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Chapter 5

Interaction of water waves with thin plates

P.A. Martin^a N.F. Parsons^b & L. Farina^a

^a*Department of Mathematics*

University of Manchester, Manchester M13 9PL, U.K.

^b*Department of Theoretical Mechanics*

University of Nottingham, Nottingham NG7 2RD, U.K.

Abstract

Various problems involving the interaction of water waves with thin plates are reduced to hypersingular boundary integral equations. Examples include scattering by submerged curved plates and by surface-piercing plates, in two dimensions; trapping of waves by submerged plates; and scattering by submerged flat plates in three dimensions. Each integral equation is solved numerically, using an expansion-collocation method; this method is effective because it incorporates the known edge behaviour of the solution and because it allows all hypersingular integrals to be evaluated analytically.

1 Introduction

Many two-dimensional problems involving thin plates or cracks can be formulated as one-dimensional hypersingular integral equations, or as systems of such equations. Examples are potential flow past a rigid plate, acoustic scattering by a hard strip, water-wave interaction with thin impermeable barriers, and stress fields around cracks. In general, the crack or plate will be curved. If we parametrise the curve, we find that scalar problems can be reduced to an equation of the form

$$\int_{-1}^1 \left\{ \frac{1}{(x-t)^2} + M(x,t) \right\} f(t) dt = p(x) \quad \text{for } -1 < x < 1, \quad (1)$$

supplemented by two boundary conditions, which are often $f(-1) = f(1) = 0$. Here, f is the unknown function, p is prescribed and the kernel M is known. The cross on the integral

sign indicates that it is to be interpreted as a two-sided finite-part integral of order two; the definition is given below by eqn (19).

Most of this paper is concerned with the derivation and solution of eqn (1), in the context of water-wave scattering by immersed thin impermeable plates. Such problems (in two dimensions) are also considered elsewhere in this volume; but our methods are quite independent of the geometry. We also describe some of our recent work in three dimensions; here, the simplest problem is scattering by a submerged flat circular plate.

We start with a brief survey of the literature on scattering by thin plates, in two dimensions. Previous work on this topic can be classified according to whether the depth of water is finite or infinite, and whether the plate is completely submerged or pierces the free surface. Moreover, most previous work assumes that the plate is flat.

The derivation of the basic integral equation is outlined in section 4. We start with the simpler problem of scattering by a submerged cylinder whose cross-section has a non-zero area (section 3). We use Green's theorem and an appropriate fundamental solution, G , to obtain an integral representation for the velocity potential, ϕ ; standard boundary integral equations (amenable to numerical solution via a boundary element method) are then easily obtained. When the cross-sectional area shrinks to zero, so that the cylinder degenerates into a thin plate, the integral equation degenerates too: this leads naturally to the introduction of a hypersingular integral equation. Actually, this is a boundary integral equation over the plate for $[\phi]$, the discontinuity in ϕ across the plate. Once the cross-sectional curve is parametrised, the equation reduces to the standard form eqn (1).

This approach has several advantages over other methods mentioned in section 2. For instance, the radiation condition is automatically satisfied by the choice of G . Similarly, the behaviour of $[\phi]$ at each edge of the plate, where there are square-root zeros, can be easily enforced. Moreover, the method is applicable to curved plates as well as to flat plates. In fact, apart from some simple quadratures, the only approximation required is that of a bounded function defined on a finite interval. We do this by choosing an appropriate set of orthogonal polynomials, namely Chebyshev polynomials of the second kind, and then using a collocation method on the governing integral equation. Similar expansion-collocation methods have been used by Frenkel [1] and by Kaya & Erdogan [2]. It is also known that these methods are convergent; see Golberg [3], [4] and Ervin & Stephan [5]. Having computed an approximation to $[\phi]$, the reflection and transmission coefficients can then be calculated directly. Several applications, published elsewhere, are described in section 9: scattering by submerged plates (both flat and curved) and by surface-piercing plates; and trapping of waves by submerged plates of various shapes.

We conclude with a description of how expansion-collocation methods can be developed for certain three-dimensional problems. An appropriate set of expansion functions is identified by studying the problem of potential flow past a flat circular disc. This leads to a good numerical method for the interaction of water waves with submerged circular plates. For non-circular plates, we map the plate onto a circular domain and then use the previous expansion-collocation method: it is essential to use a *conformal* mapping; this trick is amusing, because it is widely believed that conformal mappings are only useful for two-dimensional problems! This method is currently under development.

2 Literature survey: two dimensions

The two-dimensional scattering of linear water waves by thin rigid plates has been treated in several ways by many authors. One reason for this attention is that thin plates have been used as simple models for certain floating breakwaters; Sobhani *et al.* [6] discuss this application and give further references. Another reason is that thin plates can lead to boundary-value problems that can be solved exactly; this is very unusual in linear hydrodynamics, but is also very valuable in that such solutions provide benchmarks against which approximate solutions can be assessed. In what follows, we only refer to previous work on scattering by a single plate of finite length.

2.1 Infinite depth

Ursell [7] solved the problem of wave scattering by a fixed, surface-piercing, vertical plate. He constructed the potential on each side of the plate by using an expansion theorem due to Havelock [8]. Continuity of motion across the plane of the plate gave him an integral equation for the horizontal velocity, which he solved exactly. The reflection and transmission coefficients were obtained from the limiting forms of the potential at large distances from the plate.

John [9] considered surface-piercing plates making an angle of $\pi/2n$ to the horizontal, where n is an integer. He showed that this problem can be solved by complex function techniques. However, as n increases, the method quickly becomes unwieldy; in fact, it seems that even the case $n = 2$ has not been worked out in detail.

Evans [10] considered the scattering of surface waves by a fixed, vertical plate, submerged beneath the free surface. His method of solution is similar to that used by John, whereby a complex potential is introduced, from which a reduced potential may be defined. The choice of reduced potential ensures that the boundary conditions on the free surface and on the plate take the same form for this new problem. This simplification allows the reduced potential problem to be solved, from which the desired result follows by integration.

Burke [11] treated the problem of scattering by a fixed, submerged, horizontal plate, using the Wiener–Hopf technique. Unfortunately, no numerical results were given.

Shaw [12] has considered the problem of scattering by a surface-piercing plate, whose orientation is altered slightly from the vertical, and whose shape is slightly altered from being flat. Using perturbation techniques, Shaw found that to first order, the problem is the same as that solved by Ursell [7]. A new second-order correction is found, however, with corresponding corrections to the reflection and transmission coefficients. See also [13] and [14].

2.2 Finite depth

For water of constant finite depth, it is conventional to divide the fluid domain into three, namely a finite (rectangular) domain containing the plate, and two semi-infinite domains. In the latter, the velocity potential can be written as a series of eigenfunctions (with unknown coefficients). In the finite domain, different methods have been used. Thus, Patarapanich [15] used the finite element method and calculated the reflection and transmission coefficients

for a submerged horizontal plate. The main disadvantage of this method is that it does not readily account for the inevitable singularities at the two edges of the plate, where inviscid theory predicts infinite velocities. Moreover, care must be taken in matching with the eigenfunction expansions in the two semi-infinite domains, so as to satisfy the radiation conditions and to avoid spurious reflections. Finite elements have also been used by Sobhani *et al.* [6] in their study of inclined, surface-piercing plates, wherein the plate is hinged at the sea-floor and the effects of a mooring line are also included.

For submerged horizontal plates, one can also use eigenfunction expansions within the finite domain. This leads to the method of matched eigenfunction expansions. It has been used by McIver [16] for scattering by moored, horizontal plates, although she also computed the reflection and transmission coefficients for a fixed plate.

Liu & Abbaspour [17] have used a simple boundary integral equation method within the finite domain for inclined, surface-piercing plates. They partitioned the finite domain into two by introducing an additional boundary, extending from the lower edge of the plate to a point on the sea-floor. They then solved Laplace's equation in each sub-domain using Green's theorem and a simple ($\log R$) fundamental solution. Again, this method does not account for the plate-edge singularities in a natural way: special elements are introduced so as to incorporate the expected singular behaviour.

Finally, we mention that Hamilton [18] has given some experimental results for plates at small inclinations to the horizontal.

3 Boundary integral equations: cylinders

A cartesian coordinate system is chosen, in which y is directed vertically downwards into the fluid, the undisturbed free surface lying at $y = 0$. We choose the z -axis perpendicular to the direction of propagation of the incident wavetrain. A cylinder, lying parallel to the incident wavecrests, is introduced. For simplicity, we assume here that the cylinder is completely submerged below the free surface of the fluid, its submergence being independent of z . The problem is assumed two-dimensional, by considering the cylinder to be infinitely long in the z -direction, and the motion is taken to be simple harmonic in time. We use the assumptions of an inviscid, incompressible fluid, and an irrotational motion, to allow the introduction of a velocity potential

$$\text{Re} \left\{ \phi(x, y) e^{-i\omega t} \right\}$$

to describe the small fluid motions. The conditions to be satisfied by $\phi(x, y)$ are

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi(x, y) = 0 \quad \text{in the fluid, } D,$$

along with the free-surface condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on } y = 0,$$

where $K = \omega^2/g$ and g is the acceleration due to gravity. On the surface of the cylinder, S , the normal velocity vanishes, that is

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } S;$$

in general, S is a simple, closed smooth curve. The choice of a linear theory of water waves enables us to split the total potential ϕ into two parts,

$$\phi = \phi_{\text{sc}} + \phi_{\text{inc}}, \quad (2)$$

where ϕ_{inc} is the known incident potential and ϕ_{sc} is the scattered potential. Reformulated in terms of ϕ_{sc} , the boundary-value problem becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi_{\text{sc}} = 0 \quad \text{in the fluid,} \quad (3)$$

$$K\phi_{\text{sc}} + \frac{\partial\phi_{\text{sc}}}{\partial y} = 0 \quad \text{on } y = 0, \text{ and} \quad (4)$$

$$\frac{\partial\phi_{\text{sc}}}{\partial n} = -\frac{\partial\phi_{\text{inc}}}{\partial n} \quad \text{on } S. \quad (5)$$

The fact that ϕ_{sc} is due to the presence of the cylinder indicates the need for a radiation condition on ϕ_{sc} , that waves travel outwards towards infinity. Mathematically, this may be written as

$$\frac{\partial\phi_{\text{sc}}}{\partial r} - iK\phi_{\text{sc}} \rightarrow 0 \quad \text{as } r = (x^2 + y^2)^{\frac{1}{2}} \rightarrow \infty. \quad (6)$$

In the sequel, we use capital letters P, Q to denote points in the fluid, and lower-case letters p, q to denote points on S .

Next, we reduce the boundary-value problem for ϕ_{sc} to a boundary integral equation over S . To do this, we combine an appropriate fundamental solution with an application of Green's theorem. We use the fundamental solution

$$G(P, Q) \equiv G(x, y; \xi, \eta) = \frac{1}{2} \log\{(x - \xi)^2 + (y - \eta)^2\} + G_1(x - \xi, y + \eta), \quad (7)$$

where

$$\begin{aligned} G_1(X, Y) &= -\frac{1}{2} \log(X^2 + Y^2) - 2\Phi_0(X, Y), \\ \Phi_0(X, Y) &= \int_0^\infty e^{-kY} \cos kX \frac{dk}{k - K}. \end{aligned} \quad (8)$$

We note that Φ_0 can be computed using an expansion derived by Yu & Ursell [19]:

$$\begin{aligned} \Phi_0(X, Y) &= -e^{-KY} \{(\log KR - i\pi + \gamma) \cos KX + \beta \sin KX\} \\ &\quad + \sum_{m=1}^{\infty} \frac{(-KR)^m}{m!} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m} \right) \cos m\beta, \end{aligned} \quad (9)$$

where $\gamma = 0.5772\dots$ is Euler's constant and β is defined by $X = R \sin \beta$ and $Y = R \cos \beta$.

G satisfies eqn (3) and eqn (4), and G has a logarithmic source singularity at the point $(x, y) = (\xi, \eta)$; the integration path in eqn (8) is indented below the pole of the integrand at $k = K$ so that G also satisfies the radiation condition at infinity. Note that G_1 is non-singular everywhere in the fluid.

Applying Green's theorem to $\phi_{\text{sc}}(P)$ and $G(x, y; \xi, \eta) \equiv G(P, Q)$, we find

$$\phi_{\text{sc}}(P) = \frac{1}{2\pi} \int_S \left\{ G(P, q) \frac{\partial \phi_{\text{sc}}}{\partial n_q} - \phi_{\text{sc}}(q) \frac{\partial G(P, q)}{\partial n_q} \right\} ds_q, \quad (10)$$

where P is any point in D and $\partial/\partial n_q$ denotes normal differentiation at q on S in the direction from S into D . This is the familiar integral representation for ϕ_{sc} in the fluid in terms of ϕ_{sc} and $\partial\phi_{\text{sc}}/\partial n$ on S ; the latter is known from the boundary condition (5), whence eqn (10) becomes

$$\phi_{\text{sc}}(P) = \frac{-1}{2\pi} \int_S \left\{ G(P, q) \frac{\partial \phi_{\text{inc}}}{\partial n_q} + \phi_{\text{sc}}(q) \frac{\partial G(P, q)}{\partial n_q} \right\} ds_q. \quad (11)$$

Letting $P \rightarrow p$, a point on S , we obtain

$$\pi\phi_{\text{sc}}(p) + \int_S \phi_{\text{sc}}(q) \frac{\partial G(p, q)}{\partial n_q} ds_q = - \int_S G(p, q) \frac{\partial \phi_{\text{inc}}}{\partial n_q} ds_q,$$

which is a well-known boundary integral equation for ϕ_{sc} on S . This equation is uniquely solvable for all values of K . Once solved, ϕ_{sc} is given everywhere in the fluid by eqn (11).

4 Boundary integral equations: plates

Let the cross-section of the cylinder in the previous section shrink, so that the cylinder degenerates into a thin rigid plate. Thus, S degenerates into Γ , a finite, simple, smooth arc. Γ has two sides, Γ^+ and Γ^- . The boundary-value problem for ϕ_{sc} becomes: solve Laplace's equation (3) in D , subject to the free-surface condition (4), the radiation condition (6) and the boundary condition

$$\frac{\partial \phi_{\text{sc}}}{\partial n^\pm} = - \frac{\partial \phi_{\text{inc}}}{\partial n^\pm} \quad \text{on } \Gamma^\pm, \quad (12)$$

where $\partial/\partial n^\pm$ denote normal differentiation at a point on Γ^\pm in the direction from Γ^\pm into D ; in addition, we require that ϕ_{sc} be bounded in the neighbourhood of the two ends of Γ .

The velocity potential is discontinuous across Γ : define

$$[\phi(p)] = \phi(p^+) - \phi(p^-),$$

where p^\pm are corresponding points on Γ^\pm . However, the potential of the incident wave is continuous across the plate, so that

$$[\phi(p)] = [\phi_{\text{sc}}(p)]. \quad (13)$$

Hence, the integral representation (11) reduces to

$$\phi_{\text{sc}}(P) = \frac{-1}{2\pi} \int_\Gamma [\phi(q)] \frac{\partial G(P, q)}{\partial n_q^+} ds_q. \quad (14)$$

This equation states that $\phi_{\text{sc}}(P)$ can be represented as a double-layer potential, that is, as a distribution of normal dipoles over Γ . Note that the incident potential does not appear explicitly in eqn (14).

To use the integral representation (14), we have to find $[\phi]$. If we adopt the standard approach, that is, we let P go to p^+ and p^- in turn, and subtract the result, we obtain the nugatory result (13). Instead, we impose the boundary condition (12) on Γ^+ , giving

$$\frac{1}{2\pi} \frac{\partial}{\partial n_p^+} \int_{\Gamma} [\phi(q)] \frac{\partial G(p^+, q)}{\partial n_q^+} ds_q = \frac{\partial \phi_{\text{inc}}}{\partial n_p^+}, \quad p^+ \in \Gamma^+.$$

A similar equation is obtained if we apply the boundary condition on Γ^- ; henceforth, we shall delete the superscript $+$, whence

$$\frac{1}{2\pi} \frac{\partial}{\partial n_p} \int_{\Gamma} [\phi(q)] \frac{\partial G(p, q)}{\partial n_q} ds_q = \frac{\partial \phi_{\text{inc}}}{\partial n_p}, \quad p \in \Gamma. \quad (15)$$

This is an integro-differential equation for $[\phi(q)]$, $q \in \Gamma$. It is to be solved subject to the conditions

$$[\phi] = 0 \quad \text{at the two edges of } \Gamma; \quad (16)$$

physically, because the plate is completely submerged, we require that the discontinuity in pressure across the plate tends to zero as we approach each edge of the plate.

It is tempting simply to take the normal derivative at p in eqn (15) under the integral sign, but this leads to a non-integrable integrand. The conventional way of dealing with this difficulty is to *regularize* eqn (15); various possibilities are described in [20, 21]. Instead, we adopt a more direct approach. Thus, it can be proved [20] that interchanging the order of integration and normal differentiation at p in eqn (15) is legitimate, provided that the integral is then interpreted as a finite-part integral. By adopting this procedure, we find

$$\frac{1}{2\pi} \int_{\Gamma}^{\times} [\phi(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p, q) ds_q = \frac{\partial \phi_{\text{inc}}}{\partial n_p}, \quad p \in \Gamma, \quad (17)$$

which is to be solved for $[\phi]$, subject to eqn (16). The cross on the integral sign indicates that it is to be interpreted as a two-sided finite-part integral of order two. We digress in the next section, so as to discuss such integrals.

5 Finite-part integrals

Let f be a function of single variable, so that $f(t)$ is defined for $a \leq t \leq b$. We shall define the Cauchy principal-value integral and the Hadamard finite-part integral. Related integrals and references are given in [20].

5.1 Cauchy principal-value integral

Assume that f is a Hölder-continuous function, $f \in C^{0,\beta}$ (such functions are smoother than merely continuous functions, but they need not be differentiable). Then, the Cauchy principal-value integral of f is defined by

$$\int_a^b \frac{f(t)}{x-t} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_a^{x-\varepsilon} \frac{f(t)}{x-t} dt + \int_{x+\varepsilon}^b \frac{f(t)}{x-t} dt \right\}.$$

If we suppose that f is smoother, so that f' is continuous, the Cauchy principal-value integral can be regularized:

$$\int_a^b \frac{f(t)}{x-t} dt = f(a) \log(x-a) - f(b) \log(b-x) + \int_a^b f'(t) \log|x-t| dt.$$

If f is smoother still, so that $f' \in C^{0,\beta}$, the Cauchy principal-value integral can be differentiated:

$$\frac{d}{dx} \int_a^b \frac{f(t)}{x-t} dt = \frac{f(a)}{x-a} + \frac{f(b)}{b-x} + \int_a^b \frac{f'(t)}{x-t} dt. \quad (18)$$

5.2 Hadamard finite-part integral

Assume that $f' \in C^{0,\beta}$, that is, $f \in C^{1,\beta}$. Then, the two-sided Hadamard finite-part integral of order two is defined by

$$\int_a^b \frac{f(t)}{(x-t)^2} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_a^{x-\varepsilon} \frac{f(t)}{(x-t)^2} dt + \int_{x+\varepsilon}^b \frac{f(t)}{(x-t)^2} dt - \frac{2f(x)}{\varepsilon} \right\}. \quad (19)$$

Again, this integral can be regularized:

$$\int_a^b \frac{f(t)}{(x-t)^2} dt = -\frac{f(a)}{x-a} - \frac{f(b)}{b-x} - \int_a^b \frac{f'(t)}{x-t} dt. \quad (20)$$

Comparing eqn (18) and eqn (20), we obtain

$$\frac{d}{dx} \int_a^b \frac{f(t)}{x-t} dt = - \int_a^b \frac{f(t)}{(x-t)^2} dt; \quad (21)$$

thus, the differentiation can be taken under the integral. Note that one could take eqn (21) as the *definition* of the finite-part integral on the right-hand side.

Finite-part integrals have some unexpected behaviour. For example, by taking $f \equiv 1$, $a = -1$, $b = 1$ and $x = 0$, we find (from eqn (20)) that

$$\int_{-1}^1 \frac{dt}{t^2} = -2;$$

the integrand is positive but the integral is negative!

6 The hypersingular integral equation

Hypersingular integral equations, such as eqn (17), are unfamiliar. However, they arise naturally for many problems involving thin bodies upon which a Neumann boundary condition is imposed. They can be treated directly, or they can be rewritten in a more familiar form by a process of regularization; this may lead to a different integro-differential equation or to an equation involving tangential derivatives of $[\phi]$ (which are singular at the edges of Γ). However, the hypersingular integral equation (17) is quite general: it is valid for water of constant finite depth and in three dimensions, with an appropriate choice for G . Therefore, it

is worthwhile to develop general methods for their treatment, rather than relying on special methods that only work for special geometries.

Before developing such a method, it is convenient to first find a general expression for the kernel in our particular equation (17). This can then be used for any choice of Γ . Denote the unit normals at p and $q \in \Gamma$ by $\mathbf{n}(p) = (n_1^p, n_2^p)$ and $\mathbf{n}(q) = (n_1^q, n_2^q)$, respectively. Then, apply the formula

$$\frac{\partial^2 G}{\partial n_p \partial n_q} = n_1^p n_1^q \frac{\partial^2 G}{\partial x \partial \xi} + n_1^p n_2^q \frac{\partial^2 G}{\partial x \partial \eta} + n_2^p n_1^q \frac{\partial^2 G}{\partial y \partial \xi} + n_2^p n_2^q \frac{\partial^2 G}{\partial y \partial \eta}.$$

After differentiating, and rearranging, we find that

$$\frac{\partial^2 G}{\partial n_p \partial n_q} = -\frac{\mathcal{N}}{R^2} + \frac{2\Theta}{R^4} + \frac{\partial^2 G_1}{\partial n_p \partial n_q} \quad (22)$$

where

$$\begin{aligned} \frac{\partial^2 G_1}{\partial n_p \partial n_q} &= 2(n_1^p n_2^q - n_2^p n_1^q) \left\{ K \frac{\partial \Phi_0}{\partial X} - \frac{XY}{(X^2 + Y^2)^2} \right\} - \mathcal{N} \mathcal{K}(X, Y), \\ \mathcal{K}(X, Y) &= \frac{Y^2 - X^2}{(X^2 + Y^2)^2} + \frac{2KY}{X^2 + Y^2} + 2K^2 \Phi_0(X, Y), \end{aligned} \quad (23)$$

Φ_0 is defined by eqn (8), $X = x - \xi$, $Y = y + \eta$, $\mathcal{N} = \mathbf{n}(p) \cdot \mathbf{n}(q)$, $\Theta = (\mathbf{n}(p) \cdot \mathbf{R})(\mathbf{n}(q) \cdot \mathbf{R})$, $\mathbf{R} = (x - \xi, y - \eta)$ and $R = |\mathbf{R}|$.

The third term on the right-hand side of eqn (22) comes from differentiating G_1 in eqn (7), and so is non-singular. The first two terms on the right-hand side of eqn (22) come from differentiating the basic singularity, $\log R$:

$$\begin{aligned} \frac{\partial^2}{\partial n_p \partial n_q} \log R &= -\frac{\mathcal{N}}{R^2} + \frac{2\Theta}{R^4} \\ &= -\frac{1}{R^2} + \frac{\mathbf{n}(p) \cdot (\mathbf{n}(p) - \mathbf{n}(q))}{R^2} \\ &\quad - \left(\frac{\partial}{\partial n_p} \log R \right) \left(\frac{\partial}{\partial n_q} \log R \right); \end{aligned}$$

only the term $-1/R^2$ is singular — it displays the characteristic hypersingularity in two dimensions; the other terms are continuous for twice-differentiable curves Γ . To see this, and to convert our hypersingular integral equation (17) into the standard form (1), we parametrise the curve Γ .

Assume that Γ is defined by

$$\Gamma = \{(x, y) : x = x(s), y = y(s), -1 \leq s \leq 1\}.$$

Thus, the point $p \equiv (x, y)$ on Γ is parametrised with the parameter s whilst $q \equiv (\xi, \eta)$ has parameter t . Then,

$$\mathbf{n}(q) = (-y'(t), x'(t))/v(t)$$

where $v = \{(x')^2 + (y')^2\}^{1/2}$, and

$$\begin{aligned} R &= \{(x(t) - x(s))^2 + (y(t) - y(s))^2\}^{1/2} \\ &\sim |t - s| v \quad \text{as } |t - s| \rightarrow 0. \end{aligned}$$

Expanding for small $|t - s|$, we find that

$$\begin{aligned} \frac{\partial}{\partial n_p} \log R &\sim \frac{x'y'' - x''y'}{2v^3}, \\ \mathbf{n}(p) - \mathbf{n}(q) &\sim (t - s)v^{-3}(x'y'' - x''y')(x', y'). \end{aligned}$$

The first of these results is well known from classical potential theory (see, for example, Smirnov [22, p. 595]); the second shows that

$$\mathbf{n}(p) \cdot (\mathbf{n}(p) - \mathbf{n}(q)) = O((s - t)^2) \quad \text{as } |t - s| \rightarrow 0.$$

As $ds_q = v(t) dt$, we can write eqn (17) as

$$\int_{-1}^1 \frac{f(t)}{(s - t)^2} dt + \int_{-1}^1 f(t) M(s, t) dt = p(s), \quad -1 < s < 1, \quad (24)$$

where $f(t) = [\phi(q(t))]$ is our new unknown function representing the discontinuity in ϕ across the plate at the point q ,

$$M(s, t) = -\frac{1}{(s - t)^2} + v(t) v(s) \left\{ \frac{\mathcal{N}}{R^2} - \frac{2\Theta}{R^4} - \frac{\partial^2 G_1}{\partial n_p \partial n_q} \right\} \quad (25)$$

and

$$p(s) = -2\pi v(s) \frac{\partial \phi_{\text{inc}}}{\partial n_p}. \quad (26)$$

Note that we have isolated the hypersingular part in the first term on the left-hand side of eqn (24); the other term is an ordinary non-singular integral. For submerged plates, the equation (24) is to be solved subject to

$$f(\pm 1) = 0. \quad (27)$$

7 Method of solution

In order to solve the hypersingular integral equation (24), numerically, we use an expansion-collocation method. In such a method, one expands the unknown function as

$$f(t) = w(t) \sum_{n=0}^N a_n f_n(t) \quad (28)$$

where w is a prescribed weight function, $\{f_n\}$ is a set of expansion (basis) functions, and a_n are unknown coefficients; one then substitutes eqn (28) into the integral equation and

determines the coefficients by collocating at $N + 1$ points in the interval $-1 < s < 1$. It remains to choose these points, the weight function and the expansion functions.

Now, it can be shown [23] that any solution of eqn (24) that satisfies eqn (27) actually behaves as $f(t) \sim \sqrt{1 \mp t} f_{\pm}$ as $t \rightarrow \pm 1$, where f_{\pm} are constants. We build this into our numerical procedure for solving eqn (24) by choosing

$$w(t) = \sqrt{1 - t^2}.$$

This ensures that the edge conditions are satisfied for any bounded functions f_n .

Next, consider the hypersingular part of eqn (24). Thus, consider

$$\frac{1}{\pi} \int_{-1}^1 \frac{g(t)}{(x-t)^2} dt = h(x), \quad -1 < x < 1 \quad (29)$$

(the factor $1/\pi$ is inserted for convenience). This is known as the *dominant equation*. Its general solution (for sufficiently smooth h) is given in [24] as

$$g(x) = \frac{1}{\pi} \int_{-1}^1 h(t) \log \left(\frac{|x-t|}{1-xt + \sqrt{(1-x^2)(1-t^2)}} \right) dt + \frac{A+Bx}{\sqrt{1-x^2}}, \quad (30)$$

where A and B are arbitrary constants. The first term on the right-hand side of eqn (30) is a particular solution of eqn (29), for the given function h . The second term is the general solution of the homogeneous form of eqn (29) (that is, with $h \equiv 0$). Thus, for a unique solution of eqn (29), we need *two* supplementary conditions on g ; these are often taken to be eqn (27) whence $A = B = 0$ in eqn (30).

Let us take $g(t) = \sqrt{1-t^2} U_n(t)$, where U_n is a Chebyshev polynomial of the second kind, defined by

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.$$

If we combine formula (22.13.4) from Abramowitz & Stegun [25], namely

$$\int_{-1}^1 \frac{\sqrt{1-t^2} U_n(t)}{x-t} dt = \pi T_{n+1}(x)$$

(where T_n is a Chebyshev polynomial of the first kind), with eqn (21), we obtain

$$\frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-t^2} U_n(t)}{(x-t)^2} dt = -(n+1)U_n(x), \quad (31)$$

whereby $g(t) = \sqrt{1-t^2} U_n(t)$ is the exact bounded solution of eqn (29) when $h(x) = -(n+1)U_n(x)$, subject to the edge conditions (27).

Since the Chebyshev polynomials of the second kind form a complete set over the interval $[-1, 1]$, they are a good choice for the expansion functions f_n . Thus, we approximate $f(t)$ as follows:

$$f(t) \cong \sqrt{1-t^2} \sum_{n=0}^N a_n U_n(t) \equiv f_N(t), \quad (32)$$

say, where N is finite and the unknown coefficients a_n are to be found. Substituting eqn (32) into eqn (24), we find

$$\sum_{n=0}^N a_n A_n(s) = p(s), \quad -1 < s < 1, \quad (33)$$

where $p(s)$ is defined by eqn (26) and

$$A_n(s) = -\pi(n+1)U_n(s) + \int_{-1}^1 \sqrt{1-t^2} U_n(t) M(s,t) dt.$$

To find the unknown coefficients, we choose a family of functions $\psi_j(s)$, where $j = 0, 1, \dots, N$, called trial functions. Multiplying both sides of eqn (33) by $\psi_j(s)$ and integrating from -1 to 1 leads to the Petrov–Galerkin system

$$\sum_{n=0}^N A_{jn} a_n = p_j, \quad j = 0, 1, \dots, N,$$

where

$$A_{jn} = \int_{-1}^1 A_n(s) \psi_j(s) ds \quad \text{and} \quad p_j = \int_{-1}^1 p(s) \psi_j(s) ds.$$

One choice for $\psi_j(s)$ is $\sqrt{1-s^2}U_j(s)$, leading to a classical Galerkin method; Golberg [3], [4] has analysed the convergence of this method. A more pragmatic choice, which avoids double integrals, is $\psi_j(s) = \delta(s - s_j)$, where s_j ($j = 0, 1, \dots, N$) are points with $|s_j| < 1$. This yields

$$\sum_{n=0}^N a_n A_n(s_j) = p(s_j), \quad j = 0, 1, \dots, N, \quad (34)$$

which is a straightforward collocation scheme, with collocation points s_j . A suitable set of collocation points is

$$s_j = \cos\left(\frac{(j+1)\pi}{N+2}\right), \quad j = 0, 1, \dots, N; \quad (35)$$

these are the zeros of $U_{N+1}(s)$. This is expected to be a good choice, since, if the coefficients a_n are decaying rapidly, the error in eqn (32) is roughly proportional to $U_{N+1}(s)$; see [26, p. 228]. Another possible choice is

$$s_j = \cos\left(\frac{(2j+1)\pi}{2N+2}\right), \quad j = 0, 1, \dots, N; \quad (36)$$

these are the zeros of $T_{N+1}(s)$. Golberg [3], [4] has shown that eqn (35) and eqn (36) are both good choices: he has proved that they both yield uniformly-convergent methods,

$$\max_{-1 \leq t \leq 1} \frac{|f(t) - f_N(t)|}{\sqrt{1-t^2}} \rightarrow 0 \quad \text{as } N \rightarrow \infty;$$

see also [5]. The rate of convergence depends on the smoothness of the kernel M in eqn (24), which depends in turn on the smoothness of the plate. If the plate is infinitely smooth and completely submerged, the convergence is exponential, since M is infinitely differentiable. In all of our numerical computations, we have used the set (36).

In summary, the expansion (32) incorporates the known behaviour of f near the two ends of the plate and it enables the dominant hypersingular integral to be evaluated analytically; this leads to an effective numerical method.

8 Reflection and transmission coefficients

When an incident wave is scattered by a fixed body, some of the wave energy will be transmitted past the body, and some will be reflected back. To quantify this, we introduce the complex numbers \mathcal{R} and \mathcal{T} , which are known as the reflection and transmission coefficients, respectively. The magnitudes of \mathcal{R} and \mathcal{T} are related to the amplitude of the reflected and transmitted waves, respectively. Similarly, the arguments of \mathcal{R} and \mathcal{T} correspond to a phase shift in the scattered waves. For an incident wave travelling towards $x = +\infty$, given by

$$\phi_{\text{inc}} = e^{-Ky+iKx}, \quad (37)$$

\mathcal{R} and \mathcal{T} are defined by the asymptotic behaviour of ϕ as $|x| \rightarrow \infty$. More precisely, we have

$$\phi(x, y) \sim \begin{cases} \mathcal{T} e^{-Ky+iKx} & \text{as } x \rightarrow +\infty \\ e^{-Ky+iKx} + \mathcal{R} e^{-Ky-iKx} & \text{as } x \rightarrow -\infty. \end{cases}$$

From eqn (2), we can define \mathcal{R} and \mathcal{T} solely in terms of the scattered potential,

$$\phi_{\text{sc}}(x, y) \sim \begin{cases} (\mathcal{T} - 1) e^{-Ky+iKx} & \text{as } x \rightarrow +\infty \\ \mathcal{R} e^{-Ky-iKx} & \text{as } x \rightarrow -\infty. \end{cases} \quad (38)$$

Now, ϕ_{sc} is given by eqn (14). Since

$$G(x, y; \xi, \eta) \sim -2\pi i e^{-K(y+\eta) \pm iK(x-\xi)} \quad \text{as } x \rightarrow \pm\infty,$$

the integral representation eqn (14) gives

$$\phi_{\text{sc}}(x, y) \sim i e^{-Ky \pm iKx} \int_{\Gamma} [\phi(\xi, \eta)] \frac{\partial}{\partial n_q} e^{-K\eta \mp iK\xi} ds_q \quad \text{as } x \rightarrow \pm\infty. \quad (39)$$

Simple comparison of eqn (38) and eqn (39) now yields the formulae

$$\begin{aligned} \mathcal{R} &= i \int_{\Gamma} [\phi(q)] \frac{\partial}{\partial n_q} e^{-K\eta+iK\xi} ds_q, \\ \mathcal{T} - 1 &= i \int_{\Gamma} [\phi(q)] \frac{\partial}{\partial n_q} e^{-K\eta-iK\xi} ds_q. \end{aligned}$$

From these formulae, we see that once the discontinuity in ϕ across the plate has been found, the values of \mathcal{R} and \mathcal{T} may be found directly, without having to find ϕ_{sc} everywhere in the fluid first. Thus, parametrising as before, we find that \mathcal{R} and \mathcal{T} are given by

$$\mathcal{R} = K \int_{-1}^1 f(t) (y'(t) - ix'(t)) e^{-Ky(t)+iKx(t)} dt, \quad (40)$$

$$\mathcal{T} - 1 = -K \int_{-1}^1 f(t) (y'(t) + ix'(t)) e^{-Ky(t)-iKx(t)} dt; \quad (41)$$

These formulae may simplify for particular geometries, once the expansion (32) has been made for f .

It is known that \mathcal{R} and \mathcal{T} satisfy certain relations, for any scatterer. Let \mathcal{R}_+ and \mathcal{T}_+ be the reflection and transmission coefficients when the incident wave propagates towards $x = +\infty$. Similarly, define \mathcal{R}_- and \mathcal{T}_- for incident waves propagating towards $x = -\infty$. Then

$$\begin{aligned}\mathcal{T}_+ &= \mathcal{T}_- = \mathcal{T}, & \text{say,} \\ |\mathcal{R}_+| &= |\mathcal{R}_-| = |\mathcal{R}|, & \text{say,} \\ |\mathcal{R}|^2 + |\mathcal{T}|^2 &= 1 & \text{and} & \quad \mathcal{T}\bar{\mathcal{R}}_+ + \bar{\mathcal{T}}\mathcal{R}_- = 0,\end{aligned}$$

where the overbar denotes complex conjugation. These relationships are well known [27] and can be used as an independent check on the method of solution employed.

9 Results

The method developed above has been used for several problems. Some of the results obtained are described next.

9.1 Submerged flat plate

For a submerged flat plate of length $2a$, inclined at an arbitrary angle α to the vertical, we can take

$$x(t) = at \sin \alpha, \quad y(t) = d + at \cos \alpha, \quad -1 \leq t \leq 1,$$

where $|\alpha| \leq \pi/2$; d is the submergence of the mid-point of the plate, and satisfies $d > a \cos \alpha$ to ensure that the plate is completely submerged. It follows that $\mathbf{n}(p) = \mathbf{n}(q) = (-\cos \alpha, \sin \alpha)$, and then eqn (25) simplifies to

$$M(s, t) = a^2 \mathcal{K}(X, Y),$$

where $X = a(s - t) \sin \alpha$ and $Y = a(s + t) \cos \alpha + 2d$. Moreover, if the incident potential is given by eqn (37), then

$$p(s) = 2\pi i K a \exp \{ -K(d + as \cos \alpha) + i(Kas \sin \alpha - \alpha) \}.$$

The formulae for the reflection and transmission coefficients, eqn (40) and eqn (41), respectively, simplify:

$$\begin{aligned}\mathcal{R} &= K a e^{-Kd - i\alpha} \sum_{n=0}^N a_n [\bar{L}_n - \bar{L}_{n+2}], \\ \mathcal{T} - 1 &= -K a e^{-Kd + i\alpha} \sum_{n=0}^N a_n [L_n - L_{n+2}],\end{aligned}$$

where $L_n = (\pi/2)(-1)^n I_n(Kae^{i\alpha})$ and I_n is a modified Bessel function.

In [28], we have computed $|\mathcal{R}|$ for a submerged vertical plate (using $N = 15$), and compared our solution with the exact analytic solution obtained by Evans [10]. Excellent agreement was found.

We are not aware of any previous numerical results for a submerged horizontal plate, in deep water. However, there are results for water of finite depth, due to Patarapanich [15] and McIver [16]. Again, we found good agreement with their results when their water is sufficiently deep [28]. We remark that the main feature of the results for horizontal plates is the occurrence of zeros of the reflection coefficient as a function of frequency.

We have also obtained results for plates inclined at various angles [28]. In particular, we have considered plates inclined at small angles to the horizontal, δ . We found that the zeros of \mathcal{R} for horizontal plates ($\delta = 0$) disappear as soon as δ becomes positive. Thus, these zeros for horizontal plates probably cannot be exploited in practice, for they are destroyed by small changes in the angle of inclination.

9.2 Submerged curved plate

For a submerged curved plate, in the shape of a circular arc of radius b , we can take the parametrisation as

$$x(t) = b \sin t\vartheta, \quad y(t) = d + b - b \cos t\vartheta, \quad -1 \leq t \leq 1.$$

Thus, the plate length is $2a = 2b\vartheta$, where $|\vartheta| < \pi$, and d is the submergence of the mid-point of the plate. It follows that

$$\begin{aligned} \mathbf{n}(p) &= (n_1^p, n_2^p) = (-\sin s\vartheta, \cos s\vartheta), \\ \mathbf{n}(q) &= (n_1^q, n_2^q) = (-\sin t\vartheta, \cos t\vartheta), \\ X &= b(\sin s\vartheta - \sin t\vartheta), \quad Y = 2d + 2b - b(\cos s\vartheta + \cos t\vartheta), \\ \mathcal{N} &= \cos(s-t)\vartheta, \quad n_1^p n_2^q - n_2^p n_1^q = -\sin(s-t)\vartheta, \\ \mathbf{R} &= b(\sin s\vartheta - \sin t\vartheta, \cos t\vartheta - \cos s\vartheta), \\ \Theta &= -b^2 [1 - \cos(s-t)\vartheta]^2, \quad R^2 = 2b^2 [1 - \cos(s-t)\vartheta]. \end{aligned}$$

Hence, the kernel M is given by

$$\begin{aligned} M(s, t) &= \frac{\vartheta^2}{4} \left(\frac{1}{\sin^2 [(s-t)\vartheta/2]} - \frac{4}{(s-t)^2 \vartheta^2} \right) \\ &\quad + 2b^2 \vartheta^2 \left\{ K \frac{\partial \Phi_0}{\partial X} - \frac{XY}{(X^2 + Y^2)^2} \right\} \sin [(s-t)\vartheta] \\ &\quad + b^2 \vartheta^2 \mathcal{K}(X, Y) \cos [(s-t)\vartheta]. \end{aligned}$$

If the incident potential is given by eqn (37), the function p on the right-hand side of the integral equation (24) is given by

$$p(s) = 2\pi b \vartheta K e^{-Kd + is\vartheta} \exp \left[-Kb (1 - e^{is\vartheta}) \right].$$

In [29], we have computed $|\mathcal{R}|$ as a function of Ka for various ϑ . Comparisons with our previous results for horizontal flat plates [28] (this corresponds to letting $\vartheta \rightarrow 0$ whilst keeping a non-zero) showed that there can be abrupt changes in the value of $|\mathcal{R}|$ for small

changes in ϑ . Increasing ϑ further (for the same value of d/a) enabled us to approach the geometry of a closed circular cylinder; this corresponds to fixing the length of the plate and then bending it until it forms a circle. We found that $|\mathcal{R}|$ almost vanished in this limit, for *all* frequencies. This behaviour is expected, as Dean [30] first showed that $\mathcal{R} \equiv 0$ for a submerged circular cylinder at *any* frequency. This result was subsequently proved by Ursell, [31] using a rigorous argument. We have also obtained results by fixing the radius b , and then varying ϑ ; thus, the plate length increases and also closes up to a circular cylinder as ϑ is increased. Again, we observed the onset of zero reflection as the plate approaches a complete circle.

9.3 Surface-piercing flat plate

This problem is more difficult, because of the free-surface intersection. It is still governed by the same hypersingular integral equation for $[\phi]$, eqn (17), and $[\phi]$ has the same square-root zero at the submerged end of the plate. However, where the plate meets the free surface, there is more ambiguity. For example, it may be possible to permit $[\phi]$ to have a logarithmic singularity at the free surface. This may be used as a model for wave breaking [32]. We shall not pursue this course here, but will assume that $[\phi]$ approaches a constant as we approach the free surface; this is in accord with the exact solution of Ursell [7] for a vertical plate. It is worth remarking that similar difficulties arise in other fields, such as when a crack meets the traction-free surface of an elastic solid [33].

We parametrise the geometry in a slightly different way, using

$$x(t) = a(2t - 1) \sin \alpha, \quad y(t) = 2at \cos \alpha, \quad 0 \leq t \leq 1,$$

where $2a$ is the length of the plate and α is the angle made by the plate to the vertical. Thus, $t = 0$ corresponds to the point where the plate meets the free surface. We assume that $|\alpha| < \pi/2$, so that the plate does not lie *in* the free surface; the problem of scattering by a plate lying in the free surface is known as the *dock problem*. With this parametrisation, we find that eqn (17) can be written as

$$\oint_0^1 \frac{f(t)}{(s-t)^2} dt + 4a^2 \int_0^1 f(t) \mathcal{K}(X, Y) dt = p(s), \quad 0 < s < 1, \quad (42)$$

where \mathcal{K} is defined by eqn (23), $X = 2a(s-t) \sin \alpha$, $Y = 2a(s+t) \cos \alpha$ and

$$p(s) = 8\pi i K a \exp[-2Kas \cos \alpha + i(Ka(2s-1) \sin \alpha - \alpha)].$$

The kernel can be simplified [29]; the result is that eqn (42) can be rewritten as

$$\begin{aligned} \oint_0^1 \frac{f(t)}{(s-t)^2} dt &+ \frac{e^{2i\alpha}}{2} \int_0^1 \frac{f(t)}{(z+t)^2} dt + \frac{e^{-2i\alpha}}{2} \int_0^1 \frac{f(t)}{(\bar{z}+t)^2} dt \\ &+ 2Kae^{i\alpha} \int_0^1 \frac{f(t)}{z+t} dt + 2Kae^{-i\alpha} \int_0^1 \frac{f(t)}{\bar{z}+t} dt \\ &+ 8(Ka)^2 \int_0^1 f(t) \Phi_0(X, Y) dt = p(s), \end{aligned} \quad (43)$$

where $z = se^{2i\alpha}$ and $\bar{z} = se^{-2i\alpha}$. In view of the restriction on α ($|\alpha| < \pi/2$), we see that $-\pi < \arg z < \pi$. From eqn (43), we can see that as we collocate towards the free surface, the four integrals involving z and \bar{z} become singular; nevertheless, they can be calculated accurately using certain recurrence relations given below. The singular behaviour at $s = 0$ should also be compared with the regular behaviour of $p(s)$; the left-hand side of eqn (43) will only be regular at $s = 0$ if $f(t)$ has a certain particular behaviour as $t \rightarrow 0$. Thus, it is known that, for a plate of unit length, $f(t)$ has the asymptotic expansion [34]

$$f(t) \sim f_0(1 - Kt \sec \alpha) + f_\delta t^\delta + \frac{1}{4}f_0 K^2 t^2 \sec^2 \alpha \quad \text{as } t \rightarrow 0, \quad (44)$$

where f_0 and f_δ are constants and $\delta = 2\pi/(\pi + 2\alpha)$. The approximation (44) was derived for $|\alpha| < \pi/2$, $\alpha \neq 0$; for $\alpha = 0$ (the vertical plate) we have [23]

$$f(t) \sim f_0(1 - Kt) + (2/\pi)(Kv_0 + v_1)t^2 \log t + f_2 t^2 \quad \text{as } t \rightarrow 0,$$

where f_0 , v_0 , v_1 and f_2 are all constants (the logarithmic term is absent for a regular incident wave).

What is the best way of dealing with this problem? Is it possible to incorporate the known behaviour of $f(t)$, eqn (44), into the solution of the problem? This would give a set of constraints to be imposed on the unknown coefficients in any Chebyshev expansion used. However, this would also change the behaviour of the solution away from the point $t = 0$, and as such may not be satisfactory. In practice, it was decided not to try and impose any restriction on $f(t)$, other than the square-root behaviour at $t = 1$ and the boundedness at $t = 0$. It turns out that this approach is both simple and effective, probably because the singularity is rather weak (its effects are captured adequately by the expansion-collocation procedure). Thus, we used an identical expansion for the discontinuity in potential across the plate as that used previously for submerged plates, namely eqn (32); at the free-surface intersection point ($t = 0$), f has a constant value,

$$f(0) = \sum_{n=0}^{[N/2]} a_{2n} (-1)^n,$$

where $[x]$ represents the integer part of x . Substituting the expansion for $f(t)$ into eqn (42) gives

$$\sum_{n=0}^N a_n A_n(s) = p(s), \quad 0 < s < 1,$$

where

$$\begin{aligned} A_n(s) = & \mathcal{L}_n(s) + \frac{1}{2}e^{2i\alpha} \mathcal{L}_n(-z) + \frac{1}{2}e^{-2i\alpha} \mathcal{L}_n(-\bar{z}) \\ & - 2Ka e^{i\alpha} \mathcal{I}_n(-z) - 2Ka e^{-i\alpha} \mathcal{I}_n(-\bar{z}) \\ & + 8(Ka)^2 \int_0^1 \sqrt{1-t^2} U_n(t) \Phi_0(X, Y) dt, \end{aligned} \quad (45)$$

$$\mathcal{I}_n(z) = \int_0^1 \frac{\sqrt{1-t^2} U_n(t)}{z-t} dt \quad (46)$$

$$\mathcal{L}_n(z) = \int_0^1 \frac{\sqrt{1-t^2} U_n(t)}{(z-t)^2} dt; \quad (47)$$

We evaluate the integrals (46) and (47) using recurrence relations which are readily derived from a known recurrence relation for $U_n(t)$. Thus,

$$\mathcal{I}_{n+1} = -\mathcal{I}_{n-1} + 2z\mathcal{I}_n - \left(\frac{1}{n} + \frac{1}{n+2}\right) \sin\left(\frac{n\pi}{2}\right), \quad \text{for } n \geq 1,$$

where $\mathcal{I}_1 = 2z\mathcal{I}_0 - \frac{1}{2}\pi$ and

$$\mathcal{I}_0 = 1 + \frac{z\pi}{2} + \sqrt{1-z^2} \log\left(\frac{z}{1+\sqrt{1-z^2}}\right).$$

Similarly, \mathcal{L}_n can be determined from \mathcal{I}_n , using

$$\mathcal{L}_{n+1} = -\mathcal{L}_{n-1} + 2z\mathcal{L}_n - 2\mathcal{I}_n, \quad \text{for } n \geq 1,$$

with $\mathcal{L}_1 = 2z\mathcal{L}_0 - 2\mathcal{I}_0$ and

$$\mathcal{L}_0 = -\frac{\pi}{2} + \frac{z}{\sqrt{1-z^2}} \log\left(\frac{z}{1+\sqrt{1-z^2}}\right) - \frac{1}{z}.$$

The remaining integral in eqn (45) is well behaved as $s \rightarrow 0$, and is evaluated using the expansion for $\Phi_0(X, Y)$, eqn (9).

It remains to specify the collocation points. Previously, we used the collocation points s_j defined by eqn (36); these points lie in the range $-1 < s_j < 1$. Now, we require collocation points in the range $0 < s_j < 1$. Therefore, we apply the simple transformation $s_j \rightarrow (s_j + 1)/2$ to eqn (36), giving us the points

$$s_j = \frac{1}{2} \left\{ \cos\left(\frac{(2j+1)\pi}{2N+2}\right) + 1 \right\}, \quad j = 0, 1, \dots, N.$$

With this choice, we obtain the linear system of equations (34).

We have used the method outlined above to compute the reflection and transmission coefficients, for various angles of inclination to the vertical, α [29]. We found excellent agreement with Ursell's exact solution for a vertical barrier ($\alpha = 0$) [7].

We have also explored what happens as the plate becomes horizontal ($\alpha \rightarrow \pi/2$), so that there is a narrow wedge-shaped region above the plate: we found that the graph of $|\mathcal{R}|$ against Ka is composed of a series of spikes superimposed on an underlying smooth curve. The latter was identified as the reflection coefficient for a finite dock (where the plate lies *in* the free surface), whereas the spikes are clearly due to a resonance effect. (We solved the dock problem, numerically, by solving a well-known Fredholm integral equation of the second kind (with a logarithmic kernel) for the boundary values of ϕ_{sc} on the dock [34], [35].) It should be possible to give an asymptotic analysis of this interesting phenomenon, but we have not pursued this.

9.4 Trapping by submerged plates

Surface water waves can be trapped by a thin plate submerged in deep water. The corresponding boundary-value problem is similar to that for scattering by the plate; the differences

are: Laplace's equation is replaced by the modified Helmholtz equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - l^2 \phi = 0, \quad (48)$$

where l is a positive constant; the boundary condition on the plate is homogeneous,

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma;$$

and ϕ is required to vanish (exponentially) as $|x| \rightarrow \infty$. In general, this homogeneous problem has only the trivial solution. However, it does have non-trivial solutions for certain choices of the parameters; in particular, it is necessary that $l > K = \omega^2/g$. The search for these solutions can be reduced to finding non-trivial solutions of a homogeneous, hypersingular integral equation for $[\phi]$. This equation is precisely eqn (17), except that the right-hand side is identically zero and a different fundamental solution G is required; the appropriate singular solution of eqn (48) is [36]

$$G(P, Q) \equiv G(x, y; \xi, \eta) = K_0 \left(l \sqrt{(x - \xi)^2 + (y - \eta)^2} \right) + G_1(x - \xi, y + \eta)$$

where K_0 is a modified Bessel function,

$$\begin{aligned} G_1(X, Y) &= K_0 \left(l \sqrt{X^2 + Y^2} \right) + 2 \cot \beta \mathcal{G}(X, Y), \\ \mathcal{G} &= \int_0^\infty \frac{\sin \beta}{\cosh \mu - \cos \beta} e^{-lY \cosh \mu} \cos(lX \sinh \mu) d\mu \end{aligned}$$

and, since $l > K$, we have introduced a new variable β , defined by $K = l \cos \beta$.

We have used the expansion-collocation method to solve the hypersingular integral equation, for plates of various shapes. The method yields a homogeneous matrix equation for the unknown coefficients; a non-trivial solution is then implied by the vanishing of the determinant. Our work [37] was motivated by a paper of Linton & Evans [38]. They considered a submerged, horizontal, flat plate in water of finite depth, and used a numerical method based on matched eigenfunction expansions. They found strong numerical evidence for the existence of trapped modes above such a plate. In [37], we assumed deep water (this assumption could be relaxed by using an appropriate G), but allowed the plate to have different orientations; we also considered curved plates. We were able to confirm the results of Linton & Evans [38] for horizontal flat plates; various modes were found. Then, we followed these modes as the geometry was altered, either by rotating the plate or deforming the plate into the arc of a circle. In particular, as the arc approached a complete circle, we found agreement with the corresponding modes computed by McIver & Evans [39] for a circular cylinder (they used a multipole method, as devised by Ursell [36]). This agreement at both limits of the deformation (horizontal flat plate and solid circular cylinder), using three different numerical methods, suggests that the numerical results obtained using the hypersingular integral equation are correct.

We remark that the scattering problem for waves at oblique incidence to the plate can be solved in a similar way; this problem has $l < K$.

10 Three dimensions

Three-dimensional problems, in which water waves interact with thin plates, are also of interest. The simplest problems involve circular discs. Dock problems, where the disc is in the free surface, have been treated by a number of authors: [40]–[45]. Such problems can be reduced to Fredholm integral equations of the second kind over the wetted surface of the disc.

If the circular disc, Ω , is submerged, we can reduce the scattering or radiation problems to hypersingular integral equations over Ω . To formulate the basic scattering problem, it is convenient to introduce cartesian coordinates $Oxyz$, where z is directed vertically downwards so that $z = 0$ is the mean free surface. Then, with a given incident potential ϕ_{inc} , the scattered potential ϕ_{sc} must satisfy

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi_{\text{sc}} &= 0 \quad \text{in the fluid,} \\ K\phi_{\text{sc}} + \frac{\partial\phi_{\text{sc}}}{\partial z} &= 0 \quad \text{on } z = 0, \text{ and} \\ \frac{\partial\phi_{\text{sc}}}{\partial n} &= -\frac{\partial\phi_{\text{inc}}}{\partial n} \quad \text{on } \Omega; \end{aligned} \tag{49}$$

in addition, ϕ_{sc} must satisfy a radiation condition and be bounded everywhere.

The basic ingredient is the three-dimensional wave source, defined by

$$G(P, Q) \equiv G(x, y, z; \xi, \eta, \zeta) = \frac{1}{\sqrt{R^2 + (z - \zeta)^2}} + G_1(x - \xi, y - \eta, z + \zeta),$$

where $R = \sqrt{(x - \xi)^2 + (y - \eta)^2}$,

$$G_1 = \frac{1}{\sqrt{R^2 + (z + \zeta)^2}} + 2K \int_0^\infty e^{-k(z+\zeta)} J_0(kR) \frac{dk}{k - K}$$

and J_0 is a Bessel function. Then, an application of Green's theorem gives the integral representation

$$\phi_{\text{sc}}(P) = \frac{1}{4\pi} \int_{\Omega} [\phi(q)] \frac{\partial G(P, q)}{\partial n_q^+} ds_q$$

for ϕ_{sc} as a distribution of normal dipoles over Ω . Application of the boundary condition (49) yields

$$\frac{1}{4\pi} \frac{\partial}{\partial n_p} \int_{\Omega} [\phi(q)] \frac{\partial G(p, q)}{\partial n_q} ds_q = -\frac{\partial\phi_{\text{inc}}}{\partial n_p}, \quad p \in \Omega,$$

an integro-differential equation for $[\phi(q)]$, $q \in \Omega$. It is to be solved subject to the edge condition

$$[\phi] = 0 \quad \text{on } \partial\Omega, \tag{50}$$

where $\partial\Omega$ is the boundary of Ω .

Proceeding as in the two-dimensional case, we interchange the order of normal differentiation at p and integration over Ω to give

$$\frac{1}{4\pi} \rlap{-}\int_{\Omega} [\phi(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p, q) ds_q = -\frac{\partial \phi_{\text{inc}}}{\partial n_p}, \quad p \in \Omega, \quad (51)$$

which is to be solved for $[\phi]$, subject to eqn (50). The integral is to be interpreted as a finite-part integral.

10.1 Finite-part integrals

The hypersingular integral in eqn (51) can be defined in several equivalent ways. Let Ω be a bounded region in the xy -plane. Then, for a sufficiently smooth function w (we need $w \in C^{1,\alpha}$), one natural definition in the context of boundary-value problems is

$$\rlap{-}\int_{\Omega} w(\xi, \eta) \frac{d\Omega}{R^3} = \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \int_{\Omega} w(\xi, \eta) \left\{ \lim_{\zeta \rightarrow 0} \frac{\partial}{\partial \zeta} \left(\frac{1}{\sqrt{R^2 + (z - \zeta)^2}} \right) \right\} d\Omega,$$

where $d\Omega = d\xi d\eta$. Another (cf. eqn (19)) is

$$\rlap{-}\int_{\Omega} w(\xi, \eta) \frac{d\Omega}{R^3} = \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\Omega \setminus \Omega_{\varepsilon}} w(\xi, \eta) \frac{d\Omega}{R^3} - \frac{2\pi w(x, y)}{\varepsilon} \right\},$$

where Ω_{ε} is a small disc of radius ε centred at the singular point (x, y) . For more information on hypersingular integrals over surfaces, see [46] and [47].

11 Potential flow past a flat circular disc

So far, we have not exploited the fact that Ω is a flat circular disc; indeed, the hypersingular integral equation (51) is valid when Ω is any smooth open bounded surface. We could solve the integral equation, numerically, using a boundary element method. However, motivated by the efficacy of the expansion-collocation method for one-dimensional integral equations, we seek to develop a similar method for two-dimensional equations.

We start by assuming that Ω is a circular disc of radius a . Moreover, we concentrate on the hypersingularity by ignoring free-surface effects, so that, physically, we consider potential flow past the disc. Thus, for a disc in the xy -plane, the governing equation is

$$\frac{1}{4\pi} \rlap{-}\int_{\Omega} w(\xi, \eta) \frac{d\Omega}{R^3} = p(x, y), \quad (x, y) \in \Omega,$$

where $w = [\phi]$ and $p(x, y)$ is known; it is to be solved subject to $w = 0$ on $\partial\Omega$. Introduce plane polar coordinates (r, θ) , so that $x = r \cos \theta$, $y = r \sin \theta$ and

$$\Omega = \{(r, \theta) : 0 \leq r < a, -\pi \leq \theta < \pi\}.$$

Suppose, for simplicity, that the incident flow is symmetric about $\theta = 0$ so that we can write

$$p(x, y) = \sum_{n=0}^{\infty} p_n(r/a) \cos n\theta.$$

Then, w has a similar expansion,

$$w(x, y) = a \sum_{n=0}^{\infty} w_n(r/a) \cos n\theta.$$

It is known that w_n and p_n are related by the formula

$$w_n(r) = -\frac{4}{\pi} r^n \int_r^1 \frac{1}{t^{2n} \sqrt{t^2 - r^2}} \int_0^t \frac{p_n(s) s^{n+1}}{\sqrt{t^2 - s^2}} ds dt \quad (52)$$

for $n = 0, 1, 2, \dots$; see [48]. This formula simplifies if we expand p_n as

$$p_n(r) = r^n \sum_{j=0}^{\infty} S_j^n \frac{C_{2j+1}^{n+1/2}(\sqrt{1-r^2})}{\sqrt{1-r^2}},$$

where the coefficients S_j^n are known and $C_m^\lambda(x)$ is a Gegenbauer polynomial of degree m and index λ [49, §10.9]; these polynomials are orthogonal and satisfy

$$\int_0^1 \frac{r^{2m+1}}{\sqrt{1-r^2}} C_{2j+1}^{m+1/2}(\sqrt{1-r^2}) C_{2k+1}^{m+1/2}(\sqrt{1-r^2}) dr = h_j^m \delta_{jk},$$

where δ_{ij} is the Kronecker delta and h_j^m is a known constant. It follows from eqn (52) that

$$w_n(r) = r^n \sum_{j=0}^{\infty} W_j^n C_{2j+1}^{n+1/2}(\sqrt{1-r^2}),$$

where the coefficients W_j^n are given by

$$W_j^n = -S_j^n \frac{(n+j)! j!}{\Gamma(n+j+3/2) \Gamma(j+3/2)}.$$

This result was obtained by Krenk [50]; see also [51]. It can also be expressed in terms of associated Legendre functions or in terms of Jacobi polynomials; see [52] for further references.

The formulae above suggest introducing the functions

$$U_j^n(r, \theta) = r^n C_{2j+1}^{n+1/2}(\sqrt{1-r^2}) \cos n\theta.$$

These functions are orthogonal over the unit disc with respect to the weight $(1-r^2)^{-1/2}$; they also have square-root zeros at $r = 1$. Moreover, Krenk's formulae show that

$$\frac{1}{4\pi} \oint_{\Omega} U_j^n(\rho/a, \varphi) \frac{d\Omega}{R^3} = -A_j^n \frac{U_j^n(r/a, \theta)}{\sqrt{a^2 - r^2}}, \quad (53)$$

where $\xi = \rho \cos \varphi$, $\eta = \rho \sin \varphi$ and

$$A_j^n = \frac{\Gamma(n+j+3/2) \Gamma(j+3/2)}{(n+j)! j!}.$$

Equation (53) can be viewed as the two-dimensional analogue of eqn (31); it permits the analytical evaluation of the hypersingular integral on the left-hand side.

12 Scattering by a submerged disc

Let us now return to a water-wave problem. We consider a circular disc, submerged beneath the free surface of deep water. For simplicity, we assume that the disc is parallel to the free surface (for water of finite depth, this problem can be solved using matched eigenfunction expansions [53]). The governing integral equation is eqn (51), which we write as

$$\frac{1}{4\pi} \oint_{\Omega} w(\xi, \eta) \frac{d\xi d\eta}{R^3} + \frac{1}{4\pi} \int_{\Omega} w(\xi, \eta) M(x, y; \xi, \eta) d\xi d\eta = p(x, y) \quad (54)$$

for $(x, y) \in \Omega$, where

$$p(x, y) = -\frac{\partial \phi_{\text{inc}}}{\partial n_p} \quad \text{and} \quad M(x, y; \xi, \eta) = \frac{\partial^2}{\partial n_p \partial n_q} G_1(p, q).$$

Equation (54) is to be solved subject to $w = 0$ on $r = a$. To solve it, we expand w as

$$w(x, y) \cong a \sum_{n=0}^N \sum_{j=0}^J W_j^n U_j^n(r/a, \theta),$$

substitute into eqn (54), evaluate the hypersingular integral over Ω using eqn (53), and then collocate at $(N + 1)(J + 1)$ points on the disc; this gives an algebraic system for the coefficients W_j^n . This method has been used by Farina [54] to solve several problems involving circular discs.

13 Scattering by arbitrary flat plates

The expansion-collocation method described above is only appropriate for flat circular discs. This is in contradistinction to the situation in two-dimensions: there, a curved plate can always be parametrised so as to give a one-dimensional integral equation over a finite interval.

In order to use an expansion-collocation method in three dimensions, we map the plate onto a circular disc. However, we are not at liberty to choose any convenient mapping: typically, the mapping will modify the hypersingularity in an essential way, and this will prevent us from using the formula (53). We must use a *conformal* mapping of the plate onto a disc. This preserves the structure of the hypersingularity, allowing the use of the Fourier-Gegenbauer expansion method on the transformed integral equation. This new method is described elsewhere [55], [56].

The method requires an appropriate conformal mapping; such mappings are catalogued in [57]. In particular, the method could be applied to problems involving *square* plates; the mapped integral equation will be over a circular domain but the kernel will have ‘fixed singularities’ at four points on the circumference. One could also use a numerical method to find the conformal mapping; for example, this can be achieved by solving a certain boundary integral equation over $\partial\Omega$ [58].

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References

- [1] Frenkel, A. A Chebyshev expansion of singular integrodifferential equations with a $\partial^2 \ln |s - t|/\partial s \partial t$ kernel, *J. Comp. Phys.*, 1983, **51**, 335–342.
- [2] Kaya, A.C. & Erdogan, F. On the solution of integral equations with strongly singular kernels, *Q. Appl. Math.*, 1987, **45**, 105–122.
- [3] Golberg, M.A. The convergence of several algorithms for solving integral equations with finite-part integrals, *J. Integ. Eqns.*, 1983, **5**, 329–340.
- [4] Golberg, M.A. The convergence of several algorithms for solving integral equations with finite-part integrals. II, *J. Integ. Eqns.*, 1985, **9**, 267–275.
- [5] Ervin, V.J. & Stephan, E.P. Collocation with Chebyshev polynomials for a hypersingular integral equation on an interval, *J. Comp. & Appl. Math.*, 1992, **43**, 221–229.
- [6] Sobhani, S.M., Lee, J.J. & Wellford, L.C. Interaction of periodic waves with inclined portable barrier, *J. Waterway, Port, Coastal, & Ocean Engng.*, 1988, **114**, 745–761.
- [7] Ursell, F. The effect of fixed vertical barrier on surface waves in deep water, *Proc. Camb. Phil. Soc.*, 1947, **43**, 374–382.
- [8] Havelock, T.H. Forced surface-waves on water, *Phil. Mag.*, 1929, **8**, 569–576.
- [9] John, F. Waves in the presence of an inclined barrier, *Comm. Appl. Math.*, 1948, **1**, 149–200.
- [10] Evans, D.V. Diffraction of water-waves by a submerged vertical plate, *J. Fluid Mech.*, 1970, **40**, 433–451.
- [11] Burke, J.E. Scattering of surface waves on an infinitely deep fluid, *J. Math. Phys.*, 1964, **5**, 805–819.
- [12] Shaw, D.C. Perturbational results for diffraction of water-waves by nearly-vertical barriers, *IMA J. Appl. Math.*, 1985, **34**, 99–117.
- [13] Mandal, B.N. & Chakrabarti, A. A note on diffraction of water waves by a nearly vertical barrier, *IMA J. Appl. Math.*, 1989, **43**, 157–165.
- [14] Mandal, B.N. & Banerjea, S. A note on waves due to rolling of a partially immersed nearly vertical barrier, *SIAM J. Appl. Math.*, 1991, **51**, 930–939.
- [15] Patarapanich, M. Forces and moments on a horizontal plate due to wave scattering, *Coastal Engng.*, 1984, **8**, 279–301.
- [16] McIver, M. Diffraction of water waves by a moored, horizontal, flat plate, *J. Engng. Math.*, 1985, **19**, 297–319.
- [17] Liu, P.L.F. & Abbaspour, M. Wave scattering by a rigid thin barrier, *J. Waterway, Port, Coastal & Ocean Div.*, ASCE, 1982, **108**, 479–491.
- [18] Hamilton, W.S. Forces exerted by waves on a sloping board, *Trans. Amer. Geophys. Union*, 1950, **31**, 849–855.

- [19] Yu, Y.S. & Ursell, F. Surface waves generated by an oscillating circular cylinder on shallow water: theory and experiment, *J. Fluid Mech.*, 1961, **11**, 529–551.
- [20] Martin, P.A. & Rizzo, F.J. On boundary integral equations for crack problems, *Proc. Roy. Soc. A*, 1989, **421**, 341–355.
- [21] Martin, P.A. & Rizzo, F.J. Hypersingular integrals: how smooth must the density be?, *Int. J. Numer. Meth. Engng.*, 1996, **39**, 687–704.
- [22] Smirnov, V.I. *A Course of Higher Mathematics*, vol. IV, Pergamon, Oxford, 1964.
- [23] Martin, P.A. End-point behaviour of solutions to hypersingular integral equations, *Proc. Roy. Soc. A*, 1991, **432**, 301–320.
- [24] Martin, P.A. Exact solution of a simple hypersingular integral equation, *J. Integ. Eqns. & Applics.*, 1992, **4**, 197–204.
- [25] Abramowitz, M. & Stegun, I.A. (eds.) *Handbook of Mathematical Functions*, Dover, New York, 1965.
- [26] Atkinson, K.E. *An Introduction to Numerical Analysis*, 2nd Ed., Wiley, New York, 1989.
- [27] Martin, P.A. On the T -matrix for water-wave scattering problems, *Wave Motion*, 1985, **7**, 177–193.
- [28] Parsons, N.F. & Martin, P.A. Scattering of water waves by submerged plates using hypersingular integral equations, *Appl. Ocean Res.*, 1992, **14**, 313–321.
- [29] Parsons, N.F. & Martin, P.A. Scattering of water waves by submerged curved plates and by surface-piercing flat plates, *Appl. Ocean Res.*, 1994, **16**, 129–139.
- [30] Dean, W.R. On the reflexion of surface waves by a submerged, circular cylinder, *Proc. Camb. Phil. Soc.*, 1948, **44**, 483–491.
- [31] Ursell, F. Surface waves on deep water in the presence of a submerged circular cylinder. I, *Proc. Camb. Phil. Soc.*, 1950, **46**, 141–152.
- [32] Stoker, J.J. *Water Waves*, Interscience, New York, 1957.
- [33] Dewynne, J.N., Hills, D.A. & Nowell, D. Calculation of the opening displacement of surface-breaking plane cracks, *Comp. Meth. in Appl. Mech. & Engng.*, 1992, **97**, 321–331.
- [34] Parsons, N.F. The interaction of water waves with thin plates, PhD thesis, University of Manchester, 1994.
- [35] MacCamy, R.C. On the heaving motion of cylinders of shallow draft, *J. Ship Res.*, 1961 **5**(3), 34–43.
- [36] Ursell, F. Trapping modes in the theory of surface waves, *Proc. Camb. Phil. Soc.*, 1951, **47**, 347–358.
- [37] Parsons, N.F. & Martin, P.A. Trapping of water waves by submerged plates using hypersingular integral equations, *J. Fluid Mech.*, 1995, **284**, 359–375.

- [38] Linton, C.M. & Evans, D.V. Trapped modes above a submerged horizontal plate. *Quart. J. Mech. Appl. Math.*, 1991, **44**, 487–506.
- [39] McIver, P. & Evans, D.V. The trapping of surface waves above a submerged, horizontal cylinder, *J. Fluid Mech.*, 1985, **151**, 243–255.
- [40] MacCamy, R.C. On the scattering of water waves by a circular disk, *Arch. Rat. Mech. Anal.*, 1961, **8**, 120–138.
- [41] Kim, W.D. The pitching motion of a circular disk, *J. Fluid Mech.*, 1963, **17**, 607–629.
- [42] Miles, J. & Gilbert, F. Scattering of gravity waves by a circular dock, *J. Fluid Mech.*, 1968, **34**, 783–793.
- [43] Garrett, C.J.R. Wave forces on a circular dock, *J. Fluid Mech.*, 1971, **46**, 129–139.
- [44] Miles, J.W. A note on variational principles for surface-wave scattering, *J. Fluid Mech.*, 1971, **46**, 141–149.
- [45] Miles, J.W. On surface-wave forcing by a circular disk, *J. Fluid Mech.*, 1987, **175**, 97–108.
- [46] Krishnasamy, G., Schmerr, L.W., Rudolphi, T.J. & Rizzo, F.J. Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering, *J. Appl. Mech.*, 1990, **57**, 404–414.
- [47] Krishnasamy, G., Rizzo, F.J. & Rudolphi, T.J. Continuity requirements for density functions in the boundary integral equation method, *Computational Mech.*, 1992, **9**, 267–284.
- [48] Guidera, J.T. & Lardner, R.W. Penny-shaped cracks, *J. Elast.*, 1975, **5**, 59–73.
- [49] Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F.G. *Higher Transcendental Functions*, vol. 2, McGraw-Hill, New York, 1953.
- [50] Krenk, S. A circular crack under asymmetric loads and some related integral equations, *J. Appl. Mech.*, 1979, **46**, 821–826.
- [51] Martin, P.A. The discontinuity in the elastostatic displacement vector across a penny-shaped crack under arbitrary loads, *J. Elast.*, 1982, **12**, 201–218.
- [52] Martin, P.A. Orthogonal polynomial solutions for pressurized elliptical cracks, *Quart. J. Mech. Appl. Math.*, 1986, **39**, 269–287.
- [53] Yu, X. & Chwang, A.T. Analysis of wave scattering by submerged circular disk, *J. Eng. Mech.*, 1993, **119**, 1804–1817.
- [54] Farina, L. Interaction of water waves with thin circular plates, PhD thesis, University of Manchester, 1996.
- [55] Martin, P.A. Mapping flat cracks onto penny-shaped cracks, with application to somewhat circular tensile cracks, *Q. Appl. Math.*, 1996, **54**, 663–675.
- [56] Martin, P.A. Mapping flat cracks onto penny-shaped cracks: shear loadings, *J. Mech. Phys. Solids*, 1995, **43**, 275–294.

- [57] Kober, H. *Dictionary of Conformal Representations*, Dover, New York, 1957.
- [58] Gutknecht, M.H. Numerical conformal mapping methods based on function conjugation, *J. Comp. & Appl. Math.*, 1986, **14**, 31–77.