CHAPTER 164

Water Wave Scattering by Rows of Circular Cylinders

Robert A. Dalrymple[1], M. ASCE, Seung Nam Seo[2], Paul A. Martin[3]

Abstract

The scattering of waves by a finite number of rows of circular cylinders is examined. Reflection and transmission coefficients are obtained and compared to Kakuno's experimental data. Following Twersky (1962), the scattering from a single row of cylinders (or the single grating problem) is numerically solved. The wide-spacing approximation is used to find the effect of multiple gratings.

1 Introduction

The reflection and transmission of waves through rows of vertical cylinders, corresponding to the use of rows of piling as breakwaters (Wiegel, 1961), is examined. The physical situation also could represent the effects of piers and other pile supported structures on the wave environment.

Hayashi *et al.* (1966) and Mei *et al.* (1974) have examined several methods to calculate the wave field in the vicinity of a single row of cylinders, taking into consideration the loss of energy by the flow between the closely spaced cylinders. Ozsoy (1977) has empirically examined the Mei *et al.* solution and finds reasonable agreement with the theory; however there is considerable scatter in the data. Spring and Monkmeyer (1975) assumed that the flow was potential and examined the pressures and forces on an infinite row of cylinders.

Kakuno (1984,1986) has examined the same problem, and also the problem of a double row of vertical cylinders. He also considered potential flow with two assumptions: (1) the wave length is long compared with the pile spacing in each row, and (2) the wave length is short compared to the row spacing. Because of (1) he was able to give an analysis similar to Lamb (1898) for the analogous problem of acoustic waves passing through a row of cylinders. However, it is known that Lamb's method is deficient for this case: the exact solution is composed of an odd (dipole) part and an even (monopole) part, whereas Lamb's method only gives the odd part correctly. Recently Martin and Dalrymple (1988)

[1]Department of Civil Engineering, University of Delaware, Newark, Delaware 19716
[2]Korea Ocean Research and Development Institute, Ansan, Seoul 171-14, Korea
[3]Department of Mathematics, University of Manchester, Manchester, M13 9PL, U.K.

have shown how this deficiency can be rectified. Because of (2), Kakuno was able to use the "wide spacing approximation" for two rows of cylinders; see, e.g., Srokoscz and Evans (1979).

In this paper, we eliminate assumption (1) by giving an (in principle) exact numerical solution for a single row of circular cylinders. (Note that the asymptotic methods of Lamb and of Martin and Dalrymple work for cylinders with any cross-section.) We combine these results with the wide-spacing approximation and give a comparison with Kakuno's data. Finally, scattering from several rows of cylinders is calculated.

2 Scattering by a cylindrical grating

We first consider scattering of water waves by an infinite grating of circular cylinders, located along the y-axis. A plane wave of frequency ω over a constant depth h is incident upon the grating with angle θ_o (See Fig. 1).

The velocity potential, which must satisfy the Laplace equation, can be written as

$$\Phi(x,y,z,t) = Re\left\{\phi(x,y)\cosh k(h+z)e^{-\mathrm{i}\omega t)}\right\} \tag{1}$$

where Re denotes the real part and k is the wave number, determined from the dispersion relationship, $\omega^2 = gk\tanh kh$, which relates the angular frequency ω to the wave number, the water depth h, and the acceleration of gravity, g. Factoring out the time and depth dependency, the fluid motion is governed by the Helmholtz equation

$$(\nabla^2 + k^2)\phi = 0 \tag{2}$$

On the wall of each cylinder in the grating, the no-flux condition should be fulfilled:

$$\phi_n = 0 \tag{3}$$

where subscript n denotes the outward normal derivative. The scattered potential ϕ^{sc}, defined as the total potential ϕ minus the incident potential ϕ^{in}, must satisfy a radiation condition as $|x| \to \infty$.

Twersky (1962) used the method of separation of variables for solving scattered waves by an infinite grating of equally spaced identical circular cylinders with spacing d. (See Fig. 1) The total velocity potential by an incident wave $\phi^{\mathrm{in}} = e^{\mathrm{i}\vec{k}\cdot\vec{x}}$ can be expressed in cylindrical coordinates as,

$$\begin{aligned}\phi \;=\; & e^{\mathrm{i}kr\cos(\theta-\theta_o)} + \sum_{n=-\infty}^{\infty} e^{\mathrm{i}n(\theta+\pi/2)}\Big\{\mathrm{A}_n\mathrm{H}_n^{(1)}(kr) \\ & +\mathrm{J}_n(kr)\sum_{s=1}^{\infty}\sum_{m=-\infty}^{\infty}\mathrm{A}_m\mathrm{H}_{m-n}^{(1)}(skd)\left[e^{-\mathrm{i}skd\sin\theta_o} + (-1)^{m-n}e^{\mathrm{i}skd\sin\theta_o}\right]\Big\}\end{aligned} \tag{4}$$

where J_n and $\mathrm{H}_n^{(1)}$ denote the Bessel function of first kind order n and the Hankel function of order n. The multiple scattering coefficient A_n can be obtained by imposing no-flux condition at the cylinder wall (r=a).

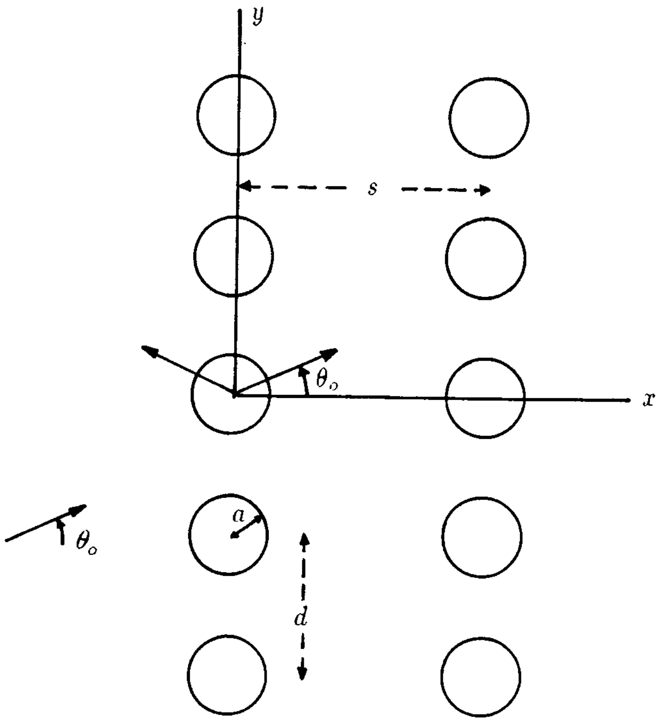

Figure 1: Schematic Diagram Showing Waves Incident on Two Rows of Cylinders

$$\mathrm{A}_n = -\frac{\mathrm{J}_n'(ka)}{\mathrm{H}_n'(ka)} \left\{ e^{-in\theta_o} + \sum_{m=-\infty}^{\infty} \mathrm{A}_m \mathrm{H}_{n-m}^{(1)}(skd) \left[e^{-iskd\sin\theta_o} + (-1)^{m-n} e^{iskd\sin\theta_o} \right] \right\} \tag{5}$$

where prime denotes differentiation.

In order to calculate transmission coefficient K_T and reflection coefficient K_R, it is convenient to express the potential as a summation of plane wave modes in Cartesian, rather than polar coordinates. By Sommerfeld's integral representation of the Hankel functions and Poisson's summation formula, Twersky (1962) obtained the following representations: For the transmitted region ($x > 0$),

$$\phi_T = e^{i\vec{k}\cdot\vec{x}} + 2 \sum_{n=-\infty}^{\infty} \mathrm{A}_n \sum_{\mu=-\infty}^{\infty} e^{in\theta_\mu} C_\mu \, e^{iky\sin\theta_\mu + ikx\cos\theta_\mu} \tag{6}$$

and for the reflected region ($x < 0$)

$$\phi_R = e^{i\vec{k}\cdot\vec{x}} + 2 \sum_{n=-\infty}^{\infty} \mathrm{A}_n \sum_{\mu=-\infty}^{\infty} e^{in(\pi-\theta_\mu)} C_\mu \, e^{iky\sin\theta_\mu + ikx\cos\theta_\mu} \tag{7}$$

with

$$C_\mu = \frac{1}{kd\cos\theta_\mu} \quad \text{for } |Re\ \theta_\mu| \le \frac{\pi}{2} \tag{8}$$

and the directions of the planar wave modes are given by

$$\sin\theta_\mu = \sin\theta_o + \frac{2\pi}{kd}\,\mu \quad \text{for } \mu = 0, \pm 1, \pm 2, \ldots \tag{9}$$

It can be seen from Eq. 7 that when $|\sin\theta_\mu| < 1$, progressive waves exist, but when $|\sin\theta_\mu| > 1$ there are evanescent modes. If $kd < \pi$, therefore, only one progressive wave exists.

Introducing a multiple scattering amplitude $G(\theta_\mu) = \sum_{n=-\infty}^{\infty} \mathrm{A}_n e^{in\theta_\mu}$, reflection and transmission coefficients for a mode μ are

$$K_{R_\mu} = 2C_\mu G(\pi - \theta_\mu) \tag{10}$$

and

$$K_{T_\mu} = \delta_{\mu o} + 2C_\mu G(\theta_\mu) \tag{11}$$

where δ denotes the Kronecker delta. From conservation of energy flux, we have

$$\sum_{\mu=\mu^-}^{\mu=\mu^+} \left(|K_{R_\mu}|^2 + |K_{T_\mu}|^2 \right) \frac{\cos\theta_\mu}{\cos\theta_o} = 1 \tag{12}$$

where

$$\mu^- = E[-(1+\sin\theta_o)\frac{kd}{2\pi}] \tag{13}$$

$$\mu^+ = E[\,(1-\sin\theta_o)\frac{kd}{2\pi}] \tag{14}$$

and E[x] stands for the integer value of x.

Slow convergence of the series in Eq. (5), which is a family of Schlömilch series, leads to a difficulty in evaluating the multiple scattering coefficients. Twersky (1961) found a way to accelerate the convergence of the series. In the present method, Shanks transformation was used to evaluate the infinite series in Twersky's representation.

Scattering coefficients A_n were obtained by retaining up to the M^{th} harmonic in Eq. (5):

$$\mathrm{A}_n = -\frac{\mathrm{J}_n'(ka)}{\mathrm{H}_n'(ka)}\left\{e^{-in\theta_o} + \sum_{m=-M}^{M} \mathrm{A}_m \sum_{s=1}^{\infty} \mathrm{H}_{n-m}(skd)\left[e^{-iskd\sin\theta_o} + (-1)^{n-m}e^{iskd\sin\theta_o}\right]\right\} \tag{15}$$

for $n = -M, \ldots, M$. (Care must be taken when Bessel function J_n is calculated using a recurrence formula, because of roundoff errors; see Abramowitz and Stegun, 1965, section 9.12.)

Twersky (1962) was able to get approximate solutions when there is only one propagating mode. Present numerical solutions of $\mathrm{K_T}$ and $\mathrm{K_R}$ were compared to Twersky's results in Figs. 2 and 3. The numerical solutions satisfied conservation of energy flux up to 10 digits for all the cases tested. Twersky's solutions (Eqs. 127 and 144) are valid only if $ka < a/d < 1$. As shown in Fig. 3, Twersky's solution does not agree to the numerical solution. For a fixed radius, longer waves propagate through the grating more easily. Normal incidence waves to the grating axis are more reflective.

3 Wide–spacing Approximation

We now move to wave scattering problems by multi-row gratings, where more than one length scale is involved. We further assume that the distance between gratings is large enough, so that gratings interact with each other via the propagating waves only; hence we can neglect the evanescent modes. For a longer wave train, $(kd < \pi)$, only one wave transmits through the grating with the given incident angle.

In the case of two gratings, the interference can be explained physically. When an incident wave train, proportional to $e^{i\ell x}$, where $\ell = k\sin\theta_o$, strikes the first grating at $x = 0$, part of the wave train is transmitted and part is reflected. Upon seeing the grating at $x = s$, the transmitted wave undergoes the same scattering process; part of the wave is transmitted to $x > s$ and part is back-reflected toward the first grating. This back and forth process of transmission and reflection is repeated infinitely. (See Fig. 4). Mathematically, we have

$$K_R = R_1 + T_1^2 R_2 e^{i2\ell s} + T_1^2 R_1 R_2^2 e^{i4\ell s} + T_1^2 R_1^2 R_2^3 e^{i6\ell s} + \cdots \tag{16}$$

$$= \frac{R_1 + R_2 e^{i2\ell s}(T_1^2 - R_1^2)}{1 - R_1 R_2 e^{i2\ell s}} \tag{17}$$

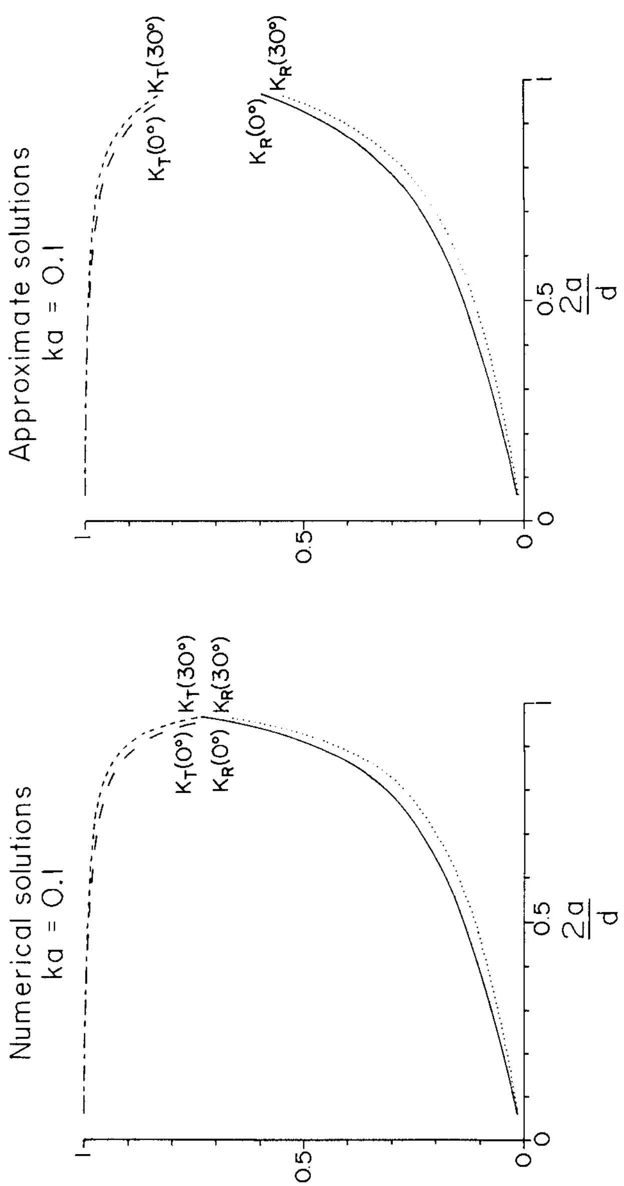

Figure 2: Comparison of Numerical and Approximate Solutions for Two Angles of Incidence for Small *ka*.

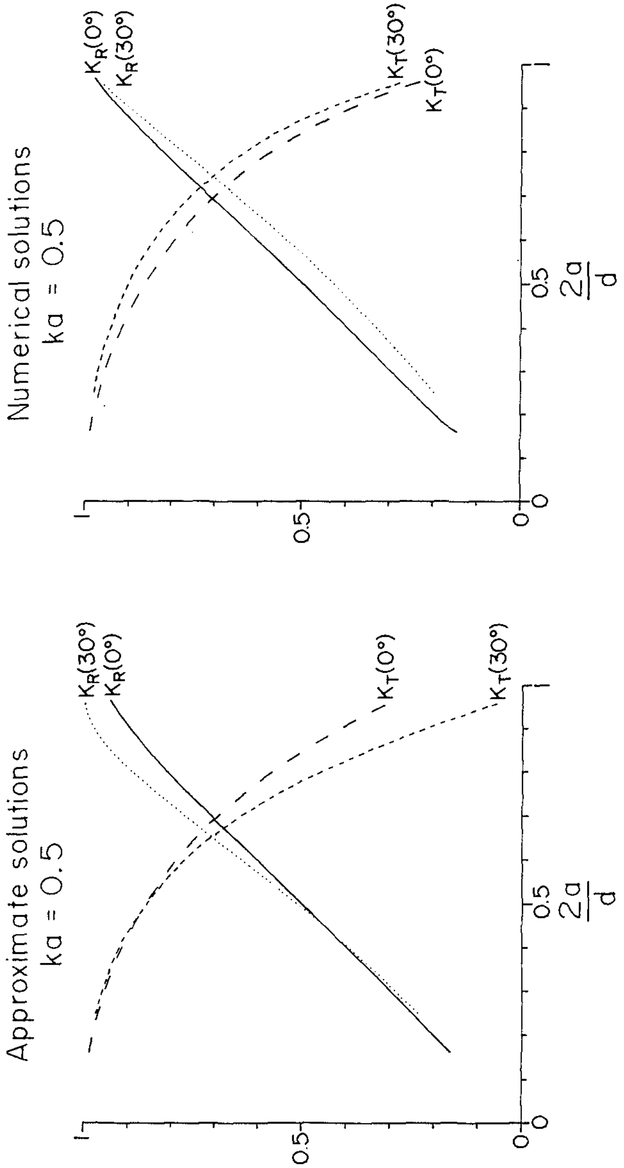

Figure 3: Comparison of Numerical and Approximate Solutions for Two Angles of Incidence for Larger ka.

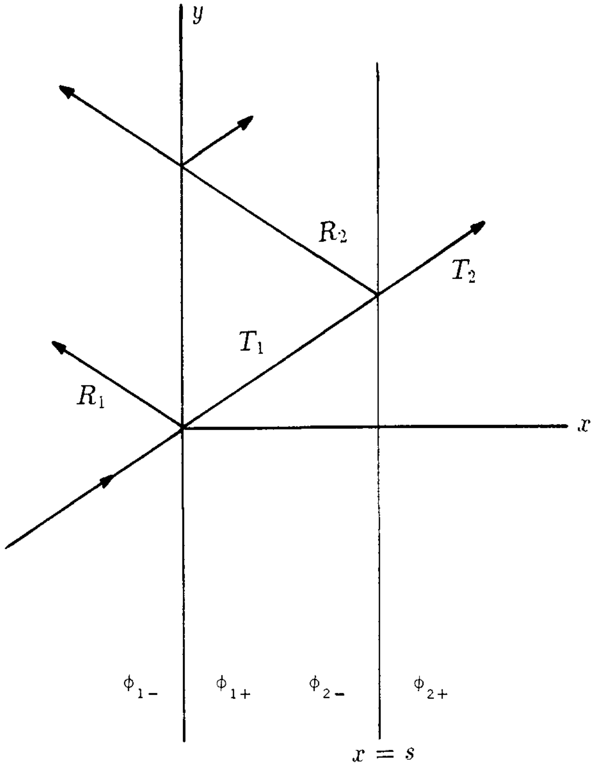

Figure 4: Reflection and Transmission by Two Gratings

$$K_T = T_1T_2 + T_1T_2R_1R_2e^{i2\ell s} + T_1T_2R_1^2R_2^2e^{i4\ell s} + \cdots \tag{18}$$

$$= \frac{T_1T_2}{1 - R_1R_2e^{i2\ell s}} \quad . \tag{19}$$

where T_i and R_i $(i = 1,2)$ are the transmission coefficient and reflection coefficient of the i^{th} grating, respectively. Stokes (1862) used this method to find light intensity in heterogeneous media. To generalize the formulation, we solve N-row grating problems using the wide-spacing approximation. Let the solution to scattering problem for the n^{th} grating by an incident wave $e^{il(x-x_n)}$ with $l = k\cos\theta_o$ be ϕ_{n-}. We define transmission and reflection coefficients, T_{n-} and R_{n-}, respectively, by

$$\phi_{n-} \sim \begin{cases} e^{i\ell(x-x_n)} + R_{n\_}e^{-i\ell(x-x_n)}, & \text{for } x \to -\infty \\ T_{n\_}e^{i\ell(x-x_n)}, & \text{for } x \to \infty \end{cases} \tag{20}$$

Similarly, let ϕ_{n+} solve the scattering problem for the n^{th} grating by an incident wave from the other direction, $e^{-i\ell(x-x_n)}$, and define T_{n+} and R_{n+} by

$$\phi_{n+} \sim \begin{cases} e^{-i\ell(x-x_n)} + R_{n_+} e^{i\ell(x-x_n)}, & \text{for } x \to \infty \\ T_{n_+} e^{-i\ell(x-x_n)}, & \text{for } x \to -\infty \end{cases} \tag{21}$$

Hence we can represent near field potential of the n^{th} grating by

$$\phi = C_n^- \phi_{n-} + C_n^+ \phi_{n+} \tag{22}$$

$$\sim \begin{cases} \left(C_n^- e^{-i\ell x_n}\right) e^{i\ell x} + \left(C_n^- R_{n\_} e^{i\ell x_n} + C_n^+ T_{n_+} e^{i\ell x_n}\right) e^{-i\ell x}, & \text{for } x \to -\infty \\ \left(C_n^- T_{n\_} e^{-i\ell x_n} + C_n^+ R_{n_+} e^{-i\ell x_n}\right) e^{i\ell x} + \left(C_n^+ e^{i\ell x_n}\right) e^{-i\ell x}, & \text{for } x \to \infty \end{cases} \tag{23}$$

where the $C_n^\pm$ denote unknown complex constants to be determined.

We now require that the near field solutions be joined smoothly in intermediate regions. For a region between the $(n-1)^{th}$ and n^{th} gratings, we have

$$\left[C_{n-1}^-(T_{n-1})_- e^{-i\ell x_{n-1}} + C_{n-1}^+(R_{n-1})_+ e^{-i\ell x_{n-1}}\right] e^{i\ell x} + \left[C_{n-1}^+ e^{i\ell x_{n-1}}\right] e^{-i\ell x} = \tag{24}$$

$$\left[C_n^- e^{-i\ell x_n}\right] e^{i\ell x} + \left[C_n^-(R_n)_- e^{i\ell x_n} + C_n^+(T_n)_+ e^{i\ell x_n}\right] e^{-i\ell x}, \quad x_{n-1} < x < x_n \tag{25}$$

Hence

$$C_{n-1}^-(T_{n-1})_- e^{i\ell\Delta_{n-1}} + C_{n-1}^+(R_{n-1})_+ e^{i\ell\Delta_{n-1}} - C_n^- = 0 \tag{26}$$

and

$$C_{n-1}^+ e^{-i\ell\Delta_{n-1}} - C_n^-(R_n)_- - C_n^+(T_n)_+ = 0 \tag{27}$$

where $\Delta_n = x_{n+1} - x_n$. This near field matching gives 2(N-1) equations for $n = 2, \mathrm{N}$.

On the other hand, the total velocity potential due to all the gratings can be expressed by

$$\phi \sim \begin{cases} e^{i\ell x} + K_R e^{-i\ell x}, & \text{for } x \to -\infty \\ K_T e^{i\ell x}, & \text{for } x \to \infty \end{cases} \tag{28}$$

Noting that $x_1 = 0$ and $C_1^- = 1$, and comparing Eqs. (23) and (28), we have

$$R_{1\_} + C_1^+ T_{1_+} = K_R \tag{29}$$

Since there is no wave incident on the N^{th} grating from the left, C_N^+ must be zero. Comparing Eqs. (23) and (28) gives

$$C_N^-(T_N)_- e^{i\ell x_N} = K_T \tag{30}$$

Finally, we have 2N unknown coefficients: $K_R, C_1^+, C_2^-, C_2^+, \cdots, C_N^-, K_T$. Eqs. (26), (27), (29) and (30) give 2N linear equations, enough to solve 2N unknowns.

In Fig. 5, we tested the present method for two identical rows of cylinders against Kakuno's experimental data. In this case, $T_{n-} = T_{n+}$ and $R_{n-} = R_{n+}$ $(n = 1, 2)$ because depth is constant. We introduce two nondimensional parameters associated with the grating geometry; spacing parameter $\alpha = s/d$ and grating parameter $\beta = 2a/d$. In the wide-spacing approximation, we can expect $\alpha \gg 1$ and shorter spacing between gratings increases

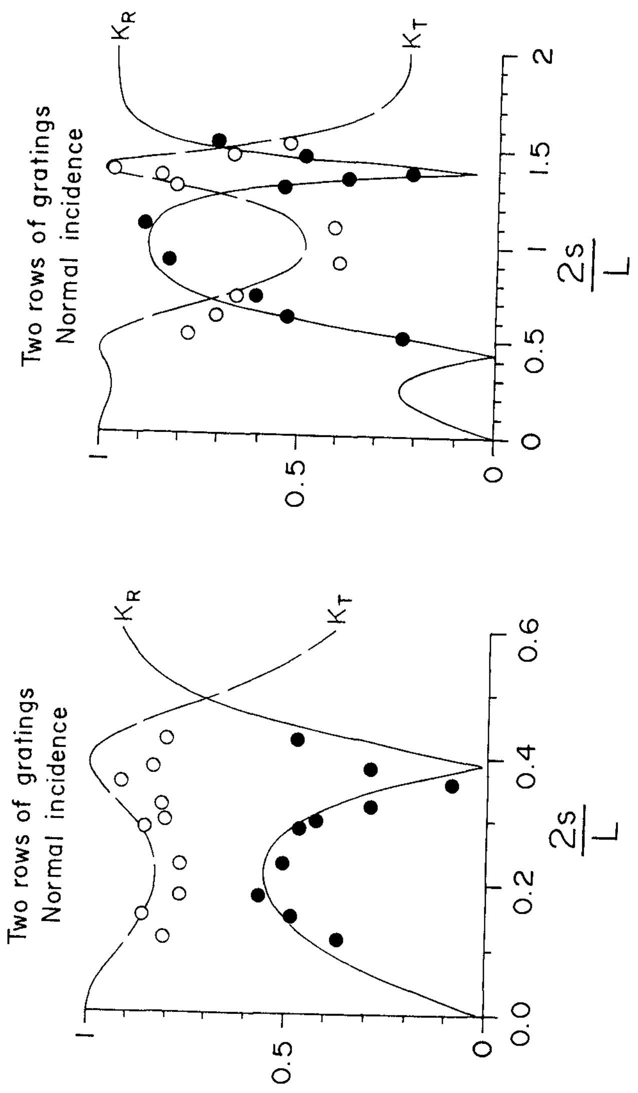

Figure 5: Comparison of Reflection and Transmission Coefficients with Kakuno's Data

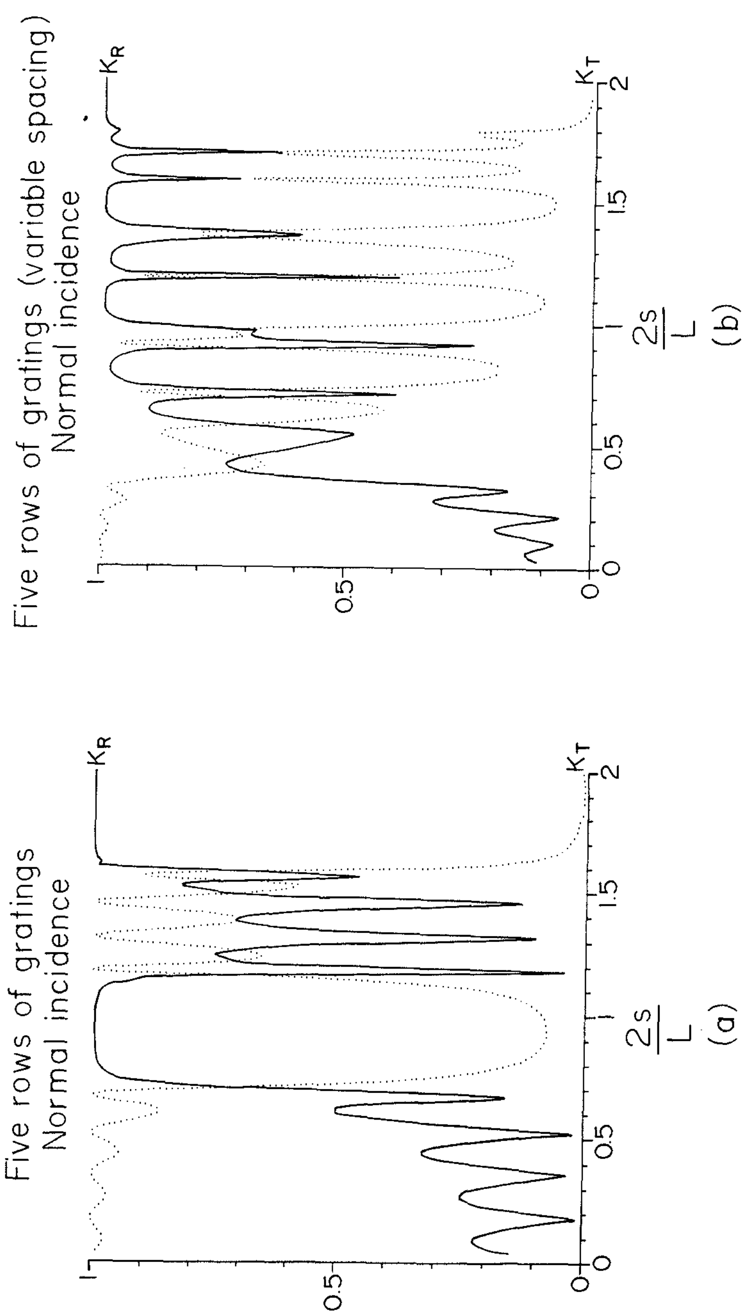

Figure 6: Reflection and Transmission Coefficients for Five Rows of Cylinders: Equal and Variable Spacing

the effect of evanescent modes. Fig. 5a shows this effect, which produces less agreeable results. For some values of α, a significant amount of waves are reflected. Under this condition, the phases of the reflected waves are the same, so that the reflected waves interfere constructively and the total amplitude is consequently the larger. This is referred to as Bragg scattering.

In Fig. 6, we presented K_R and K_T for five identical rows of cylinders. The interference of each grating is shown to be more complicated and total reflection can be observed for some values of α. In Fig. 6a, equally spaced gratings were used, while in Fig. 6b the spacing between the gratings increases from s for the first two gratings, by increments of 1.5s, to 2.5s. It is also interesting to note that some arrangements of gratings are essentially transparent to the waves.

4 Conclusion

Numerical results of wave scattering by N-row gratings are presented. Accurate numerical solutions for a single grating are constructed following Twersky (1962) procedure. For long waves ($ka << 1$), Twersky solution is identical to the numerical solution, but, as ka increases, Twersky's solution shows considerable deviation form the numerical solution. Using the wide-spacing approximation, reflection and transmission coefficients of long waves through N-row gratings are obtained.

For two identical rows of gratings, the present method shows good agreement with Kakuno's experimental data. Results for five rows of gratings are also presented.

5 Acknowledgments

Support for this research was provided by the NOAA Office of Sea Grant, Department of Commerce, under Grant NA87AA-D-SG040.

6 References

1. Abramowitz, M. and I.A. Stegun, 1965, **Handbook of Mathematical Functions**, Dover Publications, Inc., New York.

2. Hayashi, T., T. Kano and M. Shirai, 1966, "Hydraulic research on the closely spaced pile breakwater," Proc. 10th Intl. Conf. Coastal Engineering , ASCE, 873-884.

3. Kakuno, S. *et al.*, 1984, "Analysis of boundary value problems for hydraulic characteristics of vertically slotted breakwaters," Coastal Engineering in Japan (in Japanese).

4. Kakuno, S., 1986, Abstracts, 20th Int. Conf. Coastal Engineering, ASCE, Taiwan.

5. Lamb, H., 1898, "On the reflection and transmission of electric waves by a metallic grating," *Proc. London Math. Soc.*, Series 1, 29, 523-544.

6. Martin, P.A. and R.A. Dalrymple, 1988, "Scattering of long waves by cylindrical obstacles and gratings using matched asymptotic expansions," *J. Fluid Mechanics*, 188, 465-490.

7. Mei, C.C., P.L.-F. Liu, and A.T. Ippen, 1974, "Quadratic loss and scattering of long waves," *J. Waterways and Harbors Div.*, ASCE, 100, 217-239.

8. Ozsoy, E., 1977, "Dissipation and scattering by narrow openings," University of Florida, Coastal and Oceanographic Engineering Laboratory, UFL/COEL/TR-037, 137pp.

9. Spring, B.W. and Monkmeyer, P.L., 1975, "Interaction of plane waves with a row of cylinders," Proc. Civil Engineering in the Oceans III, ASCE, Newark, Delaware.

10. Srokosz, M.A. and Evans, D.V., 1979, "A theory for wave-power absorption by two independently oscillating bodies," *J. Fluid Mechanics*, 90, 337-362.

11. Stokes, G.G., 1862, "On the intensity of light reflected from or transmitted through a pile of plates," *Proc. Royal Society of London*, XI, 545-557.

12. Twersky, V., 1961, "Elementary function representations of Schlömilch series," *Arch. Rat. Mech. Anal.*, 8, 323-332

13. Twersky, V., 1962, "On the scattering of waves by the infinite grating of circular cylinders," *IRE Trans. Antennas Propag.*, 10, 737-765.

14. Wiegel, R.L., 1961, "Closely spaced piles as a breakwater," *Dock and Harbor Authority*, Sept. 1961, 150.