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Recent Advances in the Method of Fundamental Solutions

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Abstract
The aim of this paper is to describe recent developments in the method of fundamental solutions (MFS) and related methods for the numerical solution of certain elliptic boundary value problems.

Keywords: Method of Fundamental Solutions, Nonlinear Least Squares, Boundary Collocation.

Introduction

The method of fundamental solutions (MFS) is a meshless technique for the numerical solution of certain elliptic boundary value problems which falls in the class of methods generally called boundary methods. Like the boundary element method (BEM), it is applicable when a fundamental solution of the differential equation in question is known, and shares the same advantages of the BEM over domain discretization methods. Moreover, it has certain advantages over the BEM.

In the MFS, the approximate solution is expressed as a linear combination of fundamental solutions with singularities placed outside the domain of the problem. The locations of the singularities are either preassigned and kept fixed or are determined along with the coefficients of the fundamental solutions so that the approximate solution satisfies the boundary conditions as well as possible. This is usually achieved by a least squares fit of the boundary data. Early uses of the MFS were for the solution of various linear potential problems in two and three space variables. It has since been applied to a variety of situations such as plane potential problems involving nonlinear radiation-type boundary conditions, free boundary problems, biharmonic problems, problems in elastostatics and in the analysis of wave scattering in fluids and solids.

The MFS for Helmholtz problems

To illustrate the essential features of the MFS, we consider a two-dimensional exterior Helmholtz problem which is closely related to the external scattering problem for acoustic waves by a rigid obstacle. We let Ω be an unbounded domain in $\mathbb{R}^2$ and Ω^c its bounded complement in $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the problem

$$\begin{aligned} \Delta u(P) + k^2 u(P) &= 0, \quad P \in \Omega, \\ Bu(P) &= 0, \quad P \in \partial\Omega, \end{aligned}$$

where Δ denotes the Laplacian, u is the dependent variable, k a real constant, and Ω is a bounded domain in the plane with boundary $\partial\Omega$. The operator B specifies the boundary conditions (BCs). The behaviour of u at infinity must also be specified.

In the MFS, the solution u is approximated by a function of the form

$$u_N(\mathbf{c}, \mathbf{P}; Q) = \sum_{j=1}^{N} c_j G(P_j, Q), \quad Q \in \overline{\Omega},$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{C}^N$ and $\mathbf{P}$ is a $2N$-vector containing the coordinates of the singularities P_j, which lie outside $\overline{\Omega}$. The function $G(P, Q)$ is a fundamental solution of the Helmholtz equation given by $G(P, Q) = -\frac{i}{4} H_0^{(2)}(kR(P, Q))$, where $H_0^{(2)}$ is the Hankel function of the second kind of order zero and $R(P, Q)$ denotes the distance between the points P and Q. A set of observation points $\{Q_\ell\}_{\ell=1}^M$ is selected on $\partial\Omega$. When the locations of the singularities are fixed, the coefficients $\mathbf{c}$ are determined by boundary collocation leading to the equations $Bu_N(\mathbf{c}, \mathbf{P}; Q_\ell) = 0$, $\ell = 1, 2, \cdots, M$. When $M = N$ we have a linear system of N equations in N unknowns whereas when $M > N$, this yields a linear least-squares problem.

In the case of moving singularities, the $4N$ unknowns, comprising the coefficients $\mathbf{c}$ and the locations of the singularities $\mathbf{P}$, are determined by minimizing the functional $F(\mathbf{c}, \mathbf{P}) = \sum_{\ell=1}^M |Bu_N(\mathbf{c}, \mathbf{P}; Q_\ell)|^2$, which is nonlinear in the coordinates of the P_j. The minimization of this functional is done using readily available nonlinear least squares software, such as the MINPACK routines LMDIF and LMDER [16], the Harwell subroutine VA07AD [21], and the NAG routine E04UPF [36]. The relative merits of these codes are examined in [25] and [40]. The constrained optimization features of E04UPF are particularly useful for ensuring that the singularities remain outside the region.

The initial placement of the singularities can be extremely important in the convergence of a least squares routine. Usually the singularities are distributed uniformly around the domain of the problem at a fixed distance from the boundary. More details about the available least squares routines as well as the various algorithmic MFS features developed until the mid 90's may be found in [13]. More recently, the optional placement of the singularities, in problems with boundary singularities, via a simulated annealing algorithm has been the subject of studies by Cisilino and Sensale [11]. This simulated annealing algorithm involves an iterative random search with adaptive moves which enable one to avoid local minima. Saavedra and Power [45] introduce an adaptive refinement MFS algorithm in the case the singularities are fixed and their number is less than the number of boundary points. This leads to a least squares problem and in this algorithm the distribution of singularities is selectively improved. This is done by taking into account the intensities of the fundamental solutions in the MFS expansion and using them as parameters in a multiple linear regression model.

Applications

It is unclear who first used the MFS with fixed singularities; see the references in [9, 13]. The MFS with moving singularities was first proposed by Mathon and Johnston [35]. A survey of the MFS and related methods for the numerical solution of elliptic boundary value problems is presented in [13]. Also, material on the MFS may be found in the books by Golberg and Chen [17] and Kolodziej [31]. Since the approximate solution in the MFS automatically satisfies the differential equation in question, the method may also be viewed as a Trefftz method. A survey of such methods can be found in [30].

Lately, much attention has been devoted to the application of the MFS to diffusion problems [18, 19, 39], especially in conjunction with the dual reciprocity and radial basis function methods for treating the inhomogeneous terms [33, 44]. The MFS combined with radial basis functions has also been used recently by in [12] and [2, 4], for the solution of linear and nonlinear Poisson problems, respectively, whereas the MFS in conjunction with compactly supported radial basis functions has been used in [6] for the solution of Poisson problems and in [20] for the solution of three-dimensional Helmholtz type-problems. The integrations involved in the evaluation of the particular solutions in Poisson problems were treated by a quasi-Monte Carlo method in [7]. More recently, the MFS has been used for the solution of inhomogeneous problems in combination with the fundamental solutions of the modified Helmholtz equation instead of radial basis functions by Alves, Chen and Saler [1]. A comparison of the performance of the MFS combined with radial basis functions and another meshless method, called Kansa's method, is carried out in [34].

The MFS has also been used in [9, 32] to investigate the dependence of the accuracy of the solution on the position of the auxiliary boundary and the number of boundary points. In [3], the MFS was applied to singular problems governed by the modified Helmholtz equation and in [24] for the calculation of the eigenvalues of the Helmholtz equation. Cisilino and Pardo [10] used an MFS-type approximation with a functional integral method. This approach introduces a regularization parameter which can be adjusted to reduce the error. In [29], the MFS is used for the solution of anisotropic problems in elasticity. Rajamohan and Raamachandran [43], also use the MFS for the solution of anisotropic thin-plate bending problems. Fenner uses the MFS for linear isotropic elasticity problems in [15]. In the same paper, a domain decomposition technique is discussed. The MFS with moving singularities has been used recently for the solution of three-dimensional Signorini problems [41], three-dimensional elasticity problems [42] and anisotropic single material and bimaterial problems in combination with a domain decomposition technique [5].

In recent years, the MFS has also been widely used for problems in electrostatics. In particular, Ismail and Abu-Gammaz [22] apply the MFS to calculate the electric field resulting from high voltage transmission systems and Vlad et al [47] apply the MFS to calculate the electric field in plate-type electrostatic separators. In [37], Nishimura et al., investigate the positioning of the singularities via a genetic algorithm, when the MFS is applied to elastostatics problems. The same authors study the positioning of both the singularities and the boundary points in the MFS for axisymmetric elastostatics problems in [38] using the same technique. Recently, the MFS was also applied to three-dimensional shape recognition problems by Kanali, Murase and Honami [23].

When the MFS with uniformly distributed (fixed) singularities and boundary points is applied to certain problems in circular domains it leads to circulant system matrices or system matrices which may be decomposed into circulant submatrices. Ways of exploiting the properties of such systems for the efficient implementation of the MFS are investigated in [46].

Because of their advantages over domain discretization methods for the solution of scattering and radiation problems in acoustics and elastodynamics, various MFS-type formulations have been suggested for such problems. A comprehensive survey of these can be found in [14].

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