

MULTIPLE SCATTERING OF SURFACE WATER WAVES AND THE NULL-FIELD METHOD

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ABSTRACT

Two rigid cylinders of infinite length are floating in the free surface of deep water with their generators parallel. The cylinders are held fixed and a given time-harmonic wave of small amplitude is incident upon them. The corresponding linear two-dimensional boundary-value problem for a velocity potential is treated using the null-field method. This method is exact. In the resulting equations, the effects of scattering by each cylinder in isolation (via the so-called 'T-matrix' for each cylinder), and of the spacing between the cylinders, are clearly separated; computationally, this is a very desirable feature. The method is used in two ways: first, it is shown that the 'wide-spacing' approximation is recovered when the exact equations are solved in an appropriate asymptotic limit. Second, the exact equations are truncated, and numerical solutions obtained for the model problem of scattering of regular surface waves by a pair of identical half-immersed circular cylinders (i.e. a catamaran). Comparisons with the wide-spacing approximation are also given. Extensions to three dimensions and to water of constant finite depth are mentioned.

1. INTRODUCTION

In recent years, there has been much interest in multiple-scattering problems, in which surface water waves interact with two (or more) partially-immersed rigid bodies. Such problems arise when studying, e.g., the interaction between neighbouring ships, wave-power devices, or elements of a single larger structure. The corresponding linear boundary-value problem (for time-harmonic waves) is easily formulated and can be solved by integral-equation methods, but this direct approach can be computationally expensive, especially for problems involving several three-dimensional bodies. Thus, Ohkusu (1975) wrote: 'For the purpose of calculating hydrodynamic forces..., it is essential that only the hydrodynamic properties of each element be given. A method having such a merit will facilitate the calculation for a body having many elements and may be applied to the

design arrangement of the elements'. This philosophy has led to various approximate techniques for treating multiple-scattering problems, e.g. Budal's theory of 'point absorbers' (1977), Simon's 'plane-wave' approximation (1982), and Ohkusu's 'wide-spacing' approximation (1970, 1975). In the present paper, we shall describe another method that only uses solutions of single-body problems, but which is exact. This work uses the null-field method, and is an extension of the author's work on single-body problems (Martin, 1981, 1984a) and of some corresponding work in acoustics by Peterson and Ström (1974).

The plan of the paper is as follows. After a literature survey in §2, we study (in §3) the two-dimensional problem of scattering by a single rigid cylinder which is floating in the free surface of deep water; this plane problem is labelled S_1 below. We give a precise formulation of S_1 , briefly review the use of integral-equation methods for its solution, and state the well-known Kreisel-Meyer relations (these relate the complex reflection and transmission coefficients for S_1). Next, we describe the null-field method and introduce the T-matrix for S_1 ; the construction of this matrix was first discussed by Waterman (1965) in the context of electromagnetic scattering problems, and the null-field/T-matrix method is now used widely in several branches of mathematical physics (see, e.g. Varadan and Varadan, 1980).

In §4, we consider a pair of floating cylinders. We formulate the corresponding boundary-value problem (labelled S_2 below), and note the absence of a uniqueness theorem for S_2 . We derive the null-field equations for S_2 ; these can be solved for the boundary values of the potential. Under certain geometrical restrictions, the null-field equations can be reduced to matrix equations, involving the matrices Q and \hat{Q} which arise when S_1 is solved for each cylinder in isolation (see §§3.2, 3.3); they also involve a matrix S which occurs in an addition theorem for Ursell's multipole potentials (S depends only on the spacing between the cylinders). If attention is focussed on the potential in the water, the matrix equations can be recast in terms of the matrix S and two T-matrices, one for each cylinder. We

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solve this system of equations in two ways: first, we solve them asymptotically, and show that Ohkusu's wide-spacing approximation is recovered in an appropriate limit. (The wide-spacing approximation is a heuristic approach for solving S_1 , in which only wave-like interactions between the cylinders are taken into account.) Second, we solve them numerically for the model problem of scattering of regular surface waves by a pair of identical half-immersed circular cylinders, and compare our results with those obtained using the wide-spacing approximation. We conclude with a discussion of some possible extensions and generalisations of the null-field method.

2. LITERATURE SURVEY

The literature on scattering of water waves by two, or more, rigid bodies is quite extensive, but does not seem to have been surveyed previously. Here, we shall restrict ourselves to two-dimensional interactions between a pair of cylinders. (For an interesting account of the history of twin-hull ships, see Corlett, 1969.)

2.1 The Method of Multipoles

Apart from some work on scattering by two thin vertical barriers (see Evans and Morris (1972) for references), the first problem to be studied extensively was the radiation problem for two half-immersed circular cylinders. Thus, Ohkusu (1969) and Wang and Wahab (1971) extended Ursell's multipole method (1949) for one cylinder to analyse the heaving motion of a catamaran, consisting of two identical, rigidly-connected, half-immersed circular cylinders (we call this 'semicircle-catamaran problem'). For this symmetrical problem, the velocity potential at a point P in the fluid can be represented as

$$\phi(P) = \sum_{n=1}^{\infty} a_n \{ \phi_n(r_p) + (-1)^n \phi_n(r'_p) \},$$

where ϕ_n are the multipole potentials defined in Appendix A. This representation satisfies all the conditions of the problem, except the boundary condition on the wetted surfaces of the cylinders; applying this condition allows the coefficients a_n to be determined. Ohkusu (1969) and Wang and Wahab (1971) computed the wave amplitude at infinity and the virtual-mass coefficient, and found good agreement with the corresponding values obtained from their experiments. Ohkusu (1970) has also made similar calculations for the swaying and rolling motions of the same catamaran, whilst Spencer and Sayer (1981) have analysed the motions of two freely-floating identical circular cylinders.

The multipole method has also been used to treat problems involving totally-submerged circular cylinders. Thus, Wang (1970, 1981) has presented extensive numerical results for two identical cylinders, each submerged to the same depth. More recently, O'Leary (1984) has shown how the method can be used for an arbitrary number of totally-submerged circular cylinders; each cylinder can have any radius and any depth

of submergence. Essentially, her method uses an addition theorem for the submerged multipole potentials, so that the boundary condition on each cylinder can be imposed; an addition theorem for ϕ_n has previously been given by Bencheikh (1982) and Martin (1984b); see §4.2. O'Leary also gave numerical results for several configurations of two and three identical cylinders.

2.2 Integral-equation Methods

Several authors have used integral-equation methods to treat multiple-scattering problems. Most of these authors represented ϕ as a distribution of simple wave sources over the wetted surfaces of the cylinders, and then solved the corresponding integral equation of the second kind for the unknown source density (see §4.1.). Thus, Nordenström et al. (1971), Kim (1972), Lee et al. (1973) and Katory et al. (1980) have all solved the semicircle-catamaran problem for deep water, whilst Chung and Coleman (1975) have solved it for water of constant finite depth. The agreement between these solutions, those obtained using the method of multipoles, and those determined by experiment is generally very good. Other geometries have also been investigated, e.g. two different rectangles (Kim, 1972; Katory et al., 1980), two triangles (Lee et al., 1973), and bulbous-form catamarans (Kim, 1972; Maeda, 1975).

Integral-equation methods have also been used for totally-submerged cylinders. Thus, Schnute (1971) has solved an equation for the total potential (see §4.1), for two circular cylinders; he reduces it to a system of algebraic equations, but does not give numerical results. Chakrabarti (1979) has solved an equation for the diffraction potential, and has presented numerical results for several configurations of two identical circular cylinders.

Integral equations can also be obtained by applying Green's theorem to ϕ and the simple logarithmic source potential. These equations have simple kernels but the range of integration includes the free surface, the bottom, and two vertical control surfaces at some distance from the floating cylinders. The radiation condition (6), or an approximation to it, is imposed on these vertical surfaces, and the bottom is included so as to obtain a finite range of integration. For details of the method, see Mei (1978) or Yeung (1982). This method has been used by Ho and Harten (1975) to analyse the motion of one or two rectangular cylinders oscillating near a vertical wall, and by Ijima et al. (1976) to compute the transmission coefficient for the semicircle-catamaran problem.

2.3 Other Methods

Leonard et al. (1983) have used a finite-element method to solve the semicircle-catamaran problem for water of constant finite depth, and the corresponding problem with freely-floating cylinders. It may be observed that their results for the catamaran are in good qualitative agreement with those of Chung and Coleman (1975).

Two approximate methods have been used to

treat two-dimensional multiple-scattering problems. Alker (1978) has used the method of matched asymptotic expansions to study the scattering of short waves by two partially-immersed cylinders that do not make plane vertical intersections with the free surface. He has shown, e.g., that for a symmetric pair of cylinders, there is an infinite number of frequencies at which there is no reflected wave.

Secondly, Ohkusu (1970) has used a 'wide-spacing' approximation, in which only wave-like interactions between the cylinders are taken into account. This leads to an approximate solution to the multiple-scattering problem in terms of the solutions to various single-cylinder problems. An alternative treatment has been given by Srokosz and Evans (1979); see Appendix C. For the semicircle-catamaran problem, Ohkusu (1970) obtained good agreement with the exact solution (Ohkusu, 1969; Wang and Wahab, 1971), even when the assumption that the spacing between the cylinders is large compared to the wavelength is not valid. (This assumption is made precise in §4.4.) Other applications have been made by Ohkusu (1970, 1975, 1976), Ohkusu and Takaki (1971), Srokosz and Evans (1979) and Masubuchi and Shinomiya (1981).

3. SCATTERING BY ONE CYLINDER

Consider a rigid horizontal cylinder which is partially immersed in the free surface of a fluid. We suppose that the cylinder is fixed, and that the fluid is incompressible, inviscid and of infinite depth. We take Cartesian coordinates (x, y, z) , with the z -axis horizontal and the y -axis vertical (y increasing with depth) such that the free surface occupies a portion of the plane $y = 0$. For simplicity, we consider beam seas (waves with crests parallel to the z -axis) whence the fluid motion can be considered to be two-dimensional, i.e. independent of z . We assume that the fluid motion is irrotational, whence a velocity potential exists. If we further assume that the motion has a harmonic time-dependence (with radian frequency ω), then we can write the velocity potential as the real part of $\phi(P)e^{-i\omega t}$; henceforth, we shall suppress the factor $e^{-i\omega t}$. For waves of small amplitude, the total potential ϕ solves the following linear two-dimensional boundary-value problem:

Scattering boundary-value problem S_1 .

Determine a function $\phi(P)$, such that ϕ satisfies Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(P) = 0 \text{ in } D, \quad (1)$$

the free-surface condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \text{ on } F, \quad (2)$$

the boundary condition

$$\frac{\partial\phi(P)}{\partial n_p} = 0 \text{ on } \partial D, \quad (3)$$

and the condition that the fluid motion vanishes as $y \rightarrow \infty$,

$$|\text{grad } \phi| \rightarrow 0 \text{ as } y \rightarrow \infty. \quad (4)$$

In addition, if we define the diffraction potential ϕ_D by

$$\phi_D = \phi - \phi_I, \quad (5)$$

then ϕ_D must satisfy the radiation condition

$$\left(\frac{\partial}{\partial r_p} - iK\right)\phi_D \rightarrow 0 \text{ as } r_p \rightarrow \infty. \quad (6)$$

Here, we denote the fluid domain (in the xy -plane) by D , the mean free surface by F and the wetted surface of the cylinder by ∂D ; capital letters P, Q denote points of D ; lower-case letters p, q denote points of ∂D ; the origin O is assumed to lie in F_- , the portion of the line $y=0$ which is inside the body; D_- denotes the interior of the body, i.e. the region with boundary $\partial D \cup F_-$; r_p is the position vector of P with respect to O ; $r_p = |r_p|$; and $\partial/\partial n_p$ denotes normal differentiation at the point p , in the direction from ∂D into D . Also, $K = \omega^2/g$, where g is the acceleration due to gravity, and ϕ_I is the velocity potential of the given incident wave. ϕ_I satisfies (1) everywhere in $y > 0$ (except possibly at a finite number of isolated points in D) and (2) everywhere on $y = 0$. For example, ϕ_I may correspond to a radiating (line) source in D , or to a train of regular surface waves.

Let ∂D^* denote the union of ∂D and its mirror image in F . We say that ∂D has properties J if ∂D^* is convex and twice-differentiable. John (1950) has shown that if ∂D has properties J , then S_1 has precisely one solution. Actually, John's uniqueness theorem holds if

- (i) all vertical lines drawn down from F do not meet ∂D ; his existence theorem holds if, in addition to (i),
- (ii) ∂D is twice-differentiable and meets F at right angles.

More recently, Simon and Ursell (1984) have generalised John's uniqueness theorem to cover cases in which (i) is not satisfied, e.g. certain bulbous sections are now covered; if (ii) still holds, then integral-equation methods will yield a corresponding existence theorem.

In many applications, the incident potential ϕ_I corresponds to a train of regular surface waves; for such a wavetrain propagating from $x = +\infty$ towards the cylinder, we have

$$\phi_I \equiv \phi_I^+ = (gA_+/\omega)e^{-Ky - iKx}, \quad (7)$$

where $|A_+|$ is the amplitude of the wave. This wave will be partially reflected and partially transmitted; we define (complex) reflection and transmission coefficients, r_+ and t_+ respectively, by

$$\frac{\omega}{gA_+} \phi(P) \sim \begin{cases} e^{-Ky}(e^{-iKx} + r_+ e^{iKx}) & \text{as } x \rightarrow +\infty, \\ t_+ e^{-Ky - iKx} & \text{as } x \rightarrow -\infty. \end{cases} \quad (8)$$

Similarly, for a wave propagating from $x = -\infty$, we have

$$\phi_I \equiv \phi_I^- = (gA_-/\omega)e^{-Ky+iKx} \quad (9)$$

and

$$\frac{\omega}{gA_-} \phi(P) \sim \begin{cases} t_- e^{-Ky+iKx} & \text{as } x \rightarrow +\infty, \\ e^{-Ky}(e^{iKx} + r_- e^{-iKx}) & \text{as } x \rightarrow -\infty. \end{cases} \quad (10)$$

It is well known that r_{\pm} and t_{\pm} satisfy the following relations:

$$t_+ = t_- = t, \text{ say}; \quad (11)$$

$$|r_+| = |r_-| = |r|, \text{ say}; \quad (12)$$

$$|r|^2 + |t|^2 = 1; \quad (13)$$

and

$$tr_+^* + t^*r_- = 0, \quad (14)$$

where the asterisk denotes the complex conjugate. Using (12), (14) can be rewritten as

$$2 \arg(t) - \arg(r_+) - \arg(r_-) = \pi \text{ modulo } 2\pi. \quad (15)$$

Collectively, we shall call (11), (12), (13) and (14) the Kreisel-Meyer relations; they are derived systematically by Newman (1977) for water of constant finite depth. Kreisel (1949) obtained (12) and (13), and their generalisations to the situation where the asymptotic depths of water are different at $x = \pm\infty$. R. Meyer, in an appendix to a paper by Biesel and Le Méhauté (1955), proved (11), (12), (13) and (14).

3.1 Integral-equation Methods

Typically, S_1 is solved by integral-equation methods; for a summary, see, e.g. Martin (1981), Mei (1978) or Yeung (1982). To derive an integral equation, we need a fundamental solution:

$$G(P,Q) \equiv G(x,y;\xi,\eta) = \frac{1}{2} \log \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} - 2 \int_0^{\infty} e^{-k(y-\eta)} \cos k(x-\xi) \frac{dk}{k-K}. \quad (16)$$

G satisfies (1) (except at $P = Q$), (2), (4) and (6); it is the potential at P due to a wave source at Q (in the absence of the body).

If we apply Green's theorem twice, once in D to ϕ_D and G , and once in D_+ to ϕ_I and G , and add the resulting equations, we obtain (cf. Martin, 1982)

$$2\pi\phi_D(P) = - \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G(P,q) ds_q, \quad P \in D, \quad (17)$$

$$\pi\phi(P) + \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G(p,q) ds_q = 2\pi\phi_I(p), \quad p \in \partial D \quad (18)$$

and

$$2\pi\phi_I(P) = \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} G(P,q) ds_q, \quad P \in D_+. \quad (19)$$

(18) is a Fredholm integral equation of the second kind for $\phi(q)$; it is uniquely solvable except at a certain discrete set of values of the wavenumber K , called the irregular values. Irregular values are not physical (S_1 is uniquely solvable for all values of K), but are a consequence of the method of solution. They can be removed in several ways; see, e.g. Ursell (1981).

(17) is an integral representation for $\phi_D(P)$ as a distribution of wave dipoles over ∂D , whilst (19) is an integral relation satisfied by ϕ_I at all points in D_+ . We shall use (19) as the basis of our derivation of the system of null-field equations. We shall also require the following bilinear expansion of G (Ursell, 1981):

$$G(P,Q) = \sum_{m=1}^{\infty} \alpha_m(r_p) \phi_m(r_q), \quad r_p < r_q. \quad (20)$$

where the functions α_m and ϕ_m are defined in Appendix A; each is harmonic and satisfies the free-surface condition (2); α_m are regular, whilst the multipole potentials $\phi_m(r_p)$ are singular at 0 and satisfy the radiation condition (6). Henceforth, we shall use a summation convention: sum over repeated suffices from 1 to ∞ . Thus (20) becomes

$$G(P,Q) = \alpha_m(r_p) \phi_m(r_q), \quad r_p < r_q.$$

3.2 The Null-field Method

Let C_- be the inscribed semicircle to ∂D , centered on 0. If we restrict P (in D_+) to lie inside C_- , we can substitute (20) into (19) giving

$$2\pi\phi_I(P) = d_m \alpha_m(r_p), \quad (21)$$

where

$$d_m = \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} \phi_m(r_q) ds_q \quad (m=1,2,\dots). \quad (22)$$

The constants d_m are known; they are given in terms of ϕ_I by

$$\int_{\partial D} \{ \phi_I(q) \frac{\partial}{\partial n_q} \phi_m(r_q) - \phi_m(r_q) \frac{\partial}{\partial n_q} \phi_I(q) \} ds_q. \quad (23)$$

For example, if $\phi_I = \phi_I^+$, comparison of (7), (21) and (A.1) shows that

$$d_1 = -id_2 = i\pi(gA_+/\omega), \quad d_m = 0 \text{ for } m > 2. \quad (24)$$

(22) are called the null-field equations; they form an infinite system of moment-like equations from which $\phi(q)$ is to be determined. It is known that the null-field equations are uniquely solvable for all values of K —irregular values do not occur with the null-field

method (Martin, 1981).

To solve the null-field equations, we begin by choosing a basis for representing functions defined on ∂D ; let $\{\phi_n(q)\}$ ($n=1,2,\dots$) be such a basis. Thus, we may write

$$\phi(q) = \sum_n a_n \phi_n(q) \quad (25)$$

where a_n ($n=1,2,\dots$) are unknown coefficients. Substituting (25) into (22) gives

$$Q_{mn} a_n = d_m \quad (m=1,2,\dots), \quad (26)$$

where

$$Q_{mn} = \int_{\partial D} \phi_n(q) \frac{\partial}{\partial n_q} \phi_m(r_q) ds_q. \quad (27)$$

If we truncate the system (26), we obtain a numerical method for solving the null-field equations; this method has been used to solve the radiation problem associated with the heaving oscillations of a half-immersed elliptic cylinder (Martin, 1981). A different method for solving (26) has been used by Martin (1984a); this numerical method is only applicable when $\phi_I = \phi_I^+$ or $\phi_I = \phi_I^-$.

3.3 The T-matrix

Once $\phi(q)$ has been found by solving the null-field equations, $\phi_D(P)$ is given by (17). Let C_+ be the escribed semicircle to ∂D , centred on O . If we restrict P (in D) to lie outside C_+ , we can substitute (20) into (17) giving

$$2\pi\phi_D(P) = \sum_m c_m \phi_m(r_p) \quad (28)$$

where

$$c_m = - \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} \alpha_m(r_q) ds_q \quad (m=1,2,\dots). \quad (29)$$

Substituting (25) into (29), we obtain

$$c_m = - \hat{Q}_{mn} a_n \quad (m=1,2,\dots) \quad (30)$$

where

$$\hat{Q}_{mn} = \int_{\partial D} \phi_n(q) \frac{\partial}{\partial n_q} \alpha_m(r_q) ds_q. \quad (31)$$

Does the series on the right-hand side of (28) represent $\phi_D(P)$ (where $\phi_D + \phi_I$ solves S_1) throughout D (and not just outside C_+)? The assumption that it does is called the Rayleigh hypothesis in acoustic scattering. The Rayleigh hypothesis does not seem to have been investigated in the context of water-wave problems; however, we can expect that whether it is correct or not will depend on the shape of ∂D ; cf. Millar (1973), van den Berg and Fokkema (1979).

Now, the system (26) is uniquely solvable, i.e. Q^{-1} , the inverse of the infinite matrix Q , exists. Thus, eliminating a_n between (26) and (30), we obtain

$$c_m = T_{mn} d_n \quad (m=1,2,\dots) \quad (32)$$

where

$$T_{mn} = - \hat{Q}_{m\ell} Q_{\ell n}^{-1} \quad (33)$$

is known as the T-matrix. Given T , we can determine the diffraction potential, ϕ_D , outside C_+ for any given incident potential, ϕ_I , without computing the values of ϕ on ∂D .

Let us now forget about the derivation of the T-matrix, and merely assume that the coefficients c_m and d_m occurring in the representations (21) and (28), respectively, are related through a matrix T_{mn} by (32). What properties does such a matrix have? To begin with, the unique-solvability of S_1 implies that T exists and is unique. This, in turn, implies that T is independent of the choice of basis $\{\phi_n\}$, although this choice may be important in numerical calculations when T must necessarily be truncated.

Secondly, reciprocity considerations show that T is symmetric,

$$T_{mn} = T_{nm}. \quad (34a)$$

Thirdly, energy considerations show that T satisfies

$$\frac{2}{\pi} \text{Im}(T_{mn}) + T_{1m} T_{1n}^* + T_{2m} T_{2n}^* = 0. \quad (34b)$$

(34) are derived by Martin (1984c). They are useful because they provide independent checks on numerical calculations. Also, by choosing particular values for m and n , we can recover all of the Kreisel-Meyer relations, e.g., $T_{12} = T_{21}$ implies that $t_+ = t_-$; in fact, we have (see Martin, 1984c)

$$\pi T_{11} = 1 - t + \frac{1}{2}(r_- + r_+), \quad (35a)$$

$$\pi T_{22} = 1 - t - \frac{1}{2}(r_- + r_+), \quad (35b)$$

and

$$\pi T_{12} = \pi T_{21} = \frac{1}{2}(r_- - r_+). \quad (35c)$$

4. SCATTERING BY TWO CYLINDERS

Suppose, now, that a second rigid cylinder is partially immersed in the free surface, with its generators parallel to those of the first cylinder. As before, we denote the fluid domain (in the xy -plane) by D and the mean free surface by F . We distinguish all quantities pertaining to the second cylinder by a prime, e.g. $\partial D'$ denotes its wetted surface. Let $\partial D'' = \partial D \cup \partial D'$; lower-case letters p, q denote points of $\partial D''$. The analogue of S_1 for two fixed cylinders is the following problem: Scattering boundary-value problem S_2 .

Determine a function $\phi(P)$, such that ϕ satisfies Laplace's equation (1) in D , the free-surface condition (2) on F , the boundary condition

$$\frac{\partial \phi(P)}{\partial n_p} = 0 \quad \text{on } \partial D'', \quad (36)$$

and the condition (4) as $y \rightarrow \infty$. In addition, $\phi_D = \phi - \phi_I$ must satisfy the radiation condition (6).

In order to make some progress with the analysis of S_2 , we make the following Uniqueness assumption. S_2 has at most one solution, i.e. the only solution of the homogeneous problem ($\phi_I = 0$) is the trivial solution, $\phi \equiv 0$.

To the author's knowledge, this result has not been proved. John's proof (1950) for one cylinder does not seem to extend to two cylinders. A proof may also impose some restrictions on the geometry: here, we shall suppose that ∂D and $\partial D'$ each have properties J.

Although we do not have a uniqueness theorem for S_2 , uniqueness can be proved for some other configurations. Thus, John's proof succeeds for two (or more) floating three-dimensional bodies (each having a wetted surface which is bounded and has properties J); the essential difference between this problem and S_2 is the connectivity of the free surface (note that John's proof also fails for a single floating torus). We also have three results for totally-submerged bodies: Schnute (1971) has proved uniqueness for a pair of widely-spaced circular cylinders, and theorems due to Maz'ja (see Hulme, 1984) and Simon and Ursell (1984) guarantee uniqueness for any two (or more) cylinders, subject to certain geometrical restrictions.

Before describing some methods for solving S_2 , we introduce some more notation. Let F'_- denote the portion of the line $y = 0$ inside the second cylinder, and let D'_- denote the interior region bounded by F'_- and $\partial D'$. Choose a second origin $O' \in F'_-$ and let r'_p denote the position vector of P with respect to O' (see Fig.1). $\phi_m(r'_p)$ denotes the m -th multiple potential centered on (i.e. singular at) O' ; $\alpha_m(r'_p)$ is defined similarly.

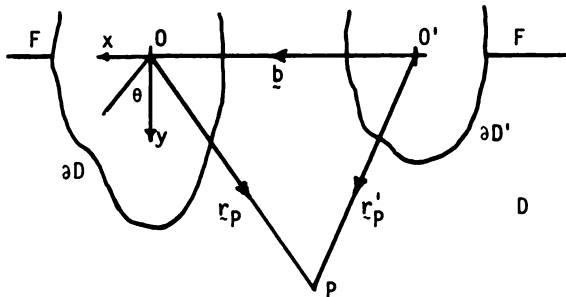


Figure 1. The two cylinders.

4.1 Integral-equation Methods

Integral equations can be, and have been, used to solve S_2 (see §2.2); one example is equation (18), with ∂D replaced by $\partial D''$ (this is the equation solved by Schnute, 1971); another is

$$\pi\mu(P) + \int_{\partial D''} \mu(q) \frac{\partial}{\partial n_p} G(p,q) ds_q = - \frac{\partial}{\partial n_p} \phi_I(P), \quad (37)$$

which is obtained by supposing that $\phi_D(P)$ can be represented as a distribution of sources over $\partial D''$,

$$\phi_D(P) = \int_{\partial D''} \mu(q) G(P,q) ds_q. \quad (38)$$

However, if the uniqueness assumption is correct, then these equations will suffer from difficulties at irregular values of K , although no such difficulties have been reported in the literature. Irregular values can be eliminated by using a different source function (Martin, 1984b), but this does not seem to be the most efficient way of solving S_2 . In fact, we shall not give further consideration to integral-equation methods.

4.2 The Null-field Method

If we apply Green's theorem thrice, once in D to ϕ_D and G , once in D_- to ϕ_I and G , and once in D'_- to ϕ_I and G , and add the resulting equations, we obtain

$$2\pi\phi_D(P) = - \int_{\partial D''} \phi(q) \frac{\partial}{\partial n_q} G(P,q) ds_q, \quad P \in D, \quad (39)$$

and

$$2\pi\phi_I(P) = \int_{\partial D''} \phi(q) \frac{\partial}{\partial n_q} G(P,q) ds_q, \quad P \in D_- \cup D'_-. \quad (40)$$

Consider (40) for $P \in D_-$. Following the derivation in §3.2, we restrict P to lie inside C_- ; we observe that for all points $q \in \partial D''$, $r_q > r_p$ and so we can use (20); and we use the representation (21) for ϕ_I to obtain

$$d_m = \int_{\partial D''} \phi(q) \frac{\partial}{\partial n_q} \phi_m(r'_q) ds_q \quad (m=1,2,\dots) \quad (41a)$$

Similarly, let C'_- be the inscribed semicircle to $\partial D'$ centred on O' . If we now consider (40) for $P \in D'_-$ and restrict P to lie inside C'_- , we obtain

$$d'_m = \int_{\partial D''} \phi(q) \frac{\partial}{\partial n_q} \phi_m(r'_q) ds_q \quad (m=1,2,\dots), \quad (41b)$$

where d'_m ($m=1,2,\dots$) are the coefficients in a second representation for ϕ_I ,

$$2\pi\phi_I(P) = d'_m \alpha_m(r'_p). \quad (42)$$

The coefficients d_m and d'_m ($m=1,2,\dots$) are assumed to be known; moreover, they are related, since (21) and (42) are two representations of the same incident potential (see, e.g. (68)).

(41) are the null-field equations for S_2 . It can be proved that if the uniqueness assumption is correct, then the null-field equations are uniquely solvable for all real values of K . To solve the null-field equations, we represent $\phi(q)$ on ∂D by (25), and on $\partial D'$ by

$$\phi(q) = a'_n \phi'_n(q), \quad (43)$$

where $\{\phi'_n(q)\}$ is a second set of basis functions and $a'_n (n=1,2,\dots)$ are unknown coefficients. Substituting (25) and (43) into (41a), we obtain

$$d'_m = Q_{mn} a'_n + a'_n \int_{\partial D'} \phi'_n(q) \frac{\partial}{\partial n_q} \phi_m(r_q) ds_q \quad (m=1,2,\dots), \quad (44)$$

where Q_{mn} is given by (27). The integral in (44) is over $\partial D'$. We would like to express it in terms of known integrals over $\partial D'$. To do this, we must replace $\phi_m(r_q)$ by functions centred on O' ; since $\phi_m(r_q)$ is regular near O' , it has an expansion in terms of $\alpha_m(r'_q)$. In fact, we have the following Addition theorem.

$$\phi_m(r_p) = S_{mn}(-b) \alpha_n(r'_p), \quad (45)$$

where $r'_p = r_p + b$ and $r'_p < b = |b|$. The matrix S is defined in Appendix B; it satisfies

$$S_{mn}(-b) = S_{nm}(b). \quad (46)$$

This theorem can be proved either by introducing complex variables (Bencheikh, 1982) or by using integral representations; both proofs are sketched by Martin (1984b).

Let C'_+ be the escribed semicircle to $\partial D'$, centred on O' . If O lies outside C'_+ , then we can substitute (45) into (44) giving

$$d'_m = Q_{mn} a'_n + S_{mk}(-b) \hat{Q}'_{kn} a'_n \quad (m=1,2,\dots) \quad (47a)$$

where

$$\hat{Q}'_{mn} = \int_{\partial D'} \phi'_n(q) \frac{\partial}{\partial n_q} \alpha_m(r'_q) ds_q. \quad (48)$$

Similarly, if O' lies outside C_+ , we can reduce (41b) to

$$d'_m = S_{mk}(b) \hat{Q}'_{kn} a'_n + Q'_{mn} a'_n \quad (m=1,2,\dots) \quad (47b)$$

where

$$Q'_{mn} = \int_{\partial D'} \phi'_n(q) \frac{\partial}{\partial n_q} \phi_m(r'_q) ds_q. \quad (49)$$

The coupled system (47) is to be solved for a'_n and $a'_n (n=1,2,\dots)$. Truncating this system leads to a numerical method for solving the null-field equations. The corresponding acoustic problem (scattering by a pair of sound-hard cylinders) has been solved in this way by Bates and Wall (1977). Note that the matrices Q and \hat{Q} (Q' and \hat{Q}') occur in the solution of S_1 for the first (second) cylinder in isolation.

4.3 The Potential in D

Once $\phi(q)$ has been found by solving the null-field equations, $\phi_D(P)$ is given by (39). If we restrict $P \in D$ to lie outside C_+ and C'_+ , we can substitute (20) into (39) giving

$$2\pi\phi_D(P) = c_m \phi_m(r_p) + c'_m \phi'_m(r'_p) \quad (50)$$

where

$$c_m = - \int_{\partial D} \phi(q) \frac{\partial}{\partial n_q} \alpha_m(r_q) ds_q \quad (51)$$

$$= - \hat{Q}_{mn} a_n = T_{mk} Q_{kn} a_n, \quad (52)$$

$$c'_m = - \int_{\partial D'} \phi(q) \frac{\partial}{\partial n_q} \alpha_m(r'_q) ds_q \quad (53)$$

$$= - \hat{Q}'_{mn} a'_n = T'_{mk} Q'_{kn} a'_n, \quad (54)$$

and T' is the T -matrix for the second cylinder. Henceforth, we shall concentrate on finding c_m and c'_m (rather than a_m and a'_m).

Multiply (47a) by $T_{\ell m}$ and sum over m to give

$$c_\ell - X_{\ell k} c'_k = t_\ell \quad (\ell=1,2,\dots), \quad (55)$$

where

$$X_{mn} = T_{mk} S_{kn}(-b) = S_{nk}(b) T_{km} \quad (56)$$

and

$$t_m = T_{mn} d'_n. \quad (57)$$

Similarly, from (47b), we obtain

$$c'_\ell - X'_{\ell k} c_k = t'_\ell \quad (\ell=1,2,\dots), \quad (58)$$

where

$$X'_{mn} = T'_{mk} S_{kn}(b) = S_{nk}(-b) T'_{km} \quad (59)$$

and

$$t'_m = T'_{mn} d'_n. \quad (60)$$

Eliminating c'_m between (55) and (58) gives

$$c_m - K_{mn} c_n = f_m \quad (m=1,2,\dots) \quad (61a)$$

where

$$K_{mn} = X_{mk} X'_{kn} \quad \text{and} \quad f_m = t_m + X_{mn} t'_n. \quad (62)$$

(61a) is an infinite system of linear algebraic equations from which $c_m (m=1,2,\dots)$ can be determined. Similarly, c'_m can be determined from

$$c'_m - X'_{mk} X_{kn} c'_n = t'_m + X'_{mn} t_n \quad (m=1,2,\dots), \quad (61b)$$

or by substituting the solution of (61a) into (58).

We remark that (61) are convenient for numerical computations. The effects of scattering by each cylinder in isolation (i.e. the matrices T and T'), and those of the spacing between the cylinders, are clearly separated: changing the spacing merely requires that S has to be recomputed. In §4.5, we solve S_2 for a semicircle-catamaran by solving (61), numerically.

In essence, the analysis of the present section follows that given by Peterson and Ström (1974) for the corresponding problem in acoustics. Similar equations have been obtained by Bencheikh (1982).

4.4 Asymptotic Solutions; Wide Spacings

(61) can also be solved asymptotically. Suppose that the second cylinder is absent: thus $T'_{mn} = 0$, whence $t'_m = 0$, $X'_{mn} = 0$, $c'_m = 0$ and $c_m = t_m = T_{mn}d_n$, in agreement with (32). This suggests that if the second cylinder is small (compared with the first cylinder and the wavelength), then approximations to c_m and c'_m could be found by substituting an appropriate approximation to T'_{mn} ; however, this will not be pursued here.

Different approximations to c_m and c'_m are obtained by assuming that the spacing between the cylinders is large, i.e. by substituting an appropriate approximation to S_{mn} into (61). Let a be a typical dimension of both cylinders, and set $\tau = a/b$ ($b = |b| =$ distance between 0 and 0'). We assume that

$$Kb \gg 1 \quad \text{and} \quad \tau \ll 1. \quad (63)$$

We have (see Appendix A)

$$\phi_1(b) = \pi e^{iKb} + O((Kb)^{-3})$$

and

$$\phi_2(b) = \pi i e^{iKb} + O((Kb)^{-2}),$$

as $Kb \rightarrow \infty$. Also,

$$a^{2m+1} \phi_{2m+1}(b) = \tau a^{2m} \phi_{2m+2}(b) = (-1)^m \tau^{2m+1}, \quad m \geq 1.$$

If we ignore all 'local' effects, i.e. we only include the wave terms in ϕ_1 and ϕ_2 , the matrix $S_{mn}(b)$ becomes very simple (see Appendix B):

$$S_{11} = S_{22} = iS_{12} = -iS_{21} = -i\lambda, \quad S_{mn} = 0 \quad m, n > 2,$$

where $\lambda = \frac{1}{2}\pi e^{iKb}$. Using these approximations, we obtain the following:

$$X_{m1} = -\lambda(iT_{m1} + T_{m2}), \quad X_{m2} = \lambda(T_{m1} - iT_{m2}),$$

$$X'_{m1} = -\lambda(iT'_{m1} - T'_{m2}), \quad X'_{m2} = -\lambda(T'_{m1} + iT'_{m2}),$$

$$X_{mn} = 0, \quad X'_{mn} = 0 \quad \text{for } n > 2,$$

$$K_{11} = iK_{12} = X_{12}(X'_{12} + X'_{21}),$$

$$K_{22} = -iK_{21} = X_{21}(X'_{12} + X'_{21}),$$

and $K_{mn} = 0$ for $n > 2$.

Substituting into (61a), we obtain

$$(1 - K_{11})c_1 - K_{12}c_2 = f_1 \quad (64a)$$

$$-K_{21}c_1 + (1 - K_{22})c_2 = f_2 \quad (64b)$$

$$c_m = K_{m1}c_1 + K_{m2}c_2 + f_m, \quad m > 2. \quad (65)$$

Thus, the infinite system (61a) reduces to a 2×2 system (64) for c_1 and c_2 , and a formula for c_m , $m > 2$. Solving (64) gives

$$\Delta c_1 = (1 - K_{22})f_1 + K_{12}f_2 \quad (66a)$$

$$\text{and } \Delta c_2 = (1 - K_{11})f_2 + K_{21}f_1, \quad (66b)$$

where

$$\begin{aligned} \Delta &= (1 - K_{11})(1 - K_{22}) - K_{12}K_{21} \\ &= 1 - (X_{12} + X_{21})(X'_{12} + X'_{21}) \\ &= 1 - r'_+ e^{2iKb}, \end{aligned} \quad (67)$$

r'_+ , t'_+ are the reflection and transmission coefficients for the second cylinder in isolation, and we have used (35).

Let us now consider a specific incident wave, namely, a regular wavetrain propagating from $x = +\infty$, with velocity potential ϕ'_+ given by (7); the coefficients d'_m are given by (24), whilst

$$d'_m = e^{iKb} d_m \quad (m = 1, 2, \dots). \quad (68)$$

Substituting into the right-hand sides of (66), we obtain

$$\mu(c_1 + ic_2) = r_+ - e^{2iKb} r'_+ (r_+ r_- + t(1-t)) \quad (69a)$$

and

$$\mu(c_1 - ic_2) = 1 - t - e^{2iKb} r'_+ r_-, \quad (69b)$$

where $\mu = \omega\Delta/(2gA_+)$. Similarly,

$$\mu(c'_1 + ic'_2) = e^{iKb} t r'_+ \quad (69c)$$

and

$$\mu(c'_1 - ic'_2) = e^{iKb} t(1-t'). \quad (69d)$$

We define reflection and transmission coefficients, R_+ and T_+ , respectively, by (8) with $r_+(t_+)$ replaced by $R_+(T_+)$; they are given by

$$R_+ = \frac{1}{2}(c_1 + ic_2 + e^{iKb}(c'_1 + ic'_2)) \quad (70a)$$

$$\text{and } T_+ = 1 - \frac{1}{2}(c_1 - ic_2 + e^{-iKb}(c'_1 - ic'_2)). \quad (70b)$$

Substituting our approximations (69) into (70), we obtain

$$\Delta R_+ = r_+ - e^{2iKb} r'_+ (r_+ r_- - t^2) \quad (71a)$$

$$\text{and } \Delta T_+ = t t'. \quad (71b)$$

Similarly, for a regular wavetrain propagating from $x = -\infty$, we can define reflection and transmission coefficients by (10); using (69), we find the approximations

$$\Delta R_- = e^{-2iKb} r'_- - r_- (r'_+ r'_- - (t')^2) \quad (71c)$$

$$\text{and } \Delta T_- = tt' \quad (71d)$$

(71) are precisely the formulae obtained using the 'wide-spacing' approximation (see Appendix C). Here, we can see precisely the assumptions that have been made, namely (63), in order to arrive at (71). The first assumption ($Kb \gg 1$) means that the wavelength is much shorter than the spacing between the cylinders. The second assumption ($\tau \ll 1$) means that each cylinder is small compared to the spacing, i.e. it is purely geometrical in nature.

Finally, we remark that it can be verified that the approximations (71) do indeed satisfy the Kreisel-Meyer relations (§3).

4.5 A Numerical Example

We consider the semicircle-catamaran problem, i.e. scattering by a pair of fixed half-immersed circular cylinders. Let each have radius a , and locate the two origins, 0 and $0'$, at their centres. For the incident wave, we chose a regular wavetrain propagating from $x = +\infty$, with velocity potential ϕ^+ given by (7); hence we non-dimensionalised ϕ^+ using gA_+/ ω . $q \in \partial D$ has coordinates $x = a \sin \theta$, $y = a \cos \theta$ with $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$. The T -matrix. The cylinders are identical, whence $T_{mn} = T'_{mn}$. We non-dimensionalised α_m and ϕ_m using powers of a . From symmetry considerations, we chose

$$\phi_{2m-1}(\theta) = \sin(2m-1)\theta \text{ and } \phi_{2m}(\theta) = \cos(2m-2)\theta \\ (m = 1, 2, \dots),$$

whence, e.g. $Q_{2m, 2n-1} = Q_{2m-1, 2n} = 0$. The non-zero elements of Q and \hat{Q} can be evaluated exactly (it is not necessary to use numerical quadrature in this simple example). Each infinite matrix was truncated to an $N \times N$ matrix (typically, $N = 24$) and an approximation to the T -matrix was computed. Two checks on this approximation were made: first, the relations (34) were verified; second, from (35), we have (by symmetry, $r_+ = r_- = r$, say)

$$t = 1 - \frac{1}{2}\pi i(T_{11} + T_{22}) \quad (72)$$

$$\text{and } r = \frac{1}{2}\pi i(T_{11} - T_{22}), \quad (73)$$

and these can be compared with other calculations, e.g. Martin and Dixon (1983) tabulate r and t for various values of Ka .

The S -matrix. The non-dimensionalisation of S is prescribed by that used for Q and \hat{Q} . The addition theorem was verified, using an $N \times N$ truncation for S . R_+ and T_+ . Approximations to c_m and c'_m ($m = 1, 2, \dots, N$) were computed by solving (61), and then R_+ and T_+ were computed from (70). It was verified that R_+ and T_+ satisfy the Kreisel-Meyer relations, which take the form

$$|R_+|^2 + |T_+|^2 = 1 \quad (74)$$

$$\text{and } \arg(R_+) - \arg(T_+) - Kb = \frac{1}{2}\pi \text{ modulo } \pi. \quad (75)$$

(Note that the geometry is symmetric about $x = -\frac{1}{2}b$, but the incident wave (7) is specified relative to 0 (cf. Srokosz and Evans, 1979); this accounts for the term Kb in (75). Thus, increasing b means that the position of the first (i.e. front) cylinder remains fixed relative to the incident wave, and the second (i.e. rear) cylinder recedes (towards $x = -\infty$.)

In Fig. 2, we plot $|R_+|$ as a function of Ka for $\tau = 0.25$ and $\tau = 0.4$; the corresponding gaps between the cylinders are $2a$ and $\frac{1}{2}a$, respectively. For comparison, we also give $|R_+|$ for a single cylinder ($\tau = 0$). It is interesting to compare the curve for $\tau = 0.25$ with a corresponding curve for scattering by two thin vertical plates (each immersed to a depth a) obtained by Srokosz and Evans (1979; Fig. 3a). This latter curve is also for a gap (between the plates) of $2a$. The two curves are broadly similar, and both have a zero near $Ka = 0.5$.

In Fig. 3, we plot $|R_+|$ as a function of τ for $Ka = 0.2$ and $Ka = 0.5$. In Table 1, we give values of $|R_+|$ and $\arg(R_+)$ for $0.1 \leq Ka \leq 1.0$ and $0 \leq \tau \leq 0.5$ (when $\tau = 0.5$, the cylinders are touching); for each pair (Ka , τ), the upper number is obtained using the present method (with $N = 24$) and the lower number is obtained using the wide-spacing approximation, (71a) (leading characters are replaced by asterisks when they agree with the upper number). The agreement is seen to be good, with the largest error ($\approx 10\%$) occurring when $Ka = 0.2$, $\tau = 0.5$.

From the wide-spacing approximation (71a), we deduce that $R_+ = 0$ whenever

$$Kb + \arg(r) = 0 \text{ modulo } \pi, \quad (76)$$

where $\arg(r)$ is a function of Ka . (We have $\Delta T_+ = (1 - |r|^2) \exp(2i \arg(t))$; when (76) holds, $\Delta = 1 - |r|^2$, whence $|T_+| = 1$ and $R_+ = 0$.) In fact, the approximation (71a) has an infinite number of zeros, both as a function of τ for fixed Ka , and as a function of Ka for fixed τ . In particular, when $Ka = 0.2$ (from the last column of Table 1, $\arg(r) = -1.768$), (71a) predicts that R_+ first vanishes at $\tau = 0.113$; this agrees well with the graph in Fig. 3. Similarly, when $Ka = 0.5$, (71a) predicts that the first two zeros of R_+ are at $\tau = 0.270$ and $\tau = 0.101$.

If τ is fixed, (76) gives an estimate for the zeros of R_+ as a function of Ka . Thus, when $\tau = 0.4$, we can use Table 1 to show that R_+ has a zero between $Ka = 0.9$ and $Ka = 1.0$; similarly, when $\tau = 0.25$, the first zero occurs between $Ka = 0.4$ and $Ka = 0.5$. Again, these observations accord with the calculations presented in Fig. 2.

The pressure on the front cylinder. In Fig. 4, we plot dimensionless forms of $|\phi(q)|$ for $q(\theta) \in \partial D$, with θ varying between 90° (front face, i.e. facing the incident wave) and -90° (rear face, i.e. facing the second cylinder). (The hydrodynamic pressure is proportional to the potential.) There are three curves, corresponding to (i) the present method, (ii) the wide-spacing approximation (given by (C.3)) and (iii) a single cylinder in isolation. We

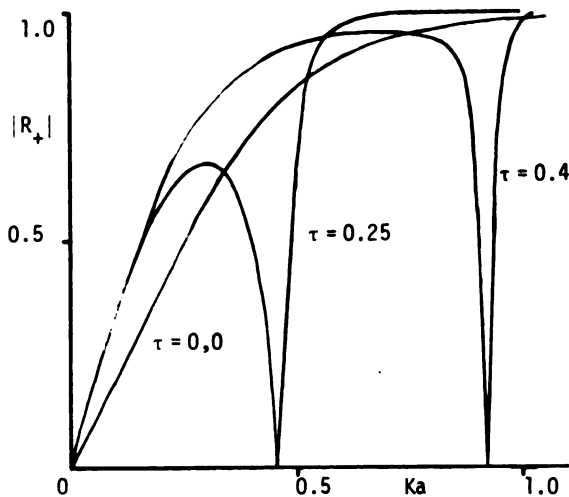


Figure 2. The reflection coefficient as a function of Ka , for $\tau=0.4$, $\tau=0.2$ and $\tau=0.0$.

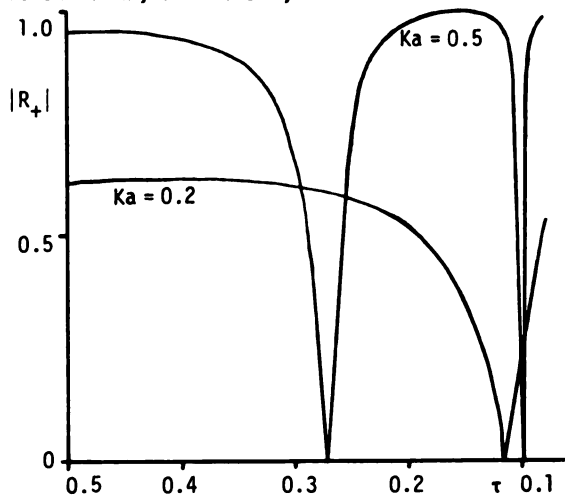


Figure 3. The reflection coefficient as a function of τ , for $Ka = 0.2$ and $Ka = 0.5$.

Table 1. The complex reflection coefficient; for each pair (Ka, τ) , the upper and lower numbers are obtained using the null-field method and the wide-spacing approximation, respectively. In the lower number, leading characters are replaced by asterisks when they agree with the upper number.

Ka	$\tau=0.5$		$\tau=0.4$		$\tau=0.3$		$\tau=0.2$		$\tau=0.1$		$\tau=0.0$	
	$ R_+ $	arg R_+	$ R_+ $	arg R_+	$ R_+ $	arg R_+	$ R_+ $	arg R_+	$ R_+ $	arg R_+	$ R_+ $	arg R_+
0.1	0.3422	-1.639	0.3466	-1.593	0.3483	-1.517	0.3393	-1.363	0.2469	-0.880	0.1942	-1.708
	736	*50	***722	***604	***681	****26	***530	****70	***539	*****3	—	—
0.2	0.6177	-1.610	0.6210	-1.536	0.6076	-1.405	0.5230	-1.115	0.2178	-3.028	0.3958	-1.768
	768	*20	***673	****44	***419	****13	***461	****21	***11	*****5	—	—
0.3	0.7980	-1.591	0.7930	-1.505	0.7494	-1.332	0.4262	-0.763	0.8613	-1.978	0.5857	-1.791
	8560	***8	**8384	****11	***851	****40	***550	*****9	***577	*****3	—	—
0.4	0.8960	-1.619	0.8840	-1.529	0.7941	-1.281	0.5330	-2.818	0.9379	-1.608	0.7369	-1.812
	9392	**29	**9195	****38	**8294	****93	***166	*****1	***47	*****5	—	—
0.5	0.9455	-1.692	0.9294	-1.598	0.6796	-1.091	0.9648	-2.086	0.0035	+2.867	0.8403	-1.855
	742	***706	***559	***611	**7309	***120	*02	*****0	***196	****77	—	—
0.6	0.9704	-1.799	0.9513	-1.693	0.6269	-2.854	0.9952	-1.986	0.9918	-2.001	0.9046	-1.928
	888	***816	***719	***712	**5839	*71	****35	*****2	****22	*****2	—	—
0.7	0.9833	-1.928	0.9591	-1.797	0.9855	-2.220	0.9990	-2.036	0.9979	-2.011	0.9427	-2.028
	950	*47	***776	***824	****13	****12	****82	*****3	****81	*****2	—	—
0.8	0.9903	-2.073	0.9485	-1.869	0.9987	-2.217	0.9997	-2.141	0.9960	-2.071	0.9650	-2.150
	****77	****91	***726	****918	****74	*****2	*****3	****39	*****4	*****3	—	—
0.9	0.9941	-2.229	0.6858	-1.506	0.9999	-2.316	0.9999	-2.273	0.9987	-2.337	0.9783	-2.288
	****89	****47	**8649	***762	*****4	*****3	*****6	*****1	*****6	*****6	—	—
1.0	0.9963	-2.392	0.9820	-2.656	1.0000	-2.451	0.9999	-2.419	0.9999	-2.443	0.9862	-2.439
	****95	****409	***754	****61	0.9998	*****0	*****7	*****7	*****7	*****7	—	—

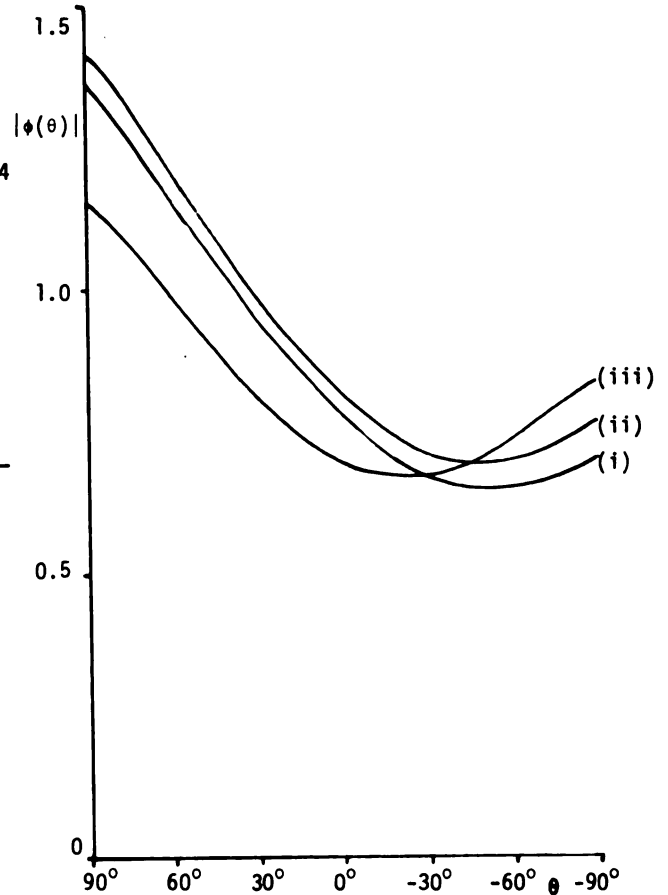


Figure 4. The normalized total potential on the front cylinder, for $Ka = 0.2$, $\tau = 0.45$. (i): the null-field method; (ii): the wide-spacing approximation; and (iii) a single cylinder in isolation.

chose $Ka = 0.2$ and $\tau = 0.45$, since, based on the results in Table 1, we expected significant discrepancies between (i) and (ii). In fact, we see that the wide-spacing approximation (ii) overestimates, with the largest error ($\approx 9\%$) occurring near $\theta = -90^\circ$, and the smallest ($\approx 4\%$) occurring near $\theta = 90^\circ$. The single-cylinder curve varies between an overestimate ($\approx 20\%$) near $\theta = -90^\circ$ to an underestimate ($\approx 15\%$) near $\theta = 90^\circ$.

If we fix Ka and reduce τ (whence Kb increases), the agreement between (i) and (ii) improves; e.g. with $Ka = 0.2$ and $\tau = 0.09$ ($Kb = 20/9$), the errors at $\pm 90^\circ$ are both less than 1%. However, if we fix τ and increase Ka (whence Kb also increases), then the agreement may or may not improve; e.g. with $Ka = 1.0$ and $\tau = 0.45$ (again, $Kb = 20/9$), the errors at -90° and $+90^\circ$ are 15% and 1%, respectively. These numerical observations verify that both assumptions (63) are important.

Finally, we mention Ohkusu's comparison (1970, 1975) between his exact solution (1969) and the wide-spacing approximation for the forced motions of a semicircle-catamaran. In the present notation, he found that they agreed 'almost completely... at least for $\tau \leq \frac{1}{3}$ unless Ka is very small'; this conclusion conforms with our analysis in §4.4.

5. DISCUSSION AND CONCLUSIONS

In this paper, we have used the null-field method to solve a multiple-scattering problem, namely, the interaction of water waves with two long horizontal cylinders, floating in the free surface of deep water. The method is exact; it yields a viable numerical approach for solving such problems; and it meets the requirements of the philosophy put forward in §1. Of course, it is not as simple as a method based on the wide-spacing approximation, although it could be useful when that approximation is not valid (see §4.4).

We conclude by noting that the null-field method could be modified in several ways:

- (i) It should be possible to extend the method to water of constant finite depth, and to three dimensions (see Martin, 1981), although the corresponding addition theorems are not yet available.
- (ii) The geometrical restriction required to reduce (41) to (47) would be altered if the multipole potentials ϕ_m were replaced by a different set, pertinent to elliptic coordinates, say (cf. Bates and Wall, 1977).
- (iii) The number of cylinders could be increased; see, e.g. Peterson and Ström (1974; §3) and Bates and Wall (1977; §7).

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APPENDIX A. Multipole Potentials

From Ursell (1981), we have (20) with

$$\begin{aligned} \phi_2(r_p) &= \int_0^\infty e^{-ky} \cos kx \frac{dk}{k-K}, \quad \phi_1(r_p) = \frac{-1}{K} \frac{\partial}{\partial x} \phi_2, \\ \phi_{2m+2}(r_p) &= \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1} \frac{\cos(2m-1)\theta}{r^{2m-1}}, \\ \phi_{2m+1}(r_p) &= \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m} \frac{\sin 2m\theta}{r^{2m}}, \\ \alpha_2(r_p) &= -2e^{-Ky} \cos Kx, \quad \alpha_1(r_p) = -2e^{-Ky} \sin Kx, \quad (A.1) \\ \alpha_{2m+2}(r_p) &= \frac{-2(2m-1)!}{K^{2m}} \sum_{q=2m}^\infty \frac{(-Kr)^q}{q!} \cos q\theta, \\ \alpha_{2m+1}(r_p) &= \frac{2(2m)!}{K^{2m+1}} \sum_{q=2m+1}^\infty \frac{(-Kr)^q}{q!} \sin q\theta, \end{aligned}$$

$m=1,2,\dots$, and the point $P \equiv (x,y)$ has circular polar coordinates given by $x = r \sin \theta$, $y = r \cos \theta$ (with $r = r_p$). Note that ϕ_{2m} and α_{2m} (ϕ_{2m-1} and α_{2m-1}) are even (odd) functions of x ($m=1,2,\dots$). Note also that $\phi_m(r_p) \equiv \phi_m(r,\theta)$ is singular at 0 , i.e. at $r=0$.

For large $|x|$, we have

$$\phi_1 \sim \pi \exp(-Ky \pm iKx), \quad \phi_2 \sim \pi i \exp(-Ky \pm iKx) \quad (A.2)$$

as $x \rightarrow \pm \infty$.

For $m > 2$, ϕ_m is a wavefree potential, i.e. it decays algebraically as $|x| \rightarrow \infty$. The precise error terms in (A.2) are given by Ursell (1961).

APPENDIX B. The Addition Theorem

$$\phi_m(r'_p) = \sum_{n=1}^\infty S_{mn}(b) \alpha_n(r_p).$$

where $r'_p = r_p + b$, $|r_p| < |b|$ and the matrix S is defined as follows.

$$\begin{aligned} S_{11} &= -\frac{1}{2} \{ \phi_2 + K^{-2} \phi_4 \}, S_{21} = -S_{12} = \frac{1}{2} \phi_1, S_{22} = -\frac{1}{2} \phi_2 \\ S_{1,2n+1} &= S_{2n+1,1} = -((n + \frac{1}{2})/K) \phi_{2n+4} \\ S_{1,2n+2} &= -S_{2n+2,1} = -(n/K) \phi_{2n+1} \\ S_{2,2n+1} &= -S_{2n+1,2} = \frac{1}{2} \phi_{2n+1} \\ S_{2,2n+2} &= S_{2n+2,2} = -\frac{1}{2} \phi_{2n+2} \\ S_{2m+1,2n+1} &= -\frac{1}{2} \{ \binom{2q+1}{2m} (2n+1) \phi_{2q+4} - \binom{2q-1}{2m} \frac{K^2}{2n} \phi_{2q+2} \} \end{aligned}$$

$$S_{2m+2,2n+2} = -\frac{1}{2} \{ \binom{2q-1}{2m-1} 2n \phi_{2q+2} - \binom{2q-3}{2m-1} \frac{K^2}{2n-1} \phi_{2q} \}$$

$$S_{2m+1,2n+2} = -S_{2n+2,2m+1}$$

$$= -\frac{1}{2} \{ \binom{2q}{2m} 2n \phi_{2q+1} - \binom{2q-2}{2m} \frac{K^2}{2n-1} \phi_{2q-1} \},$$

$q = m+n$, $m \geq 1$, $n \geq 1$, and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. All functions have argument b , i.e. $\phi_m \equiv \phi_m(b) = \phi_m(b, \frac{1}{2}\pi)$. It may be verified that (4b) is satisfied.

APPENDIX C. The Wide-spacing Approximation

The derivation below is modelled closely on that given by Srokosz and Evans (1979; §5.3). Write $S_1 \equiv S_1(\partial D; \phi_I)$ to emphasise the dependence on the particular geometry and incident potential.

Let $\phi_\pm(x,y)$ solve $S_1(\partial D; \exp(-Ky \mp iKx))$ and define corresponding reflection and transmission coefficients, r_\pm and t_\pm , by (8) and (10), with $A_\pm = \omega/g$. Similarly, let $\phi'_\pm(x,y)$ solve $S_1(\partial D'; \exp(-Ky \mp iKx))$, with corresponding coefficients r'_\pm and t'_\pm . Finally, let $\phi(x,y)$ solve S_2 , with $\phi_I = \exp(-Ky - iKx)$, and define R_+ and T_+ by

$$\phi(x,0) \sim \begin{cases} e^{-iKx} + R_+ e^{iKx}, & \text{as } x \rightarrow +\infty \quad (C.1) \\ T_+ e^{-iKx}, & \text{as } x \rightarrow -\infty. \quad (C.2) \end{cases}$$

Now, we suppose that near the first (i.e. front) cylinder (see Fig.1)

$$\phi(x,y) = \alpha \phi_+(x,y) + \beta \phi_-(x,y) \quad (C.3)$$

$$\sim \begin{cases} \alpha(e^{-iKx} + r_+ e^{iKx}) + \beta t_- e^{iKx} & \text{as } x \rightarrow +\infty \quad (C.4) \\ \alpha t_+ e^{-iKx} + \beta(e^{iKx} + r_- e^{-iKx}) & \text{as } x \rightarrow -\infty, \quad (C.5) \end{cases}$$

on $y = 0$, where α and β are unknown complex constants. Comparing (C.4) with (C.1), we find that $\alpha = 1$ and

$$R_+ = r_+ + \beta t_- \quad (C.6)$$

Also, we suppose that near the second (i.e. rear) cylinder,

$$\phi(x,y) = \gamma \phi'_+(x+b, y) \quad (C.7)$$

$$\sim \begin{cases} \gamma(e^{-iK(x+b)} + r'_+ e^{iK(x+b)}) & \text{as } x \rightarrow +\infty \quad (C.8) \\ \gamma t'_+ e^{-iK(x+b)} & \text{as } x \rightarrow -\infty \quad (C.9) \end{cases}$$

on $y = 0$, where γ is another complex constant. Comparing (C.9) with (C.2), we find that

$$T_+ = \gamma t'_+ e^{-iKb} \quad (C.10)$$

To find β and γ , we assume that there is a region between the two cylinders where (C.5) and (C.8) match; equating the coefficients of

$\exp(\pm iKx)$, we obtain

$$\beta = \gamma r_+ e^{iKb} \quad \text{and} \quad t_+ + \beta r_- = \gamma e^{-iKb}$$

i.e.

$$\Delta\beta = r_+ t_+ e^{2iKb} \quad \text{and} \quad \Delta\gamma = t_+ e^{iKb},$$

where Δ is given by (67). Substituting for β and γ in (C.6) and (C.10), we obtain the formulae (71a,b). Formulae for R_- and T_- (71c,d), can be obtained in a similar manner.