Complex Analysis

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1 Introduction

All calculus textbooks start with $f(x)$: f is a function of one (real) variable x. Topics covered include limits, continuity, differentiation, and integration, with the associated notation, such as $\mathrm{d}f/\mathrm{d}x = f'(x)$ and $\int_a^b f(x)\,\mathrm{d}x$. It is also usual to include a discussion of infinite sequences and series. The rigorous treatment of all these topics comprises *real analysis*.

Complex analysis starts with the following question. What happens if we replace x by $z = x + \mathrm{i}y$, where x and y are two independent real variables and $\mathrm{i} = \sqrt{-1}$? Answering this question leads to rich new fields of mathematics: we shall be concerned with those parts that are used in applied mathematics.

Let us begin with basic terminology and concepts. We call $z = x + \mathrm{i}y$ a *complex variable*. The *imaginary unit* i should be treated as a symbol that obeys all the usual laws of algebra together with $\mathrm{i}^2 = -1$. We call $x = \operatorname{Re} z$ the *real part* of z and $y = \operatorname{Im} z$ the *imaginary part* of z. We can identify $z = x + \mathrm{i}y$ with a point in the xy-plane (known as the *z-plane* or the *complex plane*).

The *complex conjugate* of z is $\bar{z} = x - \mathrm{i}y$: complex conjugation is reflection in the x-axis. The *absolute value* (or *modulus* or *magnitude*) of z is $|z| = +\sqrt{x^2 + y^2}$, the distance from z to the origin. Given $w = u + \mathrm{i}v$, we define $z + w = (x + u) + \mathrm{i}(y + v)$. Addition of complex quantities is therefore equivalent to addition of two-dimensional vectors. For multiplication, $zw = xu - yv + \mathrm{i}(xv + yu)$. Putting $z = w$ shows that $\operatorname{Re} z^2 = x^2 - y^2 \neq x^2$ unless z is real ($y = 0$). Also, $z\bar{z} = |z|^2$ and $z/w = z\bar{w}/|w|^2$ when $w \neq 0$.

Introducing plane polar coordinates, r and θ, we have $z = r\cos\theta + \mathrm{i}r\sin\theta = r\mathrm{e}^{\mathrm{i}\theta}$ by Euler's formula. Thus, $r = |z|$. The angle θ is called an *argument* of z, denoted by $\arg z$ or $\operatorname{ph} z$ (for *phase*). Notice that $\arg z$ is not unique, as we can always add any integer multiple of 2π; this nonuniqueness is sometimes useful and sometimes a nuisance.

If we let $r \to \infty$, the point z recedes to infinity. It is usual to state that there is a single "point at infinity," denoted by $z = \infty$, that is reached by letting $r \to \infty$ in any direction, θ. Alternatively, we can state that the formula $z = 1/w$ takes the point $w = 0$ to the point $z = \infty$.

2 Functions

A function of a complex variable, $f(z)$, is a rule: given $z = x + \mathrm{i}y$ in some set (the *domain* of f), the rule provides a unique complex number denoted by $f(z) = u + \mathrm{i}v$, say, where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are real. We write $f(z) = u(x, y) + \mathrm{i}v(x, y)$ to emphasize the dependence on x and y.

Simple examples of functions are $f(z) = z^2$ and $f(z) = \bar{z}$. Elementary functions are defined "naturally": for example, $\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x\mathrm{e}^{\mathrm{i}y}$ and $\cos z = \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})$. For powers and logarithms, we have the formulas $z^\alpha = r^\alpha\mathrm{e}^{\mathrm{i}\alpha\theta}$ (α is real) and $\log z = \log(r\mathrm{e}^{\mathrm{i}\theta}) = \log r + \mathrm{i}\theta$. Strictly, these do not define functions because of the nonuniqueness of θ: changing θ by 2π does not change z but it does change the values of z^α (unless α is an integer) and $\log z$. One response to this phenomenon is to say that $\log z$, for example, is a multivalued "function": increasing θ by 2π takes us onto another *branch* or *Riemann sheet* of $\log z$. However, in practice, it is usually better to introduce a *branch cut*, which, for $\log z$, is any line from $z = 0$ to $z = \infty$. This cut is regarded as an artificial barrier: we must not cross it. Its presence prevents us from increasing θ by 2π. For example, we could restrict θ to satisfy $-\pi < \theta < \pi$ and put the cut on the negative x-axis. Once we have restricted θ to lie in some interval of length 2π, $\log z$ and z^α become single-valued: they are now functions. We shall have more to say about branches in section 4.

There are many other ways to define functions. For example, $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is a function provided the *power series* converges at z: typically, power series converge in disks, $|z| < R$, for some $R > 0$ (R is the *radius of convergence*). The prototype power series is the *geometric series*; it converges inside the unit disk where its sum is known:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1. \tag{1}$$

For another example, take

$$f(z) = \int_0^\infty g(t)\mathrm{e}^{-zt}\,\mathrm{d}t. \tag{2}$$

This defines the *Laplace transform* of g. Typically, such integrals converge for $\operatorname{Re} z > A$, where A is a constant that depends on g. Another function defined by an integral is Euler's gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t, \quad \operatorname{Re} z > 0. \tag{3}$$

Much is known about the properties of Γ. For example, $\Gamma(n) = (n-1)!$ when n is any positive integer. There is more on this in section 13 below.

3 Analytic Functions

The notions of limit, continuity, and derivative are defined exactly as in real-variable calculus. In particular, the *derivative* of f at z is defined by

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}, \tag{4}$$

provided the limit exists. Here, h is allowed to be complex: the point $z + h$ must be able to approach the point z in any direction, and the limit must be the same. As a consequence, if $f(z) = u(x,y) + \mathrm{i}v(x,y)$ has a derivative, $f'(z)$, at z, then u and v satisfy the *Cauchy–Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{5}$$

If these are not both satisfied, then $f'(z)$ does not exist. Two examples: $f(z) = \bar{z}$ is not differentiable for any z; and any real-valued function $f(z) = u(x,y)$ is not differentiable unless u is a constant. If both Cauchy–Riemann equations (5) are satisfied and the partial derivatives in (5) are continuous functions, then f' exists.

Using (5), if f' exists, then

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} \\ &= \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \mathrm{i}\frac{\partial v}{\partial x}. \end{aligned}$$

The first equality follows by taking h to be real in (4) and the second by taking h to be purely imaginary. The four formulas for f' show that we can calculate f' from $\operatorname{Re} f$ or $\operatorname{Im} f$, or by differentiating with respect to x or y.

Differentiability is a local property, defined at a point z. Usually, we are interested in functions that are differentiable at all points in their domains. Such functions are called *analytic* or *holomorphic*. Points at which a function is not differentiable are called *singularities*.

Derivatives of higher order (such as $f''(z)$) are defined in the natural way. One surprising fact is that a differentiable function can be differentiated any number of times: once differentiable implies infinitely differentiable (see (13) below for an indication of a proof). This result is certainly not true for real functions.

If we eliminate v from (5), we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{6}$$

Thus, the real part of an analytic function, $u(x,y)$, satisfies LAPLACE'S EQUATION [**??**] (6). The imaginary part, $v(x,y)$, satisfies the same partial differential equation (PDE). This reveals a close connection between analytic functions and solutions of one particular PDE. As Laplace's equation arises in the modeling of many physical phenomena, this connection has been exploited extensively.

4 More on Branches

Let us return to log, which we can define as a (single-valued) function by

$$\log z = \log r + \mathrm{i}\theta, \quad r > 0, \quad -\pi < \theta < \pi, \tag{7}$$

with $z = r\mathrm{e}^{\mathrm{i}\theta}$. There is a branch cut along the negative x-axis, with a branch point at $z = 0$ and a branch point at $z = \infty$. Thus, our domain of definition for $\log z$ is the cut plane, i.e., the whole complex plane with the cut removed. Then, $\log z$ is analytic: it is differentiable at all points in its domain of definition. Moreover, $(\mathrm{d}/\mathrm{d}z)\log z = z^{-1}$.

According to (7), $\log z$ is not defined *on* the negative x-axis. Some authors regard this as unacceptable, and so they replace the (open) interval $-\pi < \theta < \pi$ in (7) by $-\pi < \theta \leqslant \pi$ or $-\pi \leqslant \theta < \pi$. The first choice gives, for example, $\log(-1) = \mathrm{i}\pi$ and the second gives $\log(-1) = -\mathrm{i}\pi$. Either choice enlarges the domain of definition to the whole plane with $z = 0$ removed. However, we lose analyticity: $\log z$ is not differentiable on the line $\theta = \pi$ (first choice) because points on that line are not accessible in all directions (as they must be if one wants to compute limits, as in the definition of derivative) without leaving the domain of definition. For some applications this may be acceptable but, in practice, it is usual to simply move the cut. We can therefore replace $-\pi < \theta < \pi$ in (7) by another open interval, $\theta_0 < \theta < 2\pi + \theta_0$, implying a cut along the straight half-line $\theta = \theta_0$, $r \geqslant 0$. (In fact, the cut need not be straight: any line connecting the branch point at $z = 0$ to $z = \infty$ may be used.) Then $\log z$ is analytic in a new cut plane.

Once we define a function such as $\log z$ or $z^{1/2}$ with a specified range for θ, we can say that we have defined a *principal value* of that function. Certain choices (such as $-\pi < \theta < \pi$ or $-\pi < \theta \leqslant \pi$) are common, but the reader should not overlook the option of moving cuts when it is convenient to do so.

There is another consequence of insisting on having (single-valued) functions: some standard identities, such as $\log(z^2) = 2\log z$, may no longer hold. For example, with $z = -1 + \mathrm{i}$ and the definition (7), we find $\log(z^2) = \log 2 - \frac{1}{2}\mathrm{i}\pi$ but $2\log z = \log 2 + \frac{3}{2}\mathrm{i}\pi$.

Summarizing, functions with branches are very common (for another example, see the article on THE LAMBERT W-FUNCTION [??]) but their presence often leads to complications, subtle difficulties, and calculational errors: care is always required.

5 Infinite Series

A power series about the point z_0 has the form

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n, \tag{8}$$

where the coefficients c_n are complex numbers. The series (8) converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$, where the radius of convergence, R, may be finite or infinite. (It may happen that (8) converges at $z = z_0$ only, with sum c_0.) When the series does converge, we denote its sum by $S(z)$. For an example, see the geometric series (1).

The sum $S(z)$ is analytic for $|z-z_0| < R$: power series define analytic functions.

Now we turn this around. We take an analytic function, $f(z)$, and we try to write it as a power series. Doing this is familiar from calculus, and the result is Taylor's theorem:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,$$

where $f^{(n)}$ is the nth derivative of f. The series is known as the *Taylor expansion* of $f(z)$ about z_0. It converges for $|z-z_0| < R$, where R is the distance from z_0 to the nearest singularity of $f(z)$. A Taylor expansion about the origin ($z_0 = 0$) is known as a *Maclaurin expansion*. All these expansions are the same as those occurring in the calculus of functions of one real variable. For example, (1) gives the Maclaurin expansion of $1/(1-z)$. Another familiar Maclaurin expansion is $\mathrm{e}^z = \sum_{n=0}^{\infty} z^n/n!$, which is convergent for all z.

A generalization of Taylor's theorem, Laurent's theorem, will be given in section 8.

Not all infinite series are power series. A famous series is the *Riemann zeta function*, which is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \text{for } \operatorname{Re} z > 1, \tag{9}$$

which is intimately connected with the distribution of the prime numbers.

It is possible to develop the theory of analytic functions by starting with power series: this approach, which goes back to Weierstrass, has a constructive flavor. We started with the notion of differentiability: this approach, which goes back to Riemann and Cauchy, is closer to real-variable calculus. The two approaches are equivalent: power series define analytic functions and analytic functions have power-series expansions.

6 Contour Integrals

In the calculus of functions of two real variables x and y, double integrals over regions of the xy-plane and line integrals along curves in the xy-plane are defined. In complex analysis, we are mainly concerned with integrals along curves in the z-plane. They are defined similarly to line integrals. Thus, suppose that points on a curve C are located by a parametrization,

$$C\colon z(t) = x(t) + \mathrm{i}y(t), \quad a \leqslant t \leqslant b,$$

where a and b are constants and $x(t)$ and $y(t)$ are real functions of the real variable t. As t increases from a to b, $z(t)$ moves from $z(a)$ to $z(b)$: the parametrization induces a direction or *orientation* on C. The curve C is *smooth* if $x'(t)$ and $y'(t)$ exist and are continuous. Then, if $f(z)$ is defined for all points z on a smooth curve C,

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t, \tag{10}$$

where $z'(t) = x'(t) + \mathrm{i}y'(t)$. In (10), the right-hand side defines the expression on the left-hand side as an integration with respect to the parameter t. More generally, suppose that C is a *contour*, defined as a continuous curve made from smooth pieces joined at corners. Then, to define $\int_C f\,\mathrm{d}z$, we parametrize each smooth piece separately, and then sum the contributions from each piece, ensuring that the parametrizations are such that z moves continuously along C.

If C_o is the same curve as C but traversed in the opposite direction (from $z(b)$ to $z(a)$), then $\int_{C_o} f(z)\,\mathrm{d}z = -\int_C f(z)\,\mathrm{d}z$: changing the direction changes the sign.

7 Cauchy's Theorem

A contour C is *closed* if $z(a) = z(b)$, and it is *simple* if it has no self-intersections. Cauchy's theorem can be stated as follows. Suppose that $f(z)$ is analytic inside a simple closed contour C and continuous on C. Then

$$\int_C f(z)\,\mathrm{d}z = 0. \tag{11}$$

It is worth emphasizing the hypotheses. First, we do not need to know anything about $f(z)$ outside C: Cauchy assumed stronger conditions but these were later weakened by Goursat. Second, C is a contour, so corners are allowed. Third, by requiring that "f is analytic inside C," we mean that $f(z)$ must be differentiable at all points z inside C: singularities (including branch points) are not allowed (although they may be present outside C).

There are many consequences of Cauchy's theorem. One is known as *deforming the contour*. Suppose that C_1 and C_2 are simple closed contours, both traversed in the same direction, with C_1 enclosed by C_2. Suppose that $f(z)$ is analytic in the region between C_1 and C_2 and that it is continuous on C_1 and C_2. (Note that f may have singularities inside the smaller contour C_1 or outside the larger contour C_2.) Then, $\int_{C_1} f(z)\,\mathrm{d}z = \int_{C_2} f(z)\,\mathrm{d}z$: one contour can be deformed into another without changing the value of the integral, provided the integrand is analytic between the contours. The same result is true when C_1 and C_2 are two contours with the same endpoints, provided f is analytic between C_1 and C_2. These results are useful because they may allow us to deform a complicated contour into a simpler contour (such as a circle or a straight line).

Another consequence of Cauchy's theorem is the *Cauchy integral formula*. Under the same conditions, we have

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{z - z_0}\,\mathrm{d}z, \tag{12}$$

where z_0 is an arbitrary point inside C, and C is traversed counterclockwise. This shows that we can recover the values of an analytic function inside C from its values on C.

More generally, and again under the same conditions, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z. \tag{13}$$

Formally, this can be seen as the nth derivative of the Cauchy integral formula (12), but it is deeper: it can be used to prove the existence of $f^{(n)}$, for $n = 2, 3, \dots$, assuming that f' exists. This is done using an inductive argument. We have (compare with (4))

$$f^{(n+1)}(z_0) = \lim_{h \to 0} \frac{f^{(n)}(z_0 + h) - f^{(n)}(z_0)}{h},$$

provided the limit exists. Now, on the right-hand side, use (13) twice; the limit can then be taken.

Formula (13) with $n = 1$ can be used to prove *Liouville's theorem*. Suppose that $f(z)$ is analytic everywhere in the z-plane (that is, there are no singularities): such a function is called *entire*. Suppose further that $|f(z)| < M$ for some constant M and for all z: we say that f is *bounded*. Liouville's theorem states that a bounded entire function is necessarily constant. In other words, (nonconstant) entire functions must be large somewhere in the complex plane. For example,

$$\begin{aligned} |\cos z|^2 &= \cos z\, \overline{\cos z} = \tfrac{1}{4}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})(\mathrm{e}^{\mathrm{i}\bar{z}} + \mathrm{e}^{-\mathrm{i}\bar{z}}) \\ &= \tfrac{1}{4}(\mathrm{e}^{2\mathrm{i}x} + \mathrm{e}^{-2\mathrm{i}x} + \mathrm{e}^{2y} + \mathrm{e}^{-2y}) \\ &= \tfrac{1}{2}(\cos 2x + \cosh 2y) = \cos^2 x + \cosh^2 y - 1 \end{aligned}$$

using $z + \bar{z} = 2x$ and $z - \bar{z} = 2\mathrm{i}y$. Thus, $|\cos z|$ grows rapidly as we move away from the real axis (where $y = 0$ and $\cosh 0 = 1$).

8 Laurent's Theorem

Suppose that $f(z)$ is analytic inside an annulus, $a < |z - z_0| < b$, centered at z_0. We say nothing about $f(z)$ when z is in the "hole" of radius a ($|z - z_0| < a$) or when z is outside the annulus ($|z - z_0| > b$). We then have the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \tag{14}$$

for all z in the annulus, where the coefficients are given by contour integrals,

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z, \tag{15}$$

in which C is a simple closed contour in the annulus that encircles the hole (once) in the counterclockwise direction. Note that the sum in (14) is over all n. It is often convenient to split the sum, giving, for all z in the annulus,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \tag{16}$$

where $a_n = c_n$ for $n = 0, 1, 2, \dots$ and $b_n = c_{-n}$ for $n = 1, 2, \dots$. In particular,

$$b_1 = \frac{1}{2\pi \mathrm{i}} \int_C f(z)\,\mathrm{d}z. \tag{17}$$

Suppose that $f(z)$ is also analytic in the hole, so that $f(z)$ is analytic in the disk $|z - z_0| < b$. Then $b_n = 0$ ($n = 1, 2, \dots$) by Cauchy's theorem and $a_n = f^{(n)}(z_0)/n!$ by (13): Taylor's theorem is recovered. Note that, in general, when f does have singularities in the hole, we cannot use (13) to evaluate the contour integrals defining a_n.

9 Singularities

A singularity is a point at which a function is not differentiable. There are several kinds of singularities. A point z_0 is called an *isolated singularity* if there is an annulus $0 < |z - z_0| < b$ (a "punctured disk") in which there are no other singularities. In this annulus, we have a Laurent expansion, (16). The first part (the sum over a_n) is a power series, and so it defines an analytic function on the whole disk. The singular behavior resides in the second sum (over b_n): it is called the *principal part*, $P(z)$. In practice, $P(z)$ often has a finite number of terms,

$$P(z) = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad (18)$$

with $b_n = 0$ for all $n > m$ and $b_m \neq 0$. In this situation, we say that f has a *pole of order* m at z_0. A pole of order 1 is called a *simple pole* and a pole of order 2 is called a *double pole*. For example, all the following have simple poles at $z = 0$:

$$\frac{1}{z}, \quad \frac{1+z}{z}, \quad \frac{e^z}{z}, \quad \frac{\sin z}{z^2}, \quad \frac{\pi}{\sin \pi z}; \quad (19)$$

the last in this list also has simple poles at $z = \pm 1, \pm 2, \ldots$. All the following have double poles at $z = 0$:

$$\frac{1}{z^2}, \quad \frac{1+z}{z^2}, \quad \frac{1}{z \sin z}, \quad \frac{\cos z}{z^2}, \quad \frac{1}{\sin^2 \pi z}. \quad (20)$$

If the principal part of the Laurent expansion contains an infinite number of nonzero terms, z_0 is called an *isolated essential singularity*. For example, $\exp(1/z)$ has such a singularity at $z = 0$.

The coefficient b_1 (given by (17)) will play a special role later; it is called the *residue* of f at the isolated singularity, z_0, and it is denoted by $\text{Res}[f; z_0]$.

There are also nonisolated singularities. The most common of these occur at branch points. For example, $f(z) = z^{1/2}$ has a branch-point singularity at $z = 0$. Note that any disk centered at $z = 0$ will include a piece of the branch cut emanating from the branch point: f is discontinuous across the cut so it is certainly not differentiable there, implying that $z = 0$ is not an isolated singularity.

10 Cauchy's Residue Theorem

If we want to evaluate $I = \int_C f(z)\,dz$, the basic method is to parametrize each smooth piece of C and then use the definition (10). In principle, this works for any f and for any C. However, in practice, C is often closed and f is analytic apart from some singularities. In these happy situations, we can calculate I efficiently by using Cauchy's residue theorem. Thus, suppose that $f(z)$ is analytic inside the simple closed contour C (and continuous on C) apart from isolated singularities at z_j, $j = 1, 2, \ldots, n$. (Note that f may have other singularities, including branch points, outside C, but these are of no interest here.) Then

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}[f; z_j], \quad (21)$$

where C is traversed counterclockwise. This important result is remembered as "2πi times the sum of the residues at the isolated singularities inside the contour." If there are no singularities inside, we recover Cauchy's theorem (11).

To prove the theorem, we start with the case $n = 1$. There is a Laurent expansion about the sole singularity z_1, convergent in a punctured disk $0 < |z - z_1| < b$. We deform C into a smaller contour (enclosing z_1) that is inside the disk. Then we use (17). In the general case, we deform C and "pinch off," giving a sum of n contour integrals, each one containing one singularity.

To understand the pinching-off process, suppose that $n = 2$ and deform C into a dumbbell-shaped contour, with two circles joined by two parallel straight lines, L_1 and L_2, traversed in opposite directions. The contributions from L_1 and L_2 cancel in the limit as the lines go together, leaving the contributions from disjoint closed contours around each singularity. This process is readily extended to any (finite) number of isolated singularities.

In order to exploit the residue theorem, we need efficient methods for computing residues. Recall that $\text{Res}[f; z_0]$ is the coefficient b_1 in the Laurent expansion about z_0 (see (18)). For simple poles, b_1 is the only nontrivial coefficient in the principal part; thus, at a simple pole z_0,

$$\text{Res}[f; z_0] = \lim_{z \to z_0} \{(z - z_0) f(z)\}. \quad (22)$$

Often, simple poles are characterized by writing $f(z) = p(z)/q(z)$ with $q(z_0) = 0$, $p(z_0) \neq 0$, and $q'(z_0) \neq 0$. Then

$$\text{Res}[f; z_0] = p(z_0)/q'(z_0). \quad (23)$$

For a pole of order m, we can use

$$\text{Res}[f; z_0] = \frac{1}{m!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m f(z)\}.$$

However, it is sometimes quicker to construct the Laurent expansion directly, and then to pick off b_1, the coefficient of $1/(z - z_0)$. Thus, almost by inspection, all the simple-pole examples in (19) have $\text{Res}[f; 0] = 1$. The five double-pole examples in (20) have $\text{Res}[f; 0] = 0$, 1, 0, 0, and 0, respectively.

11 Evaluation of Integrals

Cauchy's residue theorem gives a powerful method for evaluating integrals. We give a few examples.

Let C be a circle of radius a centered at the origin and traversed counterclockwise. The function e^z/z has a simple pole at $z = 0$ with residue $= 1$. Hence

$$\int_C \frac{e^z}{z}\,dz = 2\pi i. \tag{24}$$

This result could have been obtained from the Cauchy integral formula (12) with $f(z) = e^z$ and $z_0 = 0$. Note also that the value of the integral does not depend on a; this is not surprising because we know that we can deform C into a concentric circular contour (for example) without changing the value of the integral.

Now, starting from the result (24), suppose we parametrize C and then use (10); a suitable parametrization is $z(t) = ae^{it}$, $-\pi \leqslant t \leqslant \pi$. As $z'(t) = iae^{it} = iz(t)$, we obtain

$$\int_{-\pi}^{\pi} \exp(ae^{it})\,dt = 2\pi.$$

By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, the integrand is $e^{a\cos t}\cos(a\sin t) + ie^{a\cos t}\sin(a\sin t)$. The second term is an odd function of t and so it integrates to zero, leaving

$$\int_0^{\pi} e^{a\cos t}\cos(a\sin t)\,dt = \pi. \tag{25}$$

Thus, from the known value of a fairly simple contour integral, (24), we obtained the value of a complicated real integral. Notice that the formula (25) was derived by assuming that the parameter a is real and positive. In fact, it is valid for arbitrary complex a; this is an example of analytic continuation (see section 13).

We now consider doing the opposite: evaluating integrals by converting them into contour integrals, followed by use of the residue theorem.

For trigonometric integrals such as

$$I_1 = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta},$$

the substitution $z = e^{i\theta}$ will convert I_1 into a contour integral around the unit circle, $|z| = 1$. Using $d\theta/dz = 1/(iz)$ and $\cos\theta = \frac{1}{2}(z + z^{-1})$, we obtain

$$I_1 = \frac{1}{2i}\int_{|z|=1} \frac{dz}{(z+2)(z+\frac{1}{2})}.$$

The integrand is analytic apart from simple poles at $z = -2$ and $z = -\frac{1}{2}$. The latter is inside the contour; its residue is $\frac{2}{3}$ (use (22)). Cauchy's residue theorem (21) therefore gives $I_1 = \frac{2}{3}\pi$.

The method just described requires that the range of integration for θ has length 2π and that the resulting integrand has only isolated singularities (not branch points) inside $|z| = 1$.

For a second example, consider

$$I_2 = \int_{-\infty}^{\infty} f(x)\,dx \quad \text{with } f(x) = \frac{1}{x^4 + 1}.$$

In order to use the residue theorem, we need a closed contour C, so we try $\int_C f(z)\,dz$ with C consisting of a piece of the real axis from $z = -R$ to $z = R$ and a semicircle C_R in the upper half-plane of radius R and centered at $z = 0$. Then

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \times \begin{cases} \text{residues} \\ \text{at poles} \\ \text{inside } C. \end{cases} \tag{26}$$

After calculation of the residues, we let $R \to \infty$, so that the first integral $\to I_2$. We will see in a moment that the second integral $\to 0$ as $R \to \infty$.

Now, $z^4 + 1 = 0$ at $z = z_n = \exp(i(2n+1)\pi/4)$, $n = 0, 1, 2, 3$. These are simple poles of $f(z)$ with residue $1/(4z_n^3)$ (use (23) with $p = 1$, $q = z^4 + 1$). The poles z_0 and z_1 are in the upper half-plane. Hence the right-hand side of (26) is $\pi/\sqrt{2}$, and this is I_2.

For $z \in C_R$ we parametrize using $z(t) = Re^{it}$, $0 \leqslant t \leqslant \pi$. We see that $f(z)$ decays as R^{-4}, whereas the length of C_R, πR, increases; overall, $\int_{C_R} f\,dz$ decays as R^{-3}. This rough argument can be made precise.

If we replace $f(z)$ by $e^{ikz}f(z)$, we can evaluate Fourier transforms such as

$$I_3 = \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx \quad \text{with } f(x) = \frac{1}{x^2 + 1},$$

where k is a real parameter. However, some care is needed: as $e^{ikz} = e^{ikx}e^{-ky}$, we have exponential decay as $y \to \infty$ when $k > 0$, but exponential growth when $k < 0$. Therefore, we use C_R when $k > 0$ but we close using a semicircle in the lower half-plane when $k < 0$. We find that $I_3 = \pi e^{-|k|}$.

Laplace transforms (2) can be inverted using

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(z)\,e^{zt}\,dz.$$

The contour (called the *Bromwich contour*) is parallel to the y-axis in the z-plane. The constant c is chosen so that all the singularities of $f(z)$ are to the left of the contour. If $f(z)$ has poles only, the integral can be evaluated by closing the contour using a large semicircle on the left.

There are many other applications of contour-integral methods to the evaluation of integrals. They can also be used to find the sums of infinite series. Integrands containing branch points can also be considered. In all cases, one may need some ingenuity in selecting an appropriate closed contour and/or the function $f(z)$.

12 Conformal Mapping

Suppose that $f(z)$ is analytic for $z \in D$. We can regard f as a mapping, taking points $z = x + \mathrm{i}y$ to points $w = f(z) = u + \mathrm{i}v$; denote the set of such points in the uv-plane for all points $z \in D$ by R. Given f and D, we can determine R. More interestingly, given the regions D and R, can we find an analytic function f that maps D onto R? The *Riemann mapping theorem* asserts that any simply connected region D can be mapped to the unit disk $|w| < 1$. (A region bounded by a simple closed curve is simply connected if it does not contain any holes.) The analytic function f effecting the mapping is called a CONFORMAL MAPPING [**??**]: two small lines meeting at a point $z_0 \in D$ will be mapped into two small lines meeting at a point $w_0 = f(z_0) \in R$, and the angles between the two pairs of lines will be equal. The conformality property holds for all $z_0 \in D$ except for *critical points* (where $f'(z_0) = 0$ or ∞). Many conformal mappings are known (there are dictionaries of them) but constructing them for regions D with complicated shapes or holes remains a challenge. Once a conformal mapping is available, it can be used to solve boundary-value problems for Laplace's equation (6), for example.

13 Analytic Continuation

Return to the geometric series (1). Denote the infinite series on the left-hand side by $f(z)$, with domain D ($|z| < 1$). Denote the sum on the right-hand side by $g(z) = 1/(1-z)$, with domain D' ($z \neq 1$). We observe that $f(z)$ is analytic in D whereas $g(z)$ is analytic in the larger region D'. As $f(z) = g(z)$ for $z \in D$ we say that g is the *analytic continuation* of f into D'. In practice, we do not usually distinguish between f and g, we just say that $g(z)$ is analytic for $z \in D'$ and that it can be defined for $z \in D \subset D'$ using $f(z)$. This point of view is surprisingly powerful.

There are several aspects to this, and it raises several questions. To begin with, suppose we are given f and D and we want to find g outside D. There are analytical and numerical methods available for doing so. For example, we could use a chain of overlapping disks with a Taylor expansion about each center. The result will be locally unique (each step in the chain gives a unique result) but, if g has a branch point, we could step onto another branch and thus lose global uniqueness.

Often, we do not know D': typical analytic continuations will have singularities. For example, the gamma function, $\Gamma(z)$, is defined by the integral (3) for $\mathrm{Re}\, z > 0$; in this half-plane, Γ is analytic. If we continue $\Gamma(z)$ into $\mathrm{Re}\, z \leqslant 0$, we find that there are simple poles at $z = -N$, $N = 0, 1, 2, \ldots$ (so that D' is the whole complex plane with the points $z = -N$ removed). Explicitly, we can use *Hankel's loop integral*:

$$\Gamma(z) = \frac{1}{2\mathrm{i}\sin(\pi z)} \int_C t^{z-1} \mathrm{e}^t \, \mathrm{d}t.$$

This is a contour integral in the complex t-plane. There is a cut along the negative real-t axis. The branch of t^z is chosen so that $t^z = \exp(z \log t)$ when t is real and positive. The contour starts at $\mathrm{Re}\, t = -\infty$, below the cut, goes once around $t = 0$, and then returns to $\mathrm{Re}\, t = -\infty$ above the cut.

There are also loop integrals for the Riemann zeta function, $\zeta(z)$, defined initially for $\mathrm{Re}\, z > 1$ by the series (9). Thus, it turns out that $\zeta(z)$ can be analytically continued into the whole z-plane apart from a simple pole at $z = 1$.

14 Differential Equations

We usually think of a differential equation as being something to be solved for a real function of a real variable. However, it can be advantageous to "complexify" the problem. One good reason is that we may be able to construct solutions using a power-series expansion (8), and we know that the convergence of such a series is governed by singularity locations. (More generally, we could use the "method of Frobenius.") For example, one solution of *Airy's equation*, $w''(z) = zw(z)$, is

$$w(z) = 1 + \frac{1}{3!}z^3 + \frac{1 \cdot 4}{6!}z^6 + \frac{1 \cdot 4 \cdot 7}{9!}z^9 + \cdots,$$

which defines an entire function of z.

We may be able to write solutions as contour integrals, which then offers possibilities for further analysis. For example, solutions of Airy's equation can be written (or sought) in the form

$$w(z) = \int_C \mathrm{e}^{-zt+t^3/3} \, \mathrm{d}t,$$

where C is a carefully chosen contour in the complex t-plane.

The study of linear differential equations is a well-established branch of complex analysis, especially in the context of the classical SPECIAL FUNCTIONS [**??**] (e.g., Bessel functions and hypergeometric functions). Nonlinear differential equations and their associated special functions are also of interest. For example, there are the six PAINLEVÉ EQUATIONS [**??**], the simplest being $w''(z) = 6w^2 + z$; their solutions, known as *Painlevé transcendents*, have a variety of physical applications but their properties are not well-understood.

15 Cauchy Integrals

Let C be a simple closed smooth contour. Denote the interior of C by D_+ and the exterior by D_-. Define a function $F(z)$ by the *Cauchy integral*

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{g(\tau)}{\tau - z}\,\mathrm{d}\tau, \quad z \notin C, \tag{27}$$

where $g(t)$ is defined for $t \in C$. For example, if $g(t) = 1$, $t \in C$, then

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}\tau}{\tau - z} = \begin{cases} 1, & z \in D_+, \\ 0, & z \in D_-. \end{cases} \tag{28}$$

The integral in (27) is similar to that which appears in Cauchy's integral formula (12), except we are not given any information about $g(t)$ when $t \notin C$. Nevertheless, under mild conditions on g, $F(z)$ is analytic for $z \in D_+ \cup D_-$, and $F(z) \to 0$ as $z \to \infty$. What are the values of F on C? The example (28) suggests that we should expect $F(z)$ to be discontinuous as z crosses C. Therefore, we consider the limits of $F(z)$ as z approaches C (if they exist), and write

$$F_\pm(t) = \lim_{z \to t} F(z) \quad \text{with } z \in D_\pm \text{ and } t \in C. \tag{29}$$

For the example (28), $F_+(t) = 1$ and $F_-(t) = 0$.

Notice that we cannot simply put $z = t \in C$ on the right-hand side of (27): the resulting integral diverges. However, if g is differentiable at t (in fact, Hölder continuity is sufficient), we can define the *Cauchy principal-value integral*

$$⨍_C \frac{g(\tau)}{\tau - t}\,\mathrm{d}\tau = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \frac{g(\tau)}{\tau - t}\,\mathrm{d}\tau, \quad t \in C,$$

where C_ε is obtained from C as follows: draw a little circle of radius ε, centered at $t \in C$, and then remove the piece of C inside the circle.

Using this definition, define

$$F(t) = \frac{1}{2\pi \mathrm{i}} ⨍_C \frac{g(\tau)}{\tau - t}\,\mathrm{d}\tau, \quad t \in C.$$

This function is related to $F_\pm(t)$, defined by (29), by the *Sokhotski–Plemelj formula*:

$$F_\pm(t) = \pm\tfrac{1}{2} g(t) + F(t), \quad t \in C. \tag{30}$$

This describes the "jump behavior" of the Cauchy integral $F(z)$ as z crosses C. In particular,

$$F_+(t) - F_-(t) = g(t), \quad t \in C. \tag{31}$$

One elegant consequence of (30) is that the solution, w, of the *singular integral equation*

$$\frac{1}{\pi \mathrm{i}} ⨍_C \frac{w(\tau)}{\tau - t}\,\mathrm{d}\tau = g(t), \quad t \in C,$$

is given by the formula

$$w(t) = \frac{1}{\pi \mathrm{i}} ⨍_C \frac{g(\tau)}{\tau - t}\,\mathrm{d}\tau, \quad t \in C.$$

16 The Riemann–Hilbert Problem

Let $D_\pm$ and C be as in section 15. Suppose that two functions, $G(t)$ and $g(t)$, are given for $t \in C$. Then, the basic *Riemann–Hilbert problem* is to find two functions $\Phi_+(z)$ and $\Phi_-(z)$, with $\Phi_\pm$ analytic in $D_\pm$, that satisfy

$$\Phi_+(t) = G(t)\Phi_-(t) + g(t), \quad t \in C, \tag{32}$$

where $\Phi_+(t)$ and $\Phi_-(t)$ are defined as in (29); conditions on $\Phi_-(z)$ as $z \to \infty$ are usually imposed too. (There is a variant where C is not closed; in this case, the behavior near the endpoints of C plays a major role.)

When $G \equiv 1$, we can solve (32) using a Cauchy integral and (31). When $G \not\equiv 1$, we start with the following homogeneous problem ($g \equiv 0$).

Find functions K_+ and K_-, with $K_\pm$ analytic in $D_\pm$, that satisfy

$$K_+(t) = G(t)K_-(t), \quad t \in C. \tag{33}$$

Suppose we can find such functions and that they do not vanish. Then, eliminating G from (32) gives

$$\frac{\Phi_+(t)}{K_+(t)} - \frac{\Phi_-(t)}{K_-(t)} = \frac{g(t)}{K_+(t)}, \quad t \in C,$$

which, again, we can solve using a Cauchy integral and (31).

The problem of finding $K_\pm$ is more delicate. At first sight, we could take the logarithm of (33), giving $\log K_+ - \log K_- = \log G$. This looks similar to (31) but it usually happens that $\log G(t)$ is not continuous for all $t \in C$, which means that we cannot use (30). However, this difficulty can be overcome.

The problem of finding $K_\pm$ such that (33) is satisfied is also the key step in the *Wiener–Hopf technique* (a method for solving linear PDEs with mixed boundary conditions and semi-infinite geometries). In that context, a typical problem would be: factor a given function $L(z)$ as $L(z) = L_+(z)L_-(z)$, where $L_+(z)$ is analytic in an upper half-plane, $\operatorname{Im} z > a$, $L_-(z)$ is analytic in a lower half-plane, $\operatorname{Im} z < b$, and $a < b$ so that the two half-planes overlap. There are also related problems where L is a 2×2 or 3×3 matrix; it is not currently known how to solve such *matrix Wiener–Hopf problems* except in some special cases.

17 Closing Remarks

Complex analysis is a rich, deep, and broad subject with a history going back to Cauchy in the 1820s. Inevitably, we have omitted some important topics, such as approximation theory in the complex plane and analytic number theory. There are numerous fine textbooks, a few of which are listed below. However, do not get the impression that complex analysis is a dead subject: it is not. In this article we have tried to cover the basics, with some indications of where problems and opportunities remain.

Further Reading

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