Some hydrodynamic applications of hypersingular boundary integral equations*

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1 Introduction

Many two-dimensional problems involving thin plates or cracks can be formulated as one-dimensional hypersingular integral equations, or as systems of such equations. Examples are potential flow past a rigid plate, acoustic scattering by a hard strip, water-wave interaction with thin impermeable barriers, and stress fields around cracks. In general, the crack or plate will be curved. If we parametrise the curve, we find that scalar problems can be reduced to an equation of the form

$$\oint_{-1}^{1} \left\{ \frac{1}{(x-t)^2} + M(x,t) \right\} f(t)\, dt = p(x) \quad \text{for } -1 < x < 1, \tag{1}$$

supplemented by two boundary conditions, which are often $f(-1) = f(1) = 0$. Here, f is the unknown function, p is prescribed and the kernel M is known. The cross on the integral sign indicates that it is to be interpreted as a two-sided finite-part integral of order two. This integral is defined by

$$\oint_{-1}^{1} \frac{f(t)}{(s-t)^2}\, dt = \lim_{\varepsilon \to 0} \left\{ \int_{-1}^{s-\varepsilon} \frac{f(t)}{(s-t)^2}\, dt + \int_{s+\varepsilon}^{1} \frac{f(t)}{(s-t)^2}\, dt - \frac{2f(s)}{\varepsilon} \right\}. \tag{2}$$

for sufficiently smooth functions f: we need $f' \in C^{0,\beta}$, the set of Hölder continuous functions (this smoothness restriction is essential, and should be reflected in any discretisation procedure [6]). This definition should be compared with the more familiar Cauchy principal-value integral of f, which is defined by

$$⨍_{-1}^{1} \frac{f(t)}{s-t}\, dt = \lim_{\varepsilon \to 0} \left\{ \int_{-1}^{s-\varepsilon} \frac{f(t)}{s-t}\, dt + \int_{s+\varepsilon}^{1} \frac{f(t)}{s-t}\, dt \right\} \tag{3}$$

for $f \in C^{0,\beta}$. In fact, if $f' \in C^{0,\beta}$, these two integrals are related by

$$-\frac{d}{ds} ⨍_{-1}^{1} \frac{f(t)}{s-t}\, dt = \oint_{-1}^{1} \frac{f(t)}{(s-t)^2}\, dt. \tag{4}$$

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An effective way of solving (1), numerically, is provided by expanding f using Chebyshev polynomials of the second kind, U_n:

$$f(t) \simeq \sqrt{1-t^2} \sum_{n=0}^{N} f_n \, U_n(t),$$

where $f_0, f_1, \ldots, f_N$ are coefficients to be found; note that this expansion incorporates the known behaviour of f near the two ends of the plate [7]. The method is effective because, with this expansion, the hypersingular integral can be evaluated analytically:

$$⨎_{-1}^{1} \frac{\sqrt{1-t^2}\, U_n(t)}{(x-t)^2} \, dt = -\pi(n+1)\, U_n(x).$$

One can then develop a Galerkin-type method or a collocation method for determining the unknown coefficients. Such methods have been used by many authors; examples are [2], [5] and [10]. Moreover, the convergence of the collocation scheme has been proved by Golberg [3], [4] and by Ervin & Stephan [1].

We have used this method for various problems in the theory of small-amplitude water waves in two dimensions. We are also developing related methods for analogous three-dimensional problems. This work is outlined below.

2 Interaction with submerged plates

A cartesian coordinate system is chosen, in which y is directed vertically downwards into the fluid, the undisturbed free surface lying at $y = 0$. We choose the z-axis perpendicular to the direction of propagation of the incident wavetrain. A plate, lying parallel to the incident wavecrests, is introduced below the free surface of the fluid, the submergence of the plate being independent of z. The problem is assumed two-dimensional, by considering a plate infinitely long in the z-direction, and the motion is taken to be simple harmonic in time. We use the assumptions of an inviscid, incompressible fluid, and an irrotational motion, to allow the introduction of a velocity potential $\mathrm{Re}\,\{\phi(x,y)e^{-i\omega t}\}$ to describe the small fluid motions. The conditions to be satisfied by $\phi(x,y)$ are

$$\nabla^2 \phi(x,y) = 0 \qquad \text{in the fluid,} \tag{5}$$

along with the free-surface condition

$$K\phi + \partial\phi/\partial y = 0 \qquad \text{on } y = 0, \tag{6}$$

where $K = \omega^2/g$ and g is the acceleration due to gravity. On the plate, the normal velocity vanishes, that is

$$\partial\phi/\partial n = 0 \qquad \text{on } \Gamma; \tag{7}$$

in general, Γ is a finite, simple, smooth arc. The choice of a linear theory of water waves enables us to split the potential into two parts,

$$\phi = \phi_{\text{sc}} + \phi_{\text{inc}}, \tag{8}$$

where ϕ_{inc} is the known incident potential and ϕ_{sc} is the scattered potential; ϕ_{sc} satisfies a radiation condition at infinity.

We formulate the problem as an integral equation by choosing an appropriate fundamental solution with an application of Green's theorem. We use the fundamental solution

$$G(x,y;\xi,\eta) = \frac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K}, \tag{9}$$

which satisfies (5) and (6); G has a logarithmic source singularity at the point $(x,y)=(\xi,\eta)$; the integration path is indented below the pole of the integrand at $k=K$ so that G also satisfies the radiation condition. Applying Green's theorem to $\phi_{\text{sc}}(P)$ and $G(x,y;\xi,\eta)\equiv G(P,Q)$, we find

$$\phi_{\text{sc}}(P) = \frac{1}{2\pi}\int_\Gamma [\phi(q)]\,\frac{\partial G(P,q)}{\partial n_q}\,ds_q, \tag{10}$$

where P is any point in the fluid, $\partial/\partial n_q$ represents normal differentiation at q on Γ, and $[\phi(q)]$ is the discontinuity in the potential across the plate at the point q. The potential defined by (10) satisfies (5), (6) and the radiation condition. It remains to impose the boundary condition on the plate:

$$\frac{1}{2\pi}\frac{\partial}{\partial n_p}\int_\Gamma [\phi(q)]\,\frac{\partial G(p,q)}{\partial n_q}\,ds_q = -\frac{\partial\phi_{\text{inc}}}{\partial n_p}, \qquad p\in\Gamma. \tag{11}$$

This is an integro-differential equation for $[\phi(q)]$, $q\in\Gamma$. It is to be solved subject to the conditions

$$[\phi]=0 \qquad \text{at the two edges of } \Gamma; \tag{12}$$

physically, because the plate is completely submerged, we expect the discontinuity in pressure across the plate to tend to zero as we approach each edge of the plate.

Let us now interchange the order of integration and normal differentiation at p in (11). Although this leads to a non-integrable integrand, Martin & Rizzo[8] have shown that this procedure is legitimate, provided that the integral is then interpreted as a finite-part integral. By adopting this procedure, we find

$$\frac{1}{2\pi}\,⨎_\Gamma [\phi(q)]\,\frac{\partial^2}{\partial n_p\partial n_q}G(p,q)\,ds_q = -\frac{\partial\phi_{\text{inc}}}{\partial n_p}, \qquad p\in\Gamma, \tag{13}$$

which is to be solved for $[\phi]$, subject to (12).

Parametrising Γ, we obtain (1); the kernel M is complicated but readily computable [10]. We have used the collocation-expansion method described above to compute the reflection coefficient R for submerged plates of various shapes. In particular, we have shown [9] that R approaches zero as the plate is bent into the shape of a circle, in agreement with the well-known fact that a submerged circular cylinder has $R=0$ exactly (for all depths of submergence and for all frequencies).

Parsons [9] has also obtained results for surface-piercing plates, and for the trapping of waves by submerged plates. This latter problem involves replacing Laplace's equation by a modified Helmholtz equation (so that the corresponding G is more complicated), and then seeking non-trivial solutions of the homogeneous form of (1).

Finally, we are currently developing analogous methods for three-dimensional problems. A typical problem is the scattering of water waves by a submerged circular disc, for which good expansion methods are available; the difficulty is to extend these methods to non-circular plates. Alternatively, we have various boundary element methods at our disposal.

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