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Exact Green's functions and the boundary element method

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Abstract

Boundary value problems for linear elliptic partial differential equations may be solved by constructing an exact Green's function for the domain involved. Alternatively, an integral equation defined on the boundary of the domain, with unprescribed boundary data as an unknown, may be solved. It is easily argued that both approaches must be equivalent in the broadest sense. In this paper, the precise equivalence between an exact Green's function and the solution of the boundary integral equation is made explicit.

Introduction

It is well known, e.g. Kellogg [1], Webster [2], that an exact Green's function G^* exists and may be used, in principle, to construct the solution of a boundary value problem governed by a linear elliptic partial differential equation. Alternatively, the solution of the problem may be obtained via the boundary integral equation (BIE) formalism, where the BIE employs only the free-space Green's function G, or fundamental solution of the differential equation. Since it is evident that both approaches to the solution must be equivalent [3], one may conjecture that, using the BIE , one must have done the equivalent of constructing the exact Green's function G^*.

Indeed, in this paper (see also [4],[5],[6]) it is explicitly shown that G^* and the unknown boundary variable in the BIE method satisfy the same BIE, but with different right-hand sides. As a consequence, the representation integral for the BIE solution of the boundary value problem may be written in a form which contains a precise expression for G^*. The equivalence between the BIE process and constructing G^* is thus made explicit.

A number of ingredients in the boundary element method (BEM) may now be interpreted as numerical approximations to exact Green's functions. Some strategies for creating a library of such functions for repeated use are suggested.

Exact Green's Functions and the BIE Process

The essential aspects of the following arguments hold for linear elliptic boundary value problems (BVP's); however, to fix ideas, consider finding a time-harmonic acoustic field u which exists in the region D exterior to a single finite volume V with closed surface S. The field u satisfies the scalar wave (Helmholtz) equation in D and satisfies a radiation condition for indefinitely large distance R from V. On S we assume $\partial u/\partial n = f$ where f is a prescribed function.

A representation integral for u may be written

$$2u(P) = \int_S [f(q)G(q,P) - u(q)\frac{\partial}{\partial n_q}G(q,P)]dS_q \tag{1}$$

wherein

$$G(P,Q) = G(Q,P) = \frac{-e^{ikR}}{2\pi R} + w(P,Q) \tag{2}$$

is a Green's function with $R = |Q - P|$, where P,Q are arbitrary points in D and p,q are arbitrary points on S; k is the acoustic wavenumber, w is an arbitrary regular solution to the wave equation, and the normal n points into D at q. Equation (1) is readily obtained by inserting u and G into Green's reciprocal identity.

Now suppose that G in eqn (1) is an exact Green's function G^* defined such that

$$\frac{\partial G^*(q,P)}{\partial n_q} = 0 \quad \text{or} \quad \frac{\partial}{\partial n_q}(\frac{-e^{ikR}}{2\pi R}) + \frac{\partial w(P,q)}{\partial n_q} = 0 \tag{3}$$

whenever $q \in S$. Using G^* instead of G, representation (1) simplifies considerably to

$$2u(P) = \int_S f(q)G^*(q,P)dS_q \tag{4}$$

which is now the explicit solution to the posed BVP rather than a mere representation, if G^* is assumed known, since unprescribed $u(q)$ does not appear in (4).

Note that finding G^* is tantamount to finding w which satisfies the wave equation subject to the boundary condition (3). This task is comparable in difficulty to finding u itself subject to $\partial u/\partial n = f$. This is why, no doubt, the idea of an exact Green's function has not received more attention for practical problems.

Instead, the BIE/BEM has been the method of choice for many problems of the present type, and the method, in essence, proceeds as follows. Choose the simplest w in (2), namely $w = 0$, and take the limit in representation (1) as $P \rightarrow p$. The familiar result is the BIE

$$u(p) + \int_S u(q)\frac{\partial G(q,p)}{\partial n_q}dS_q = \int_S f(q)G(q,p)dS_q . \tag{5}$$

Symbolically, eqn (5) may be written

$$Au = Bf \tag{6}$$

where A and B are the indicated integral operators. The unknown function u on S may be obtained formally as the solution of the BIE, namely

$$u = A^{-1}Bf . \tag{7}$$

Thus, using (7) we may write the solution for $u(P)$ as

$$2u(P) = \int_S f(q)G(q,P)dS_q - \int_S \underbrace{A^{-1}Bf}_{u}(q)\frac{\partial G(q,P)}{\partial n_q}dS_q . \tag{8}$$

Comparing (8) with (4), we find two representations for the solution to our boundary value problem; (8) explicitly involves the inverse operator A^{-1} acting upon the function Bf, whereas, (4) explicitly involves the exact Green's function G^*.

To more closely see the equivalence between (8) and (4), it is instructive to reintroduce the integral form of the operator Bf into (8), and in the process interchange the order of the inner integration with the operation A^{-1}. The result is

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$$2u(P) = \int_S f(q)G(q,P)dS_q - \int_S f(q)\left\{\int_S A^{-1}G(q,l)\frac{\partial G(l,P)}{\partial n_l}dS_l\right\}dS_q \qquad (9)$$

where $l \in S$. Next, factoring out a common $f(q)$ we have

$$2u(P) = \int_S f(q)\left\{G(q,P) - \int_S A^{-1}G(q,l)\frac{\partial G(l,P)}{\partial n_l}dS_l\right\}dS_q. \qquad (10)$$

Now if (10) and (4) are both correct, the term in brackets in (10) must be G^*.

To see that the term in brackets is, in fact, G^*, apply Green's reciprocal theorem to G and G^*, to get

$$2G^*(P,Q) = 2G(P,Q) - \int_S G^*(l,P)\frac{\partial G(l,Q)}{\partial n_l}dS_l \qquad (11)$$

where we recall that $\partial G^*(l,P)/\partial n = 0$. Next take the limit in (11) as $Q \rightarrow s \in S$ to get (cf. Boley [5])

$$G^*(P,s) + \int_S G^*(l,P)\frac{\partial G(l,s)}{\partial n_l}dS_l = 2G(P,s) \qquad (12)$$

or symbolically

$$AG^* = 2G. \qquad (13)$$

From (12) (and (13)) and (6) (and (7)) we see that both u and G^* satisfy the same BIE with different right hand sides. Solving (13) for G^* as (cf. Tewary [6])

$$G^* = 2A^{-1}G \qquad (14)$$

and substituting under the integral sign in (11), we obtain, after interchanging P with Q (or q)

$$G^*(P,q) = G(P,q) - \int_S A^{-1}G(l,q)\frac{\partial G(l,P)}{\partial n_l}dS_l. \qquad (15)$$

Expression (15) for G^* is precisely that in brackets in equation (10) such that (10) and (4) are identical.

It is explicit, therefore, that in using the BIE method to solve a given boundary value problem for the scalar wave equation, one has in fact constructed the Green's function for the domain. The key

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ingredient in both methods is the solution to essentially the same BIE, which is expressible as A^{-1}.

Some Approximate Forms and Solution Strategies

From the observations above, the boundary element method may be thought of as a systematic way of approximating the BIE (6) by systems of algebraic equations. In so doing, A and B may be interpreted as (square) matrix approximations to the integral operators based on a discretization of the domain surface S, with u and f familiar (column matrix) numerical approximations to the continuous boundary variables.

Therefore, it is clear that for a given discretization, we may form and invert a matrix A, and via (14), we would have an approximate representation for $G^*(q_N,P)$ for a given choice of surface nodes q_N. To use this $G^*(q_N,P)$ to get the solution $u(P)$ based on (4), it would be necessary to get representations for $G^*(q_G,P)$ at Gaussian quadrature points q_G, in order to do the quadrature indicated in (4) numerically. That quadrature is expressible in the form

$$2u(P) = G^*(P,q_G)f(q_G). \tag{16}$$

In (16) $f(q_G)$ is a column of discrete values of f at the Gauss points q_G on S and $G^*(P,q_G)$ is a row matrix of values of Gauss-weighted G^* evaluated at the same q_G for chosen P. How to best provide the mentioned representations for $G^*(P,q_G)$,based on $G^*(q_N,P)$, with sufficient accuracy for a given f and S, is an interesting study in itself. Some studies like this are underway, and preliminary findings will be reported at the conference.

As a more familiar alternative to (16), we have the approximate form of (1) that comes directly from the BEM as usually coded, i. e.,

$$2u(P) = G(P,q_G)f(q_G) - G^n(P,q_G)u(q_G) \tag{17}$$

where the superscript n indicates the normal derivative of G and, wherein,

$$u(q_N) = A^{-1}Bf(q_N). \tag{18}$$

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In (17) function values at q_G are given in terms of values at q_N via the shape functions. Thus, it is possible to factor out f in (17), to write

$$2u(P) = \left\{G(P,q_G) - G^n(P,q_G)CA^{-1}BC^T\right\}f(q_G) \tag{19}$$

where C and C^T are rectangular matrices dependent upon the shape functions and coordinates q_G, and where we can identify an approximate form of G^* as the term in brackets in (19) , as we did with the comparable analytical expressions.

The bracket-term in (19) is equivalent in character to G^* in (16), but there is an important strategic difference: to get G^* values at q_G (for chosen P) in (16) requires some kind of approximate representation of G^* over S, as mentioned above; whereas comparable G^* values via (19) or (17) require no such representation. Indeed, since both G and G^n in (19) and (17) have analytical form, each may readily be evaluated anywhere. Specifically, since with the conventional BEM, $u(q_N)$ is obtained via (18), only $u(q_G)$ need be expressed as usual, with standard shape functions, in terms of nodal values. Thus (17) rather than (19) is usually used by the BEM community to get $u(P)$.

To exploit the apparent simplicity of (16), with its need for perhaps special representations of G*, versus the more complicated (17) or (19), emanating from the standard BIE with no such need, deserves more study. Either way, it is possible, with today's technology, to take the following somewhat radical point of view.

The Library Idea

The key and most computationally-intensive ingredient in the usual BEM solution of a BVP of the present type is the construction of A^{-1}. This requires a mesh, a code to form A and B, and then the effort to find A^{-1} (or, equivalently, the LU decomposition of A). The rest of the solution process involves mainly matrix multiplications based on formulas for numerical quadrature. Therefore, why not consider forming and storing at least A^{-1}, and possibly B (depending on the tradeoff on using (16) versus (17) or (19)), for common and/or important shapes S? In effect, why not create a library of numerical approximations to exact Green's functions for repeated use? Modern technology for storage of massive amounts of data, on CDs or on central storage, accessible via networks, would suggest that at least some heavy computing could be 'done in advance', the results of which could be made available to non expert users.

Some details for the formation of a Green's function library may be found in [4] and [7]. Also, one of us (LP) has constructed a library of A^{-1} matrices for elastodynamic scattering [8] from families of oblate-spheroidal voids, of various eccentricities, for waves of different frequencies. With this library, the elastodynamic scattered field at arbitrary points, from shapes and frequencies in the library, due to arbitrary incident waves, is just a matter of matrix multiplications. Library entries for other scatterers, e.g., cracks, inclusions, are already in existence or are being formed - all of which find use by physicists engaged in nondestructive evaluation at Iowa State University. Partially-exact Green's functions, which model only (a common or especially complicated) part of a surface are also being formed, for repeated use, for acoustic and electromagnetic field problems. Much unnecessary duplication in computing can be avoided in this way.

Finally, one of us (DZ), is working on the capability [9] to compute not only elastic fields, but also the sensitivities of these fields to changes in a geometric (e.g., shape) parameter. This is being done for both conventional and hypersingular BIE's. With this capability, fewer entries in a library would be needed, to cover a spectrum of geometrical shapes, since a given A^{-1} could provide not only the field, but also its sensitivity with respect to a change in shape closer to the next library entry. More significantly, with the hypersingular capability, which is totally new, sensitivities for problems with cracks and cracklike shapes may be addressed, quite apart from any library use.

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