ON THE SCATTERING OF SPHERICAL ELECTROMAGNETIC WAVES BY A PENETRABLE CHIRAL OBSTACLE[1]

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Abstract

The problem of scattering of spherical electromagnetic waves by a bounded chiral obstacle is considered. General scattering theorems, relating the far–field patterns due to scattering of waves from a point source put in any two different locations (the reciprocity principle, the optical theorem, etc), and mixed scattering relations (relating the scattered fields due to a point source and a plane wave) are established. Further, in the case of a spherical chiral scatterer, the exact Green's function and the electric far–field pattern of the problem are constructed.

1 Introduction

The interaction of an incident wavefield with a bounded 3-dimensional obstacle is a classic problem in scattering theory. The vast majority of the literature is concerned with incident plane waves. However, in some recent papers Dassios and his co–workers, as well as three and all of the present authors, see [1], [2], [3], [5], have studied incident waves generated by a point source in the vicinity of the scatterer. Point sources have been used by Potthast [8] to solve standard inverse problems. For related work by other authors see the bibliographies of all the previous references. In this work we consider the problem of scattering of spherical electromagnetic waves by a bounded chiral obstacle. General scattering theorems, relating the far–field patterns due to scattering of waves from a point source put in any two different locations (the reciprocity principle, the optical theorem, etc), and mixed scattering relations (relating the scattered fields due to a point source and a plane wave) are established. Further, in the case of a *spherical* chiral scatterer, the exact Green's function and the electric far–field pattern of the problem are constructed, using spherical vector wave functions. These results generalize related properties of the problem where the scatterer is achiral [1], [3].

2 Formulation

Let Ω_c be a bounded three–dimensional obstacle with a smooth closed boundary S, the scatterer. The exterior Ω is an infinite homogeneous isotropic achiral medium with electric permittivity ε, magnetic permeability μ and conductivity $\sigma = 0$. The scatterer Ω_c is filled with a chiral homogeneous isotropic medium with corresponding electromagnetic parameters ε_c, μ_c

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and chirality measure β_c. The parameters ε and μ are assumed to be real constants and ε_c, μ_c and β_c to be complex constants.

We consider an incident spherical electromagnetic wave due to a point source located at a point with position vector $\boldsymbol{a}$, with respect to the origin O. Suppressing the time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$, ω being the angular frequency, the incident wave $(\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{H}^{\mathrm{i}}_a)$ has the form [1], [3]

$$\boldsymbol{E}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}) = \frac{a\,\mathrm{e}^{-\mathrm{i}ka}}{\mathrm{i}k}\,\nabla\times\left(\frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{a}|}}{|\boldsymbol{r}-\boldsymbol{a}|}\,\widehat{\boldsymbol{a}}\times\widehat{\boldsymbol{p}}\right), \quad \boldsymbol{H}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}) = \frac{1}{\mathrm{i}k\eta}\,\nabla\times\boldsymbol{E}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}), \tag{2.1}$$

where $\widehat{\boldsymbol{p}}$ is a constant unit vector with $\widehat{\boldsymbol{p}}\cdot\boldsymbol{a} = 0$, $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance, $k = \omega\sqrt{\varepsilon\mu} > 0$ is the free–space wave number, and $a = |\boldsymbol{a}|$. Physically $(\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{H}^{\mathrm{i}}_a)$ represents the field generated by a magnetic dipole with dipole moment $\widehat{\boldsymbol{a}}\times\widehat{\boldsymbol{p}}$; see p.163 of [4], or p.23 of [5]. The coefficient $a\,\mathrm{e}^{-\mathrm{i}ka}/(\mathrm{i}k)$ in (2.1) assures that when the point source recedes to infinity the spherical wave reduces to a plane electric wave with direction of propagation $-\widehat{\boldsymbol{a}}$ and polarization $\widehat{\boldsymbol{p}}$. The total exterior electric field $\boldsymbol{E}^{\mathrm{t}}_a$ is given by

$$\boldsymbol{E}^{\mathrm{t}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}) = \boldsymbol{E}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}) + \boldsymbol{E}^{\mathrm{s}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}), \quad \boldsymbol{r}\in\Omega\setminus\{\boldsymbol{a}\}, \tag{2.2}$$

where $\boldsymbol{E}^{\mathrm{s}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}})$ is the scattered electric field, which is assumed to satisfy the Silver–Müller radiation condition

$$\lim_{r\to\infty}(\widehat{\boldsymbol{r}}\times\nabla\times\boldsymbol{E}^{\mathrm{s}}_a + \mathrm{i}kr\boldsymbol{E}^{\mathrm{s}}_a) = \boldsymbol{0}, \tag{2.3}$$

uniformly in all directions $\widehat{\boldsymbol{r}}\in S^2$, where S^2 is the unit sphere.

The behaviour of $\boldsymbol{E}^{\mathrm{s}}_a$ in the radiation zone is given by

$$\boldsymbol{E}^{\mathrm{s}}_a(\boldsymbol{r}) = h_0(kr)\,\boldsymbol{g}_a(\widehat{\boldsymbol{r}}) + O(r^{-2}), \quad r\to\infty, \tag{2.4}$$

where $h_0(x) = \mathrm{e}^{\mathrm{i}x}/(\mathrm{i}x)$ is the spherical Hankel function of the first kind and order zero, and $\boldsymbol{g}_a(\widehat{\boldsymbol{r}})$ is the electric far–field pattern.

The total exterior electric field solves the equation

$$\nabla\times\nabla\times\boldsymbol{E}^{\mathrm{t}}_a - k^2\boldsymbol{E}^{\mathrm{t}}_a = \boldsymbol{0} \quad \text{in } \Omega. \tag{2.5}$$

We note that the incident electric field satisfies the radiation condition (2.3), and hence the total electric field also satisfies (2.3).

The incident electromagnetic waves are transmitted into the chiral scatterer. Let $\boldsymbol{E}^{\mathrm{c}}_a$ be the total electric field in the interior. Then $\boldsymbol{E}^{\mathrm{c}}_a$ satisfies [6]

$$\nabla\times\nabla\times\boldsymbol{E}^{\mathrm{c}}_a - 2\beta_c\gamma^2\nabla\times\boldsymbol{E}^{\mathrm{c}}_a - \gamma^2\boldsymbol{E}^{\mathrm{c}}_a = \boldsymbol{0} \quad \text{in } \Omega_c\ , \tag{2.6}$$

where $\gamma^2 = k_c^2(1-k_c^2\beta_c^2)^{-1}$ and $k_c^2 = \omega^2\varepsilon_c\mu_c$. On the surface of the scatterer we have the following transmission conditions:

$$\left.\begin{array}{c}\widehat{\boldsymbol{n}}\times\boldsymbol{E}^{\mathrm{t}}_a = \widehat{\boldsymbol{n}}\times\boldsymbol{E}^{\mathrm{c}}_a \\ \widehat{\boldsymbol{n}}\times\nabla\times\boldsymbol{E}^{\mathrm{t}}_a = \mathcal{B}_1\widehat{\boldsymbol{n}}\times\nabla\times\boldsymbol{E}^{\mathrm{c}}_a + \mathcal{B}_2\widehat{\boldsymbol{n}}\times\boldsymbol{E}^{\mathrm{c}}_a\end{array}\right\} \quad \text{on } S \tag{2.7}$$

where $\mathcal{B}_1 = (\mu/\mu_c)k_c^2/\gamma^2$ and $\mathcal{B}_2 = -(\mu/\mu_c)\beta_c k_c^2$.

3 The general scattering theorem

In the sequel, for an incident time–harmonic spherical wave $\boldsymbol{E}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}})$ due to a point source located at $\boldsymbol{a}$, we will denote the total field in Ω, the scattered field and the far–field pattern by writing $\boldsymbol{E}^{\mathrm{t}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}})$, $\boldsymbol{E}^{\mathrm{s}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}})$ and $\boldsymbol{g}_a(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}})$, respectively, indicating the dependence on the position $\boldsymbol{a}$ of the point source and the polarization $\widehat{\boldsymbol{p}}$. Also, the total electric field in Ω_c will be denoted by $\boldsymbol{E}^{\mathrm{c}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}})$.

We are interested in relations between these fields. We consider a point source at $\boldsymbol{a}$ with polarization $\widehat{\boldsymbol{p}}_1$ and another point source at $\boldsymbol{b}$ with polarization $\widehat{\boldsymbol{p}}_2$. For a shorthand notation, we use

$$\left\{\boldsymbol{E},\boldsymbol{E}'\right\}_{\mathcal{S}}=\int_{\mathcal{S}}\left[(\widehat{\boldsymbol{n}}\times\overline{\boldsymbol{E}})\cdot(\nabla\times\boldsymbol{E}')-(\widehat{\boldsymbol{n}}\times\boldsymbol{E}')\cdot(\nabla\times\overline{\boldsymbol{E}})\right]ds,$$

where the overbar denotes complex conjugation.

Let S_r denote a large sphere of radius r, surrounding the points $\boldsymbol{a}$ and $\boldsymbol{b}$, and let $S_{a,\epsilon}=\{\boldsymbol{r}\in\mathbb{R}^3:\ |\boldsymbol{a}-\boldsymbol{r}|=\epsilon\}$ surrounding the point $\boldsymbol{a}$. Then we have the following Lemma from [1].

Lemma 1 *Let $\boldsymbol{E}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)$ be a point source at $\boldsymbol{a}$. Let $\boldsymbol{E}^{\mathrm{i}}_b(\boldsymbol{r};\widehat{\boldsymbol{p}}_2)$ be a point source at $\boldsymbol{b}$, with corresponding scattered field $\boldsymbol{E}^{\mathrm{s}}_b(\boldsymbol{r};\widehat{\boldsymbol{p}}_2)$ and far–field pattern $\boldsymbol{g}_b(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}}_2)$. Then*

$$\lim_{\epsilon\to 0}\left\{\boldsymbol{E}^{\mathrm{i}}_a(\,\cdot\,;\widehat{\boldsymbol{p}}_1),\boldsymbol{E}^{\mathrm{s}}_b(\,\cdot\,;\widehat{\boldsymbol{p}}_2)\right\}_{S_{a,\epsilon}}-\lim_{r\to\infty}\left\{\boldsymbol{E}^{\mathrm{i}}_a(\,\cdot\,;\widehat{\boldsymbol{p}}_1),\boldsymbol{E}^{\mathrm{s}}_b(\,\cdot\,;\widehat{\boldsymbol{p}}_2)\right\}_{S_r}=\frac{4\pi a}{\mathrm{i}k}\,\widehat{\boldsymbol{p}}_1\cdot\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2),$$

where

$$\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)=\mathrm{e}^{\mathrm{i}ka}\boldsymbol{a}\times\left[\nabla\times\boldsymbol{E}^{\mathrm{s}}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)-\frac{\mathrm{i}k}{2\pi}\int_{S^2}\widehat{\boldsymbol{r}}\times\boldsymbol{g}_b(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}}_2)\,\mathrm{e}^{\mathrm{i}k\widehat{\boldsymbol{r}}\cdot\boldsymbol{a}}ds(\widehat{\boldsymbol{r}})\right] \tag{3.1}$$

is a spherical far–field pattern generator.

Now, the general scattering theorem, [1], for spherical electric waves scattered by a chiral obstacle is formulated as follows.

Theorem 2 *For any two point–source locations in Ω, $\boldsymbol{a}$ and $\boldsymbol{b}$, and for any polarizations, $\widehat{\boldsymbol{p}}_1$ and $\widehat{\boldsymbol{p}}_2$, we have*

$$\widehat{\boldsymbol{p}}_1\cdot\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)+\widehat{\boldsymbol{p}}_2\cdot\overline{\boldsymbol{G}_a(\boldsymbol{b};\widehat{\boldsymbol{p}}_1)}+\frac{1}{2\pi}\int_{S^2}\boldsymbol{g}_b(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}}_2)\cdot\overline{\boldsymbol{g}_a(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}}_1)}\,ds(\widehat{\boldsymbol{r}})=\mathcal{E}_{a,b}(\widehat{\boldsymbol{p}}_1;\widehat{\boldsymbol{p}}_2) \tag{3.2}$$

where

$$\begin{aligned}\mathcal{E}_{a,b}(\widehat{\boldsymbol{p}}_1;\widehat{\boldsymbol{p}}_2) \;=\; & \frac{k}{2\pi}\Bigg\{\mathrm{Im}\,(\mathcal{B}_1)\int_{\Omega_c}(\nabla\times\overline{\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)})\cdot(\nabla\times\boldsymbol{E}_b(\boldsymbol{r};\widehat{\boldsymbol{p}}_2))\,dv\\ & -\mathrm{Im}\,(\mathcal{B}_1\gamma^2)\int_{\Omega_c}\overline{\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)}\cdot\boldsymbol{E}_b(\boldsymbol{r};\widehat{\boldsymbol{p}}_2)\,dv\\ & +\mathrm{i}[\beta_c\gamma^2\mathcal{B}_1+\mathrm{Re}\,(\mathcal{B}_2)]\int_{\Omega_c}\overline{\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)}\cdot(\nabla\times\boldsymbol{E}_b(\boldsymbol{r};\widehat{\boldsymbol{p}}_2))\,dv\\ & -\mathrm{i}[\overline{\beta_c\gamma^2\mathcal{B}_1}+\mathrm{Re}\,(\mathcal{B}_2)]\int_{\Omega_c}\boldsymbol{E}_b(\boldsymbol{r};\widehat{\boldsymbol{p}}_2)\cdot(\nabla\times\overline{\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)})\,dv\Bigg\}.\end{aligned} \tag{3.3}$$

Proof. In view of the relations $\boldsymbol{E}^{\mathrm{t}}_{\alpha} = \boldsymbol{E}^{\mathrm{i}}_{\alpha} + \boldsymbol{E}^{\mathrm{s}}_{\alpha}$, $\alpha = a, b$, we have

$$\{\boldsymbol{E}^{\mathrm{t}}_a, \boldsymbol{E}^{\mathrm{t}}_b\} = \{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{i}}_b\} + \{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{s}}_b\} + \{\boldsymbol{E}^{\mathrm{s}}_a, \boldsymbol{E}^{\mathrm{i}}_b\} + \{\boldsymbol{E}^{\mathrm{s}}_a, \boldsymbol{E}^{\mathrm{s}}_b\}. \tag{3.4}$$

We use the transmission conditions (2.7) and apply the divergence theorem in Ω_c; this gives

$$\{\boldsymbol{E}^{\mathrm{t}}_a, \boldsymbol{E}^{\mathrm{t}}_b\} = \frac{4\pi \mathrm{i}}{k} \mathcal{E}_{a,b}(\widehat{\boldsymbol{p}}_1; \widehat{\boldsymbol{p}}_2). \tag{3.5}$$

Since $\overline{\boldsymbol{E}^{\mathrm{i}}_a}$ and $\boldsymbol{E}^{\mathrm{i}}_b$ are entire solutions of (2.5), the vector Green's second theorem gives

$$\{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{i}}_b\} = 0. \tag{3.6}$$

For the other terms in (3.4), we consider two small spheres, S_{a,ϵ_1} and S_{b,ϵ_2}, centred at $\boldsymbol{a}$ and $\boldsymbol{b}$ with radii ϵ_1 and ϵ_2, respectively, with $S_{a,\epsilon_1} \cap S_{b,\epsilon_2} = \emptyset$, as well as a large sphere S_R centred at the origin, surrounding the whole system of the scatterer and the two small spheres. Since $\overline{\boldsymbol{E}^{\mathrm{i}}_a}$ and $\boldsymbol{E}^{\mathrm{s}}_b$ are solutions of (2.5) for $\boldsymbol{r} \neq \boldsymbol{a}, \boldsymbol{b}$, the vector Green's second theorem gives

$$\{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{s}}_b\} = \{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{s}}_b\}_{S_R} - \{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{s}}_b\}_{S_{a,\epsilon_1}} - \{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{s}}_b\}_{S_{b,\epsilon_2}}. \tag{3.7}$$

The last term in (3.7) is zero because $\overline{\boldsymbol{E}^{\mathrm{i}}_a}$ and $\boldsymbol{E}^{\mathrm{s}}_b$ are regular solutions of (2.5) in the interior of S_{b,ϵ_2}. Then letting $R \to \infty$ and $\epsilon_1 \to 0$, using Lemma 1, we obtain

$$\{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{s}}_b\} = \frac{4\pi \mathrm{i}}{k} \widehat{\boldsymbol{p}}_1 \cdot \boldsymbol{G}_b(\boldsymbol{a}; \widehat{\boldsymbol{p}}_2). \tag{3.8}$$

As $\{\boldsymbol{E}^{\mathrm{s}}_a, \boldsymbol{E}^{\mathrm{i}}_b\} = -\overline{\{\boldsymbol{E}^{\mathrm{i}}_a, \boldsymbol{E}^{\mathrm{s}}_b\}}$, we easily deduce that

$$\{\boldsymbol{E}^{\mathrm{s}}_a, \boldsymbol{E}^{\mathrm{i}}_b\} = \frac{4\pi \mathrm{i}}{k} \widehat{\boldsymbol{p}}_2 \cdot \overline{\boldsymbol{G}_a(\boldsymbol{b}; \widehat{\boldsymbol{p}}_1)}. \tag{3.9}$$

Finally, in view of the regularity of $\overline{\boldsymbol{E}^{\mathrm{s}}_a}$ and $\boldsymbol{E}^{\mathrm{s}}_b$ in the region exterior to S, we have

$$\{\boldsymbol{E}^{\mathrm{s}}_a, \boldsymbol{E}^{\mathrm{s}}_b\} = \{\boldsymbol{E}^{\mathrm{s}}_a, \boldsymbol{E}^{\mathrm{s}}_b\}_{S_R}. \tag{3.10}$$

Then, letting $R \to \infty$, we pass to the radiation zone and thus using (2.4) we get

$$\{\boldsymbol{E}^{\mathrm{s}}_a, \boldsymbol{E}^{\mathrm{s}}_b\} = \frac{2\mathrm{i}}{k} \int_{S^2} \overline{\boldsymbol{g}_a(\widehat{\boldsymbol{r}}; \widehat{\boldsymbol{p}}_1)} \cdot \boldsymbol{g}_b(\widehat{\boldsymbol{r}}; \widehat{\boldsymbol{p}}_2)\, ds(\widehat{\boldsymbol{r}}). \tag{3.11}$$

Substituting (3.5), (3.6), (3.8), (3.9), and (3.11) in (3.7) gives (3.3), and the theorem is proved.

4 Reciprocity

In [5] a reciprocity relation for spherical waves scattered by an achiral obstacle has been proved. The same relation also holds for a penetrable chiral scatterer.

Theorem 3 *For any two point–source locations in Ω, $\boldsymbol{a}$ and $\boldsymbol{b}$, for any polarizations, $\widehat{\boldsymbol{p}}_1$ and $\widehat{\boldsymbol{p}}_2$, and for a penetrable chiral scatterer, we have*

$$h_0(ka)\, (\widehat{\boldsymbol{b}} \times \widehat{\boldsymbol{p}}_2) \cdot \left(\nabla \times \boldsymbol{E}^{\mathrm{s}}_a(\widehat{\boldsymbol{b}}; \widehat{\boldsymbol{p}}_1)\right) = h_0(kb)\, (\widehat{\boldsymbol{a}} \times \widehat{\boldsymbol{p}}_1) \cdot (\nabla \times \boldsymbol{E}^{\mathrm{s}}_b(\widehat{\boldsymbol{a}}; \widehat{\boldsymbol{p}}_2))\,. \tag{4.1}$$

Proof. Using the transmission conditions (2.7) and applying the divergence theorem in Ω_c, we obtain $\{\overline{\boldsymbol{E}^{\mathrm{t}}_a}, \boldsymbol{E}^{\mathrm{t}}_b\}_S = 0$. Also, using some asymptotics we get $\{\overline{\boldsymbol{E}^{\mathrm{s}}_a}, \boldsymbol{E}^{\mathrm{s}}_b\}_S = 0$. Now $\{\overline{\boldsymbol{E}^{\mathrm{i}}_a}, \boldsymbol{E}^{\mathrm{s}}_b\}_S = 0$, since the corresponding integral on the large sphere S_R vanishes due to the asymptotic form (2.4) and $\boldsymbol{E}^{\mathrm{i}}_a(\widehat{\boldsymbol{r}}; \widehat{\boldsymbol{p}}_1) = h_0(kr)\, \boldsymbol{g}^{\mathrm{i}}_a(\widehat{\boldsymbol{r}}; \widehat{\boldsymbol{p}}_1) + O(r^{-2})$ as $r \to \infty$, where $\boldsymbol{g}^{\mathrm{i}}_a(\widehat{\boldsymbol{r}}; \widehat{\boldsymbol{p}}_1) = \mathrm{i}ka \mathrm{e}^{-\mathrm{i}ka(1+\widehat{\boldsymbol{r}} \cdot \boldsymbol{a})}\, (\widehat{\boldsymbol{r}} \times (\boldsymbol{a} \times \widehat{\boldsymbol{p}}_1))\,$. Combining the above we finally obtain (4.1).

5 The optical theorem

We define the scattering and absorption cross–sections due to a point source at $\boldsymbol{a}$, [5], as

$$\sigma_a^{\mathrm{s}} = \frac{1}{k^2}\int_{S^2} |\boldsymbol{g}_a(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}})|^2\, ds(\widehat{\boldsymbol{r}}) \quad \text{and} \quad \sigma_a^{\mathrm{a}} = \frac{1}{k}\,\mathrm{Im}\int_S \widehat{\boldsymbol{n}}\cdot(\boldsymbol{E}_a^{\mathrm{t}}\times\nabla\times\overline{\boldsymbol{E}_a^{\mathrm{t}}})\, ds,$$

respectively, and the extinction cross–section by $\sigma_a^{\mathrm{e}} = \sigma_a^{\mathrm{s}} + \sigma_a^{\mathrm{a}}$. If we put $\boldsymbol{a} = \boldsymbol{b}$ and $\widehat{\boldsymbol{p}}_1 = \widehat{\boldsymbol{p}}_2 = \widehat{\boldsymbol{p}}$ in Theorem 2, we obtain

$$\sigma_a^{\mathrm{s}} = -4\pi k^{-2}\,\mathrm{Re}\,[\widehat{\boldsymbol{p}}\cdot\boldsymbol{G}_a(\boldsymbol{a};\widehat{\boldsymbol{p}})] + 2\pi k^{-2}\mathcal{E}_{a,a}(\widehat{\boldsymbol{p}};\widehat{\boldsymbol{p}}). \tag{5.1}$$

From the above definitions and (3.3) we have

$$\sigma_a^{\mathrm{a}} = -2\pi k^{-2}\mathcal{E}_{a,a}(\widehat{\boldsymbol{p}};\widehat{\boldsymbol{p}}). \tag{5.2}$$

Hence, adding (5.1) and (5.2), the definition (3.1) gives

$$\sigma_a^{\mathrm{e}} = -4\pi k^{-2}\,\mathrm{Re}\,[\widehat{\boldsymbol{p}}\cdot\boldsymbol{G}_a(\boldsymbol{a};\widehat{\boldsymbol{p}})]\,. \tag{5.3}$$

The value of $\mathcal{E}_{a,a}(\widehat{\boldsymbol{p}};\widehat{\boldsymbol{p}})$ is given in Theorem 2; it depends on the scatterer's properties.

6 Mixed scattering relations

Let $\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}}) = \widehat{\boldsymbol{p}}\,\exp\{\mathrm{i}k\widehat{\boldsymbol{d}}\cdot\boldsymbol{r}\}$ be an incident time–harmonic plane electric wave, where the unit vector $\widehat{\boldsymbol{d}}$ describes the direction of propagation and the unit vector $\widehat{\boldsymbol{p}}$ gives the polarization. We will indicate the dependence of the total field in Ω, the total field in Ω_c, the scattered field and the electric far–field pattern on the incident direction $\widehat{\boldsymbol{d}}$ and the polarization $\widehat{\boldsymbol{p}}$ by writing $\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}})$, $\boldsymbol{E}^{-}(\boldsymbol{r};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}})$, $\boldsymbol{E}^{\mathrm{s}}(\boldsymbol{r};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}})$ and $\boldsymbol{g}(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{d}},\widehat{\boldsymbol{p}})$, respectively.

Here, we consider mixed situations, and relate fields due to one spherical electric wave $\boldsymbol{E}_a^{\mathrm{i}}(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)$ and one plane electric wave $\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)$; we do this by letting $b\to\infty$ in our previous results.

Using the asymptotic forms $|\boldsymbol{r}-\boldsymbol{a}| = r - \widehat{\boldsymbol{r}}\cdot\boldsymbol{a} + O(r^{-1})$ and $|\boldsymbol{r}-\boldsymbol{a}|^{-1} = r^{-1} + O(r^{-2})$, we can easily show that for the spherical electric wave (2.1) we have $\lim_{b\to\infty}\boldsymbol{E}_b^{\mathrm{i}}(\boldsymbol{r};\widehat{\boldsymbol{p}}) = \boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}})$, that is the spherical electric wave, when the point source goes to infinity, reduces to a plane electric wave with direction of propagation $-\widehat{\boldsymbol{b}}$ and polarization $\widehat{\boldsymbol{p}}$. Similarly, we have $\boldsymbol{E}_b^{\mathrm{t}}(\boldsymbol{r};\widehat{\boldsymbol{p}})\to\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}})$, $\boldsymbol{E}_b^{\mathrm{s}}(\boldsymbol{r};\widehat{\boldsymbol{p}})\to\boldsymbol{E}^{\mathrm{s}}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}})$ and $\boldsymbol{g}_b(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}})\to\boldsymbol{g}(\widehat{\boldsymbol{r}};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}})$ as $b\to\infty$.

Next, let $b\to\infty$ in Lemma 1 to give the following result.

Lemma 4 *Let $\boldsymbol{E}_a^{\mathrm{i}}(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)$ be an incident spherical electric wave and let $\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)$ be an incident plane electric wave. Then*

$$\lim_{\varepsilon\to 0}\Big\{\boldsymbol{E}_a^{\mathrm{i}}(\,\cdot\,;\widehat{\boldsymbol{p}}_1),\boldsymbol{E}^{\mathrm{s}}(\,\cdot\,;-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)\Big\}_{S_{a,\varepsilon}} - \lim_{r\to\infty}\Big\{\boldsymbol{E}_a^{\mathrm{i}}(\,\cdot\,;\widehat{\boldsymbol{p}}_1),\boldsymbol{E}^{\mathrm{s}}(\,\cdot\,;-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)\Big\}_{S_r} = \frac{4\pi a}{\mathrm{i}k}\widehat{\boldsymbol{p}}_1\cdot\boldsymbol{G}(\boldsymbol{a};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)$$

where

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{a};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2) &= \lim_{b\to\infty}\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)\\ &= \mathrm{e}^{\mathrm{i}ka}\boldsymbol{a}\times\Big[\nabla\times\boldsymbol{E}^{\mathrm{s}}(\boldsymbol{a};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2) - \frac{\mathrm{i}k}{2\pi}\int_{S^2}\widehat{\boldsymbol{r}}\times\boldsymbol{g}(\widehat{\boldsymbol{r}};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)\,\mathrm{e}^{\mathrm{i}k\widehat{\boldsymbol{r}}\cdot\boldsymbol{a}}ds(\widehat{\boldsymbol{r}})\Big]\end{aligned}$$

is a plane far–field pattern generator.

For the generators $\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)$ and $\boldsymbol{G}(\boldsymbol{a};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)$ we have the following limiting values [8].

Theorem 5 *For two incident point–source electric waves, $\boldsymbol{E}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)$ and $\boldsymbol{E}^{\mathrm{i}}_b(\boldsymbol{r};\widehat{\boldsymbol{p}}_2)$, we have*

$$\lim_{a\to\infty}\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)=\boldsymbol{g}_b(-\widehat{\boldsymbol{a}};\widehat{\boldsymbol{p}}_2) \tag{6.1}$$

and

$$\lim_{a\to\infty}\boldsymbol{G}(\boldsymbol{a};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)=\boldsymbol{g}(-\widehat{\boldsymbol{a}};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2). \tag{6.2}$$

We can now let $b\to\infty$ in the general scattering theorem, Theorem 1. The proof of the following result is similar to that of Theorem 2 and is omitted for the sake of brevity.

Theorem 6 *Let $\boldsymbol{E}^{\mathrm{i}}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)$ be an incident spherical electric wave and let $\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)$ be an incident plane electric wave. Then*

$$\widehat{\boldsymbol{p}}_1\cdot\boldsymbol{G}(\boldsymbol{a};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)+\widehat{\boldsymbol{p}}_2\cdot\overline{\boldsymbol{g}_a(\widehat{\boldsymbol{b}};\widehat{\boldsymbol{p}}_1)}+\frac{1}{2\pi}\int_{S^2}\boldsymbol{g}(\widehat{\boldsymbol{r}};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)\cdot\overline{\boldsymbol{g}_a(\widehat{\boldsymbol{r}};\widehat{\boldsymbol{p}}_1)}\,ds(\widehat{\boldsymbol{r}})=\mathcal{M}_a(-\widehat{\boldsymbol{b}};\widehat{\boldsymbol{p}}_1,\widehat{\boldsymbol{p}}_2),$$

where

$$\begin{aligned}\mathcal{M}_a(-\widehat{\boldsymbol{b}};\widehat{\boldsymbol{p}}_1,\widehat{\boldsymbol{p}}_2) &= \lim_{b\to\infty}\mathcal{E}_{a,b}(\widehat{\boldsymbol{p}}_1;\widehat{\boldsymbol{p}}_2)\\ &= \frac{k}{2\pi}\Bigg\{\mathrm{Im}\,(\mathcal{B}_1)\int_{\Omega_c}(\nabla\times\overline{\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)})\cdot(\nabla\times\boldsymbol{E}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2))\,dv\\ &\quad-\mathrm{Im}\,(\gamma^2\mathcal{B}_1)\int_{\Omega_c}\overline{\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)}\cdot\boldsymbol{E}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)\,dv\\ &\quad+\mathrm{i}[\beta_c\gamma^2+\mathrm{Re}\,(\mathcal{B}_2)]\int_{\Omega_c}\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1)\cdot(\nabla\times\boldsymbol{E}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2))\,dv\\ &\quad-\mathrm{i}[\overline{\beta_c\gamma^2\mathcal{B}_1}+\mathrm{Re}\,(\mathcal{B}_2)]\int_{\Omega_c}(\nabla\times\boldsymbol{E}_a(\boldsymbol{r};\widehat{\boldsymbol{p}}_1))\cdot\boldsymbol{E}(\boldsymbol{r};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)\,dv\Bigg\}.\end{aligned} \tag{6.3}$$

To conclude, we note that we also have

$$\lim_{a\to\infty}\lim_{b\to\infty}\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)=\lim_{b\to\infty}\lim_{a\to\infty}\boldsymbol{G}_b(\boldsymbol{a};\widehat{\boldsymbol{p}}_2)=\boldsymbol{g}(-\widehat{\boldsymbol{a}};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2). \tag{6.4}$$

This can be used to verify that the known scattering relations for plane–wave incidence [4], [5] are recovered when $a\to\infty$ and $b\to\infty$. Furthermore, (6.2) and the reciprocity principle for plane waves [5] give the following limiting property:

$$\lim_{a\to\infty}\widehat{\boldsymbol{p}}_1\cdot\boldsymbol{G}(\boldsymbol{a};-\widehat{\boldsymbol{b}},\widehat{\boldsymbol{p}}_2)=\lim_{b\to\infty}\widehat{\boldsymbol{p}}_2\cdot\boldsymbol{G}(-\boldsymbol{b};\widehat{\boldsymbol{a}},\widehat{\boldsymbol{p}}_1). \tag{6.5}$$

7 Exact Green's function for a chiral dielectric sphere

Consider a spherical scatterer of radius a. Take spherical polar coordinates (r,θ,ϕ) with the origin at the centre of the sphere, so that the point source is at $r=r_0$, $\theta=0$, and so that the polarization vector $\widehat{\boldsymbol{p}}$ is in the x–direction. Thus, $\boldsymbol{r}_0=r_0\widehat{\boldsymbol{z}}$ and $\widehat{\boldsymbol{b}}=\widehat{\boldsymbol{x}}$, where $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{z}}$ are unit vectors in the x and z directions, respectively.

Using spherical vector wave functions, and in particular (13.3.68), (13.3.69), (13.3.70) of [7] we obtain the following expansion for the incident field, see [3]

$$\boldsymbol{E}^{\text{inc}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{x}}) = \frac{\mathrm{i}}{h_0(kr_0)}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left\{h_n(kr_0)\,\boldsymbol{N}^1_{e1n}(\boldsymbol{r}) - \widetilde{h}_n(kr_0)\,\boldsymbol{M}^1_{o1n}(\boldsymbol{r})\right\} \tag{7.1}$$

for $r < r_0$, where $h_n \equiv h_n^{(1)}$ is a spherical Hankel function, $\widetilde{h}_n(x) = x^{-1}h_n(x) + h_n'(x) = x^{-1}[xh_n(x)]'$, and $\boldsymbol{M}^1_{\sigma 1n}$ and $\boldsymbol{N}^1_{\sigma 1n}$, for $n = 1, 2, \dots$ and $\sigma = e$ or o are the spherical vector wave functions of the first kind.

The scattered field due to a chiral sphere does not have the same ϕ–dependence as the incident wave, so it has the following general form, [6], p. 394,

$$\boldsymbol{E}^{\text{s}}_{\boldsymbol{r}_0}(\boldsymbol{r};\widehat{\boldsymbol{x}}) = \frac{\mathrm{i}}{h_0(kr_0)}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left\{a_n\boldsymbol{N}^3_{e1n}(\boldsymbol{r}) - b_n\boldsymbol{M}^3_{o1n}(\boldsymbol{r}) + c_n\boldsymbol{M}^3_{e1n}(\boldsymbol{r}) - d_n\boldsymbol{N}^3_{o1n}(\boldsymbol{r})\right\},$$

for $r > a$. In order to evaluate the electric field in the interior of the chiral sphere, we consider the Bohren decomposition, [6]

$$\boldsymbol{E}^{\text{c}}(\boldsymbol{r}) = \boldsymbol{Q}_L(\boldsymbol{r}) - \sqrt{\mu_c/\varepsilon_c}\,\boldsymbol{Q}_R(\boldsymbol{r}) \quad \text{and} \quad \boldsymbol{H}^{\text{c}}(\boldsymbol{r}) = -\mathrm{i}\sqrt{\varepsilon_c/\mu_c}\,\boldsymbol{Q}_L(\boldsymbol{r}) + \boldsymbol{Q}_R(\boldsymbol{r}), \quad \boldsymbol{r}\in\Omega_c,$$

where $\boldsymbol{Q}_L(\boldsymbol{r})$ and $\boldsymbol{Q}_R(\boldsymbol{r})$ are Beltrami fields, satisfying $\nabla\times\boldsymbol{Q}_L = \gamma_L\boldsymbol{Q}_L$ and $\nabla\times\boldsymbol{Q}_R = -\gamma_R\boldsymbol{Q}_R$. We employ the following expansions for the Beltrami fields, [6], p.395

$$\boldsymbol{Q}_L(\boldsymbol{r}) = \frac{\mathrm{i}}{h_0(kr_0)}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left\{A_n\left[\boldsymbol{M}^1_{o1n}(\boldsymbol{r}) + \boldsymbol{N}^1_{o1n}(\boldsymbol{r})\right] + B_n\left[\boldsymbol{M}^1_{e1n}(\boldsymbol{r}) + \boldsymbol{N}^1_{e1n}(\boldsymbol{r})\right]\right\}$$

$$\boldsymbol{Q}_R(\boldsymbol{r}) = \frac{\mathrm{i}}{h_0(kr_0)}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left\{C_n\left[\boldsymbol{M}^1_{o1n}(\boldsymbol{r}) - \boldsymbol{N}^1_{o1n}(\boldsymbol{r})\right] + D_n\left[\boldsymbol{M}^1_{e1n}(\boldsymbol{r}) - \boldsymbol{N}^1_{e1n}(\boldsymbol{r})\right]\right\}$$

Using the transmission conditions on $r = a$, we obtain

$$a_n = \frac{S_{nL}W_{nR} + S_{nR}W_{nL}}{V_{nL}W_{nR} + V_{nR}W_{nL}}, \quad b_n = \frac{T_{nL}V_{nR} + T_{nR}V_{nL}}{V_{nL}W_{nR} + V_{nR}W_{nL}}$$

$$c_n = -\frac{\widetilde{h}_n(kr_0)}{h_n(kr_0)}d_n = \frac{\widetilde{h}_n(kr_0)}{h_n(kr_0)}\frac{S_{nL}V_{nR} - S_{nR}V_{nL}}{V_{nL}W_{nR} + V_{nR}W_{nL}}$$

where, for $A = L, R$,

$$W_{nA} = \widetilde{h}_n(ka)j_n(\gamma_A a) - (\eta/\eta_c)\,h_n(ka)\widetilde{j}_n(\gamma_A a)$$

$$V_{nA} = h_n(ka)\widetilde{j}_n(\gamma_A a) - (\eta/\eta_c)\,\widetilde{h}_n(ka)j_n(\gamma_A a)$$

$$S_{nA} = h_n(kr_0)\left[j_n(ka)\widetilde{j}_n(\gamma_A a) - (\eta/\eta_c)\,\widetilde{j}_n(ka)j_n(\gamma_A a)\right]$$

$$T_{nA} = \widetilde{h}_n(kr_0)\left[(\eta/\eta_c)\,j_n(ka)\widetilde{j}_n(\gamma_A a) - \widetilde{j}_n(ka)j_n(\gamma_A a)\right],$$

$\widetilde{j}_n(x) = x^{-1}j_n(x) + j_n'(x) = x^{-1}\left[xj_n(x)\right]'$,

$$A_n = \frac{1}{2j_n(\gamma_L a)}\left\{-h_n(ka)\left[\frac{\eta_c}{\eta}c_n + b_n\right] - \widetilde{h}_n(kr_0)j_n(ka)\right\} \tag{7.2}$$

$$B_n = \frac{1}{2j_n(\gamma_L a)} \left\{ h_n(ka) \left[\frac{\eta_c}{\eta} a_n + c_n \right] + \frac{\eta_c}{\eta} h_n(kr_0) j_n(ka) \right\} \tag{7.3}$$

$$C_n = \frac{1}{2\mathrm{i}\eta_c j_n(\gamma_R a)} \left\{ h_n(ka) \left[-\frac{\eta_c}{\eta} d_n + b_n \right] + \widetilde{h}_n(kr_0) j_n(ka) \right\} \tag{7.4}$$

$$D_n = \frac{1}{2\mathrm{i}\eta_c j_n(\gamma_R a)} \left\{ h_n(ka) \left[\frac{\eta_c}{\eta} a_n - c_n \right] + \frac{\eta_c}{\eta} h_n(kr_0) j_n(ka) \right\} \tag{7.5}$$

and $\eta_c = \sqrt{\mu_c/\varepsilon_c}$ is the interior intrinsic impedance.

Let us calculate the electric far–field pattern. Since $h_n(x) \sim (-\mathrm{i})^n h_0(x)$ and $h_n'(x) \sim (-\mathrm{i})^{n-1} h_0(x)$ as $x \to \infty$, and using (13.3.68) and (13.3.69) of [7] we find that

$$\begin{aligned} \boldsymbol{M}^3_{\sigma 1n}(\boldsymbol{r}) &= \sqrt{n(n+1)}\, h_n(kr)\, \boldsymbol{C}_{\sigma 1n}(\widehat{\boldsymbol{r}}) \\ &\sim \sqrt{n(n+1)}\, (-\mathrm{i})^n\, h_0(kr)\, \boldsymbol{C}_{\sigma 1n}(\widehat{\boldsymbol{r}}) \end{aligned} \tag{7.6}$$

and

$$\begin{aligned} \boldsymbol{N}^3_{\sigma 1n}(\boldsymbol{r}) &= n(n+1)\, (kr)^{-1}\, h_n(kr)\, \boldsymbol{P}_{\sigma e1n}(\widehat{\boldsymbol{r}}) + \sqrt{n(n+1)}\, \widetilde{h}_n(kr)\, \boldsymbol{B}_{\sigma 1n}(\widehat{\boldsymbol{r}}) \\ &\sim \sqrt{n(n+1)}\, (-\mathrm{i})^{n-1}\, h_0(kr)\, \boldsymbol{B}_{\sigma 1n}(\widehat{\boldsymbol{r}}) \end{aligned} \tag{7.7}$$

as $kr \to \infty$, where $\boldsymbol{C}_{\sigma 1n}(\widehat{\boldsymbol{r}})$, $\boldsymbol{P}_{\sigma 1n}(\widehat{\boldsymbol{r}})$ and $\boldsymbol{B}_{\sigma 1n}(\widehat{\boldsymbol{r}})$ can be found e.g. in the Appendix of [3]. Therefore for the electric far–field pattern, we have

$$\boldsymbol{g}_{\boldsymbol{r}_0}(\boldsymbol{r}; \widehat{\boldsymbol{x}}) = -\frac{1}{h_0(kr_0)} \sum_{n=1}^{\infty} \frac{(2n+1)(-\mathrm{i})^n}{\sqrt{n(n+1)}} \left\{ a_n\, \boldsymbol{B}_{e1n}(\widehat{\boldsymbol{r}}) + \mathrm{i} b_n\, \boldsymbol{C}_{o1n}(\widehat{\boldsymbol{r}}) + \mathrm{i} c_n\, \boldsymbol{C}_{e1n}(\widehat{\boldsymbol{r}}) + d_n\, \boldsymbol{B}_{o1n}(\widehat{\boldsymbol{r}}) \right\}. \tag{7.8}$$

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