On the Added Mass of Wrinkled Discs*

P. A. Martin
Department of Mathematics, University of Manchester
Manchester M13 9PL, U.K.

1. Introduction

Ideal flow past a rigid sphere of radius a is a textbook boundary-value problem (BVP). It can be solved exactly using separation of variables. However, this method is not immediately applicable when the sphere is perturbed so that the new boundary is given by

$$S : r = a(1 + \varepsilon\, f(\theta, \varphi)), \quad 0 \le \theta \le \pi,\ 0 \le \varphi < 2\pi,$$

where (r, θ, φ) are spherical polar coordinates, f is a given function and ε is a small parameter. Nevertheless, such problems can be solved approximately, exploiting the size of ε. Conventionally, this is done by the 'boundary perturbation technique' [4], in which the boundary condition on S is Taylor-expanded about the unperturbed boundary $r = a$. Then, an expansion of the potential as

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$$

leads to a sequence of BVPs for ϕ_n in the unperturbed domain ($r > a$); ϕ_0 is the unperturbed solution and subsequent ϕ_n are forced by ϕ_m with $m < n$.

The idea behind the boundary perturbation technique is old and familiar. For example, it is used in the theory of small-amplitude water waves, wherein the nonlinear boundary conditions on the (unknown) free surface $z = F(x, y)$ are 'linearized' about the mean free surface $z = 0$. The work of G.H. Darwin is also of interest. He wrote five papers in 1879 on the motions of an incompressible viscous fluid inside a perturbed sphere He neglected the inertia terms in the Navier–Stokes equations (Stokes approximation) giving a linear interior BVP, which he solved for 'small deviations from sphericity', to first order in ε. He argued that one could take account of these deviations by imposing certain tractions on $r = a$, and then solved the corresponding BVP inside the sphere. His method is applicable to any f although he was mainly interested in spheroidal surfaces.

In this paper, we describe an alternative method. First, we reduce the BVP to a boundary integral equation (BIE) over S. We rewrite this equation by projecting onto the unperturbed (reference) surface. At this stage, we have an exact reformulation of the original BVP. Next, we introduce perturbation expansions, leading to a sequence of BIEs, $L\phi_n = b_n$, $n = 0, 1, 2, \ldots$. Each BIE involves the same operator L but different forcing functions b_n; L corresponds to the unperturbed BVP. Any convenient method can be used to invert L.

Our development is concerned with problems where the obstacle is *thin*. Thus, we replace the closed surface S by an open surface Ω. We suppose that Ω is a non-planar perturbation of a circular disc D. Incomplete analyses of related problems have been published [1, 5].

In one sense, we can view the perturbation as a singular perturbation: the place where ϕ is singular (namely the edge) has moved, and so this makes perturbing the BVP itself somewhat difficult. But in another sense, the perturbation is regular: the potential *on* Ω will not be too different from the potential at nearby points on D. This means that it will be relatively straightforward to work with the associated BIEs.

So, we proceed by reducing the exact BVP to a hypersingular integral equation for $[\phi]$, the discontinuity in the potential across Ω. After projection, we obtain a sequence of hypersingular integral equations of the form $Hw_n = b_n$ where Ω is given by

$$\Omega : z = \varepsilon\, f(x, y), \quad (x, y) \in D,$$

$$[\phi] = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots$$

and H corresponds to potential flow past a rigid circular disc. We can derive an explicit closed-form expression for the first-order correction w_1. Note that, for some problems (such as axial motion of an axially-perturbed disc), the first-order correction to the added mass is identically zero.

To verify the method, we have derived explicit results for w_0, w_1 and w_2 for two problems, namely, an inclined flat elliptical screen and a spherical cap. In particular, we have calculated the added mass for these flows, and find agreement with known exact solutions. Moreover, our result for uniform flow past a shallow spherical cap in any direction interpolates between the known results for axial flow and flow perpendicular to the axis of the cap.

*Appeared in *EMAC '98, 3rd Biennial Engineering Mathematics and Applications Conference* (ed. E. O. Tuck and J. A. K. Stott), Institution of Engineers, Australia, 1998, pp. 351–354.

The use of hypersingular integral equations leads to a simpler formulation than would follow from the use of regularized integral equations. This advantage should make the treatment of problems in Stokes' flow and crack problems in elasticity feasible. Xu, Bower and Ortiz [10] have used a perturbation theory based on a regularized integral equation for a dislocation density (analogous to the tangential gradient of $[\phi]$), but they were only able to find the first-order correction (w_1) for a semi-infinite crack. We are currently extending the method to non-planar perturbations of a penny-shaped crack. We have considered in-plane perturbations of a penny-shaped crack elsewhere [6, 7].

2. Formulation

We consider potential flow past a thin rigid screen Ω. The problem is to solve Laplace's equation in three dimensions, $\nabla^2\phi = 0$, subject to

$$\partial\phi/\partial n + \partial\phi_0/\partial n = 0 \quad \text{on } \Omega \qquad (1)$$

and $\phi = O(r^{-1})$ as $r \to \infty$, where $r^2 = x^2+y^2+z^2$, $\partial/\partial n$ denotes normal differentiation, and ϕ_0 is the velocity potential of the given ambient flow, which we take to be uniform:

$$\phi_0(x, y, z) = U(x \sin\beta - z\cos\beta).$$

Denote the two sides of Ω by Ω^+ and Ω^-. Define the unit normal vector on Ω to point from Ω^+ into the fluid, and define the discontinuity in ϕ across Ω by

$$[\phi(q)] = \lim_{Q\to q^+} \phi(Q) - \lim_{Q\to q^-} \phi(Q)$$

where $q \in \Omega$, $q^\pm \in \Omega^\pm$ and Q is a point in the fluid. Then the added mass is given by

$$M = \frac{\rho}{U^2}\int_\Omega [\phi]\,\frac{\partial\phi_0}{\partial n}\,dS,$$

where ρ is the fluid's density. M is known explicitly for flat circular and elliptical discs, and for spherical caps.

3. Governing integral equation

For a thin screen Ω, we can write the potential at P as a distribution of normal dipoles:

$$\phi(P) = \frac{1}{4\pi}\int_\Omega [\phi(q)]\,\frac{\partial}{\partial n_q} G(P, q)\,dS_q. \qquad (1)$$

Suppose that the surface Ω is given by

$$\Omega : z = F(x, y), \quad (x, y) \in D,$$

where D is the unit disc in the xy-plane. Define a normal vector to Ω by

$$\mathbf{N} = (-\partial F/\partial x, -\partial F/\partial y, 1),$$

and then $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ is a unit normal vector. Suppose that P and $q \in \Omega$ are at (x_0, y_0, z_0) and (x, y, z), respectively. Let $[\phi(q)] = w(x, y)$. Then, we find that (1) becomes

$$\phi(x_0, y_0, z_0) = \frac{1}{4\pi}\int_D w(x, y)\,\mathbf{N}(q)\cdot\mathbf{R}_2\,\frac{dA}{R_2^3}$$

where $\mathbf{R}_2 = (x_0 - x, y_0 - y, z_0 - F(x, y))$, $R_2 = |\mathbf{R}_2|$ and $dA = dx\,dy$. This is what we mean by 'projecting onto D'.

Application of the boundary condition (1) to (1) gives

$$\frac{1}{4\pi}\,⨍_\Omega [\phi(q)]\,\frac{\partial^2}{\partial n_p \partial n_q} G(p, q)\,dS_q = -\frac{\partial\phi_0}{\partial n_p} \qquad (2)$$

for $p \in \Omega$, where the integral must be interpreted in the finite-part sense. Equation (2) is the governing hypersingular integral equation for $[\phi]$; it is to be solved subject to the edge condition $[\phi(q)] = 0$ for all $q \in \partial\Omega$, the edge of Ω.

Projecting onto D, (2) becomes

$$\frac{1}{4\pi}\,⨍_D K(x_0, y_0; x, y)\,w(x, y)\,dA = b(x_0, y_0) \qquad (3)$$

for $(x_0, y_0) \in D$, where

$$K = \frac{\mathbf{N}(p)\cdot\mathbf{N}(q)}{R_1^3} - 3\frac{(\mathbf{N}(p)\cdot\mathbf{R}_1)\,(\mathbf{N}(q)\cdot\mathbf{R}_1)}{R_1^5},$$

$\mathbf{R}_1 = (x - x_0, y - y_0, F(x, y) - F(x_0, y_0))$, $R_1 = |\mathbf{R}_1|$ and

$$b(x, y) = U\left(\cos\beta + (\partial F/\partial x)\sin\beta\right).$$

Equation (3) is to be solved subject to the edge condition $w(x, y) = 0$ for $r = \sqrt{x^2 + y^2} = 1$.

Let

$$F_1 = \partial F/\partial x \quad \text{and} \quad F_2 = \partial F/\partial y \qquad (4)$$

evaluated at (x, y), with F_1^0 and F_2^0 being the corresponding quantities at (x_0, y_0). Let $R = \{(x - x_0)^2 + (y - y_0)^2\}^{1/2}$ and $\Lambda = \{F(x, y) - F(x_0, y_0)\}/R$. Also, define the angle Θ by

$$x - x_0 = R\cos\Theta \quad \text{and} \quad y - y_0 = R\sin\Theta,$$

whence $\mathbf{R}_1 = R(\cos\Theta, \sin\Theta, \Lambda)$. Hence

$$K = \frac{1}{R^3}\left\{\frac{1 + F_1F_1^0 + F_2F_2^0}{(1 + \Lambda^2)^{3/2}} - \frac{3YY^0}{(1 + \Lambda^2)^{5/2}}\right\}.$$

where $Y = F_1\cos\Theta + F_2\sin\Theta - \Lambda$ and $Y^0 = F_1^0\cos\Theta + F_2^0\sin\Theta - \Lambda$. This formula is exact. Of course, one can attempt to solve (3) numerically, but here we are interested in asymptotic approximations for almost-flat discs.

4. Wrinkled discs

Suppose that

$$F(x,y) = \varepsilon\, f(x,y)$$

where ε is a small dimensionless parameter and f is independent of ε. Setting

$$\Lambda = \varepsilon\lambda \quad \text{with} \quad \lambda = \{f(x,y) - f(x_0,y_0)\}/R$$

we find that

$$K = R^{-3}\{1 + \varepsilon^2 K_2 + O(\varepsilon^4)\} \quad \text{as } \varepsilon \to 0,$$

where

$$K_2 = f_1 f_1^0 + f_2 f_2^0 - \tfrac{3}{2}\lambda^2 - 3\mathcal{Y}\mathcal{Y}^0$$

$\mathcal{Y} = f_1\cos\Theta + f_2\sin\Theta - \lambda$, $\mathcal{Y}^0 = f_1^0\cos\Theta + f_2^0\sin\Theta - \lambda$ and f_j, f_j^0 are defined similarly to F_j, F_j^0; see (4).

We expand b similarly, obtaining

$$b(x,y) = b_0 + \varepsilon\, b_1(x,y)$$

where $b_0 = U\cos\beta$ and $b_1 = Uf_1\sin\beta$.

Then, if we expand w as

$$w(x,y) = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots,$$

we find from (3) that

$$Hw_0 = b_0, \quad Hw_1 = b_1, \quad \text{and} \quad Hw_2 = -\mathcal{K}_2 w_0,$$

where

$$(Hw)(x_0,y_0) = \frac{1}{4\pi} \oint_D w(x,y)\, \frac{dA}{R^3}$$

is the basic hypersingular operator for potential flow past a rigid circular disc and

$$(\mathcal{K}_2 w)(x_0,y_0) = \frac{1}{4\pi} \oint_D K_2(x,y;x_0,y_0)\, w(x,y) \frac{dA}{R^3}.$$

As b_0 is a constant, we can determine w_0 immediately by solving $Hw_0 = b_0$:

$$w_0(x,y) = -(4/\pi) b_0 \sqrt{1-r^2}.$$

Next, we calculate w_1 by solving $Hw_1 = b_1$. General methods for solving $Hw = b$ are available (see [7] for references), and these can be used to solve for w_1 for any ambient flow and any disc perturbation f. Thus, introduce plane polar coordinates on D, so that $x = r\cos\theta$ and $y = r\sin\theta$, and then expand b as

$$b(r,\theta) = B_0(r) + \sum_{n=1}^{\infty} \left\{ B_n(r)\, \cos n\theta + \widetilde{B}_n \sin n\theta \right\}.$$

Then the solution of $Hw = b$ is given by

$$w(r,\theta) = W_0 + \sum_{n=1}^{\infty} \left\{ W_n \cos n\theta + \widetilde{W}_n \sin n\theta \right\}$$

where

$$W_n(r) = \frac{-4}{\pi} r^n \int_r^1 \frac{Q_n(t)}{t^{2n}\sqrt{t^2-r^2}}\, dt,$$

$$Q_n(t) = \int_0^t \frac{B_n(s)\, s^{n+1}}{\sqrt{t^2-s^2}}\, ds,$$

with a similar relation between $\widetilde{W}_n$ and $\widetilde{B}_n$. Replacing b by b_1 gives the first-order correction w_1.

For w_2, we can foresee that the most difficult part of the calculation will involve the evaluation of $\mathcal{K}_2 w_0$. The simplest results obtain when f is a polynomial. Some sample results will be described below.

The added mass is given by

$$M = -\frac{\rho}{U} \int_D w(x,y)\, \{\cos\beta + F_1 \sin\beta\}\, dA.$$

So, writing $M = M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \cdots$ gives

$$M_0 = -\frac{\rho}{U}\cos\beta \int_D w_0\, dA = \tfrac{8}{3}\rho\cos^2\beta,$$

$$M_1 = -\frac{\rho}{U} \int_D \{w_1 \cos\beta + f_1 w_0 \sin\beta\}\, dA, \quad (1)$$

$$M_2 = -\frac{\rho}{U} \int_D \{w_2 \cos\beta + f_1 w_1 \sin\beta\}\, dA. \quad (2)$$

M_0 is the added mass for a flat circular disc of unit radius. Note that if the disc is wrinkled axisymmetrically so that $f = f(r)$, then $M_1 \equiv 0$.

4.1. Example 1: inclined ellipse. Suppose that Ω is an ellipse on the plane $z = x\tan\gamma$. Let X and Y be Cartesian coordinates on this plane, so that $X = x\cos\gamma + z\sin\gamma$, $Y = y$ and $Z = z\cos\gamma - x\sin\gamma$, where Z is a coordinate perpendicular to the plane. Then, the ellipse Ω with $\partial\Omega$ given by

$$X^2\cos^2\gamma + Y^2 = 1$$

can be specified by

$$z = F(x,y) = x\tan\gamma, \quad (x,y) \in D.$$

For small inclinations of the ellipse to the plane $z = 0$, set $\varepsilon = \tan\gamma$ and $f(x,y) = x$, whence $K_2 = \frac{1}{4}(1 - 3\cos 2\Theta)$, $b_1 = U\sin\beta$, $w_1 = -(4/\pi) b_1 \sqrt{1-r^2}$ and

$$M_1 = \tfrac{8}{3}\rho\sin 2\beta.$$

It turns out that [8] $Hw_2 = -\frac{1}{4}b_0$ and so $w_2 = -\frac{1}{4}w_0$. Substituting into (2) then gives

$$M_2 = \rho\left(1 - \tfrac{5}{3}\cos 2\beta\right).$$

Thus, we find that the added mass is given by

$$M = \tfrac{8}{3}\rho\left\{\cos^2\beta + \varepsilon\sin 2\beta + \varepsilon^2\left(\tfrac{3}{4}\cos^2\beta - \cos 2\beta\right)\right\} + O(\varepsilon^3).$$

This agrees with the known exact solution.

4.2. Example 2: spherical cap. Consider a spherical cap given by

$$z = F(x,y) = a - \sqrt{a^2 - x^2 - y^2}, \quad (x,y) \in D,$$

where a is the radius of the sphere. The cap subtends a solid angle of $2\pi(1-\cos\alpha)$ at the centre of the sphere, where $\sin\alpha = a^{-1}$.

We consider a shallow spherical cap, given approximately by $z = \varepsilon\, f(x,y)$ with

$$f(x,y) = \tfrac{1}{2}(x^2+y^2) \quad \text{and} \quad \varepsilon = a^{-1} = \sin\alpha.$$

This is an example of a rippled surface. We have $f_1 = x$, $f_2 = y$, $b_1 = Ux\sin\beta$ and

$$w_1(x,y) = -\tfrac{8}{3}\pi^{-1}Ux\sin\beta\,\sqrt{1-r^2}.$$

Thus, from (1),

$$M_1 = \frac{10}{3\pi}\rho\sin 2\beta \int_D x\sqrt{1-r^2}\,dA = 0,$$

as expected. So, $M = M_0 + O(\varepsilon^2)$, for all β.

The second-order correction is given by (2); write it as

$$M_2 = M_2^{(1)} + M_2^{(2)}$$

where

$$\begin{aligned} M_2^{(1)} &= -\frac{\rho}{U}\sin\beta\int_D f_1 w_1\,dA \\ &= \tfrac{16}{45}\rho\sin^2\beta \end{aligned}$$

and

$$M_2^{(2)} = -\frac{\rho}{U}\cos\beta\int_D w_2\,dA. \qquad (3)$$

Next, we solve $Hw_2 = -\mathcal{K}_2 w_0$ for w_2. Direct calculation gives [8]

$$K_2 = P_0 + P_c\cos 2\Theta + P_s\sin 2\Theta$$

where P_0, P_c and P_s are quadratic polynomials:

$$\begin{aligned} P_0 &= \tfrac{9}{16}(x^2+y^2+x_0^2+y_0^2) - \tfrac{7}{8}(xx_0+yy_0), \\ P_c &= -\tfrac{3}{16}(x^2-y^2+2xx_0-2yy_0+x_0^2-y_0^2), \\ P_s &= -\tfrac{3}{8}(xy+xy_0+yx_0+x_0y_0). \end{aligned}$$

Next, we use known results for penny-shaped cracks [6, 7] to evaluate $\mathcal{K}_2 w_0$; this gives

$$Hw_2 = -\mathcal{K}_2 w_0 = \tfrac{3}{32}b_0(2-x_0^2-y_0^2).$$

Solving this equation then gives

$$w_2(x,y) = -\tfrac{1}{6}(b_0/\pi)(4-r^2)\sqrt{1-r^2}$$

for the second-order solution. Integrating over the unit disc D, (3) gives

$$M_2^{(2)} = \tfrac{2}{5}\rho\cos^2\beta.$$

Finally, we find that

$$M = \rho\left\{\tfrac{8}{3}\cos^2\beta + \varepsilon^2\left(\tfrac{16}{45}\sin^2\beta + \tfrac{2}{5}\cos^2\beta\right)\right\},$$

correct to second order in ε. This formula agrees with the known exact results for the axisymmetric problem ($\beta = 0$) [2] and for flow perpendicular to the axis of the cap ($\beta = \tfrac{1}{2}\pi$) [3].

5. Conclusions

We have described a method for calculating the added mass of wrinkled discs in potential flow. The general methodology (reduce the BVP to a BIE, project onto a reference surface—these are both exact steps—and then introduce perturbation expansions) has wider applicability: we are currently using it for crack problems in elasticity theory.

For axisymmetric problems, one can make progress by performing the azimuthal integrationin (3). As is well known for axisymmetric potential theory, this leads to one-dimensional integral equations with kernels expressed in terms of complete elliptic integrals. These equations can then be solved as before. Some details are given in [9].

References

[1] Beom, H.G. and Earmme, Y.Y., Axisymmetric crack with a slightly non-flat surface, *Int. J. Fract.*, **58**, 1992, 115–136.

[2] Collins, W.D., On the solution of some axisymmetric boundary value problems by means of integral equations II. Further problems for a circular disc and a spherical cap, *Mathematika* **6**, 1959, 120–133.

[3] Collins, W.D., On some dual series equations and their application to electrostatic problems for spheroidal caps, *Proc. Camb. Phil. Soc.* **57**, 1961, 367–384.

[4] Erma, V.A., Perturbation approach to the electrostatic problem for irregularly shaped conductors, *J. Math. Phys.*, **4**, 1963, 1517–1526.

[5] Jansson, P.-Å., Acoustic scattering from a rough circular disk, *J. Acoust. Soc. Amer.*, **99**, 1996, 672–681.

[6] Martin, P.A., Mapping flat cracks onto penny-shaped cracks: shear loadings, *J. Mech. & Phys. of Solids*, **43**, 1995, 275–294.

[7] Martin, P.A., Mapping flat cracks onto penny-shaped cracks, with application to somewhat circular tensile cracks, *Quart. Appl. Math.*, **54**, 1996, 663–675.

[8] Martin, P.A., On potential flow past wrinkled discs, *Proc. Roy. Soc.* A, **454**, 1998, 2209–2221.

[9] Martin, P.A., On the added mass of rippled discs, *J. Engng. Math.*, **33**, 1998, 421–435.

[10] Xu, G., Bower, A.F. and Ortiz, M., An analysis of non-planar crack growth under mixed mode loading, *Int. J. Solids & Struct.*, **31**, 1994, 2167–2193.