS have been found for the case of constant potential, $V(x) = V_0$. These solutions have the form

$$\psi(x) = \sqrt{\rho(x)} \cdot e^{i\phi(x)},$$

where $\rho(x)$ is the particle density and $\phi(x)$ is the phase. The density

$$\rho_c(x) = \mathcal{A} \cdot \text{sn}^2[\beta(x + x_s)|m] + B$$

is an exact solution to the NLS for constant potential. In Eq. (6), the function $\text{sn}(x|m)$ is a Jacobi elliptic function, $\mathcal{A}$ is the amplitude of oscillations, $\beta$ is the translational scaling,
$x_s$ is the translational offset, $B$ is the density offset, and $m$ is the elliptic parameter. The Jacobi elliptic functions are described in more detail in Appendix A.

Thus, we may assume that the Bloch wave function has the constant-potential form given by Eq. (5). That is, both equations (3) and (5) are valid descriptions of the Bloch wave function. Equating these two expressions, and solving for $f_q(x)$, we find

$$f_q(x) = \sqrt{\rho(x)} \cdot e^{i[\varphi(x) - qx]}.$$  \hfill (7)

From Eq. (7) and the requirement that $f_q$ must have the same periodicity as the lattice, we find that the period of $\rho(x)$ is also equal to the period of the lattice. Together with boundary conditions and normalization, this requirement constrains the free parameters $x_s$, $\beta$, and $m$, leaving only one free parameter, $B$. We can then obtain plots of the energy as a function of the wavenumber for several values of $B$. The value of $B$ determines the energy band. \hfill (5, 6)

It is important to fully understand the Bloch-wave case before moving on to the much more difficult problem of a gap soliton. Thus, I will carefully reproduce the results in Seaman et al. \hfill (6) as a first step. This work is already partially completed.

After completing the verification of the Bloch-wave results, I will move on to the gap soliton. I have already noted that a gap soliton looks like a Bloch wave multiplied by an envelope function. Thus, as an ansatz for the gap soliton wave function, we take

$$\psi(x) = \psi_0(x) \cdot \psi_{\text{Bloch}}(x),$$  \hfill (8)

where

$$\psi_0(x) = A \cdot \operatorname{sech}[b(x + x_0)],$$  \hfill (9)

and $\psi_{\text{Bloch}}(x)$ is given by Equations (3) and (5). The hyperbolic-secant envelope modulates the wave function so that its peaks lie on a curve as in Figure 3, rather than having a constant amplitude as in Figure 2.

### 3 Timeline

Table 1 shows milestones for this project and approximate dates of completion.

<table>
<thead>
<tr>
<th>Task</th>
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<tbody>
<tr>
<td>Reproduce energy plots in Seaman’s paper (to verify mathematical methods) \hfill (6)</td>
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<tr>
<td>Determine parameter constraints for gap soliton ansatz via boundary conditions</td>
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<tr>
<td>Determine whether physically acceptable solutions of the form of Eq. (8) exist</td>
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<tr>
<th>Approximate Completion Date</th>
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<tr>
<td>Nov. 26</td>
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<tr>
<td>Dec. 3</td>
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<tr>
<td>Jan. 15</td>
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If physically acceptable solutions of the form given by Eq. (8) do exist, then the next step will be to compare these solutions to experimentally observed gap solitons, and test whether Eq. (8) is an accurate description of real gap solitons. If such solutions do not exist, or if they do not match the experimental observations, then we will need to modify the ansatz and repeat the analysis. In this case, we may need to consider the possibility of quasi-bound states.

If an analytic form which agrees with experiment can be found, then the remainder of the spring semester will be devoted to refining the theory and using it to make further predictions. Specifically, we will need to compare the analytic result with many experiments to ensure that the agreement is good. Also, we must ensure that the theory yields the results of linear quantum mechanics in the limit that the nonlinearity $g$ becomes 0. In addition, we will explore different physical regimes of the problem and predict the outcomes using this method. For example, we may vary the lattice spacing, the nonlinearity, or the potential strength. Each of these factors may lead to different physical regimes.

However, the problem may turn out to be intractable; i.e., we may find that the gap soliton cannot be described analytically. If this occurs, I will focus on either higher-dimensional cases, or $N$-periodic Bloch waves. An $N$-periodic Bloch wave is a waveform that repeats at every $N$th lattice site. Both of these situations require much of the same physics and mathematics that are used for the gap soliton, so that the work already done will still be applicable.

4 Costs

This project is a theoretical project and requires no sophisticated equipment. *Mathematica* and other analysis tools are available for use on campus. Therefore, there is no expected cost for this project.
References


Appendix A: Jacobi Elliptic Functions

There are 12 Jacobi elliptic functions in all, of the form $pq(z|m)$, where $p, q \in \{s, c, d, n\}$, with $p \neq q$. These functions can be defined by an inverse integral, or by the locations of their poles in the complex plane when $z \in \mathbb{C}$, the set of all complex numbers. The Jacobi elliptic functions are an interpolation between the more familiar circular functions and hyperbolic functions. The parameter $m$ is called the \textit{elliptic parameter} and takes values between 0 and 1. When $m = 0$, the Jacobi elliptic functions reduce to the circular trigonometric functions. When $m = 1$, the Jacobi functions reduce to the hyperbolic functions. For values of $m$ between 0 and 1, the Jacobi functions are still periodic, but their shape depends sensitively on the value of $m$. 