Integral Equations for Crack Scattering*

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1 Introduction

This chapter is concerned with *cracks*. Real cracks in solids are complicated: they are thin cavities, their two faces may touch, and the faces may be rough. We consider ideal cracks. By definition, such a crack is modelled by a smooth open surface Ω (such as a disc or a spherical cap); the elastic displacement is discontinuous across Ω, and the traction vanishes on both sides of Ω (so that the crack is seen as a cavity of zero volume).

We suppose that we have one crack with a smooth edge, $\partial\Omega$, embedded in an infinite, unbounded, three-dimensional solid. Extensions to multiple cracks, to cracks in two dimensions, to cracks in half-spaces or in bounded domains, or to cracks with less smoothness may be made, with varying degrees of difficulty. For a variety of applications, see the book by Zhang and Gross [16].

In fact, to keep the analysis relatively simple, we shall focus on analogous scalar problems coming from acoustics. Thus, we suppose that Ω is a thin *screen* in a compressible fluid. The screen is *hard* (or rigid), which means that we have a Neumann boundary condition. See Section 2 for details.

We are interested in scattering time-harmonic waves by the screen. Much is known about how to calculate scattering from objects of non-zero volume [9]. Except in a few special cases (such as scattering by a sphere), it is usual to derive and solve (numerically) a boundary integral equation over the boundary of the scatterer. However, special methods are needed for zero-volume obstacles such as cracks and screens. In particular, the Neumann boundary condition means that it is inevitable that we shall encounter hypersingular boundary integral equations over the screen. These equations can be tackled directly (using boundary elements, perhaps), or they may be recast into other equivalent forms. For example, if the screen is flat, various simplifications can be made. Integral equations can also be used as the basis for various approximation schemes.

After formulating our scattering problem in the next section, we give the governing hypersingular integral equation in Section 3. This equation is solved approximately for long waves (low-frequency scattering) in Section 4. The approach used is elevated to a well-known 'strategy' in Section 5 prior to further applications.

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For flat cracks and screens, we can simplify the governing hypersingular integral equation. This is done in Section 6. Alternatively, we can use a direct approach, using Fourier transforms; see Section 7. Methods for solving the resulting equations are discussed in Section 8. The final section is concerned with curved cracks and screens. Results for cracks that are almost flat are described.

2 Formulation

We consider acoustic scattering by a thin rigid screen Ω surrounded by a compressible fluid; we model the screen as a smooth simply-connected bounded surface with a smooth edge $\partial\Omega$. We write the scattered field as $\mathrm{Re}\,\{u_{\mathrm{sc}}\,\mathrm{e}^{-\mathrm{i}\omega t}\}$, where ω is the circular frequency. Then, u_{sc} solves the Helmholtz equation in three dimensions,

$$\nabla^2 u_{\mathrm{sc}} + k^2 u_{\mathrm{sc}} = 0, \quad \text{in the fluid,} \tag{1}$$

the Sommerfeld radiation condition at infinity, and the boundary condition

$$\frac{\partial u_{\mathrm{sc}}}{\partial n} + \frac{\partial u_{\mathrm{in}}}{\partial n} = 0 \quad \text{on } \Omega. \tag{2}$$

In addition u_{sc} is required to be bounded everywhere: we do not permit sources on $\partial\Omega$. Here, $k = \omega/c$, c is the constant speed of sound, u_{in} is the given incident field and $\partial/\partial n$ denotes normal differentiation. The total field is $u = u_{\mathrm{sc}} + u_{\mathrm{in}}$ so that (2) gives $\partial u/\partial n = 0$ on Ω.

Denote the two sides of Ω by Ω^+ and Ω^-, and define the unit normal vector on Ω, $\mathbf{n}$, to point from Ω^+ into the fluid. Then, define the discontinuity in u across Ω by

$$[u(q)] = \lim_{Q\to q^+} u(Q) - \lim_{Q\to q^-} u(Q)$$

where $q \in \Omega$, $q^\pm \in \Omega^\pm$ and Q is a point in the fluid. Notice that $[u] = [u_{\mathrm{sc}}]$ as $[u_{\mathrm{in}}] = 0$.

The scattered field has the integral representation

$$u_{\mathrm{sc}}(P) = \frac{1}{4\pi}\int_\Omega [u(q)]\,\frac{\partial}{\partial n_q} G(P,q)\,\mathrm{d}S_q, \tag{3}$$

where

$$G(P,q) = \mathcal{R}^{-1}\exp{(\mathrm{i}k\mathcal{R})} \tag{4}$$

is the free-space Green's function and $\mathcal{R}$ is the distance between P and $q \in \Omega$.

To be more explicit, we introduce Cartesian coordinates $Oxyz$ and suppose that the surface Ω is given by

$$\Omega\ :\ z = S(x,y), \quad (x,y)\in D,$$

where D is a region in the xy-plane with edge ∂D. We define a normal vector to Ω by

$$\mathbf{N} = (-\partial S/\partial x,\ -\partial S/\partial y,\ 1),$$

and then $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ is a unit normal vector.

Also, if u_{in} represents an incident plane wave, then

$$u_{\text{in}}(x, y, z) = \mathrm{e}^{\mathrm{i}k(x\alpha_1 + y\alpha_2 + z\alpha_3)}, \tag{5}$$

where $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$.

Suppose that P has position vector $\mathbf{r} = (x_0, y_0, z_0)$ and $q \in \Omega$ has position vector $\mathbf{q} = (x, y, S(x, y))$. Let

$$[u(q)] = w(x, y).$$

Then, we find that (3) becomes, exactly,

$$u_{\text{sc}}(x_0, y_0, z_0) = \frac{1}{4\pi} \int_D w(x, y) \, (\mathbf{N}(q) \cdot \mathbf{R}_2)(1 - \mathrm{i}kR_2) \, \mathrm{e}^{\mathrm{i}kR_2} \, \frac{\mathrm{d}A}{R_2^3}$$

where $\mathrm{d}A = \mathrm{d}x \, \mathrm{d}y$, $\mathbf{R}_2 = \mathbf{r} - \mathbf{q}$ and $R_2 = |\mathbf{R}_2|$.

In the far field, we have

$$u_{\text{sc}}(P) \sim r^{-1} \mathrm{e}^{\mathrm{i}kr} \, f(\hat{\mathbf{r}}) \quad \text{as } r \to \infty,$$

where $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/r$ and

$$f(\hat{\mathbf{r}}) = \frac{k}{4\pi\mathrm{i}} \int_\Omega [u(q)] \, \{\hat{\mathbf{r}} \cdot \mathbf{n}(q)\} \, \exp\left(-\mathrm{i}k\mathbf{q} \cdot \hat{\mathbf{r}}\right) \mathrm{d}S_q \tag{6}$$

$$= \frac{k}{4\pi\mathrm{i}} \int_D w(x, y) \, \{\hat{\mathbf{r}} \cdot \mathbf{N}(q)\} \, \exp\left(-\mathrm{i}k\mathbf{q} \cdot \hat{\mathbf{r}}\right) \mathrm{d}A; \tag{7}$$

f is the *far-field pattern*.

The formula (7) is exact. Although the integration is over a flat region, the geometry of the screen enters through w, $\mathbf{N}$ and $\mathbf{q}$. Thus, we can expect that reasonable approximations to w will generate good approximations to f.

For example, in some applications, a static approximation to w (or $[u]$) could be used; this idea, leading to low-frequency approximations, will be developed in Section 4.

For another example, we might be able to use high-frequency approximations for $[u]$. The most popular of these is the *Kirchhoff approximation*, $[u] \simeq u^{\mathrm{K}}$, where $u^{\mathrm{K}}(q)$ is the total field at q when the incident field is reflected by an infinite flat plane perpendicular to $\mathbf{n}(q)$. This approximation is used widely in models of ultrasonic nondestructive evaluation; see, for example, [12, Chapter 10].

3 A hypersingular integral equation

Application of the boundary condition (2) to (3) gives

$$\frac{1}{4\pi} \oint\!\!\!\!\times_\Omega [u(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p, q) \, \mathrm{d}S_q = -\frac{\partial u_{\text{in}}}{\partial n_p}, \quad p \in \Omega, \tag{8}$$

where the integral must be interpreted in the finite-part sense. Equation (8) is the governing hypersingular integral equation for $[u]$; it is to be solved subject to the edge condition

$$[u(q)] = 0 \text{ for all } q \in \partial\Omega. \tag{9}$$

Equation (8) could be solved numerically, using boundary elements combined with regularization techniques. Indeed, this is probably the only option if Ω has a complicated, three-dimensional shape. As a sample, we cite a paper by Tada *et al.* [13], in which a formulation for transient elastodynamics and non-planar cracks is developed. The literature on methods for solving hypersingular equations, numerically, is extensive.

The right-hand side of (8) reduces to $-\mathrm{i}ku_{\mathrm{in}}\hat{\boldsymbol{\alpha}}\cdot\mathbf{n}(p)$ when u_{in} is given by (5); here,

$$\hat{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2, \alpha_3)$$

is a unit vector giving the direction in which the incident plane wave is propagating.

What does 'hypersingular' mean? This question can be answered precisely, using the notion of pseudodifferential operators acting between function spaces. However, for many purposes, it is enough to gain intuition through simple examples. Suppose we have an expression Lf, where L is a linear operator and f is a function. If L is an integral operator with a weakly-singular kernel, Lf will be smoother than f: we usually think of integration as a smoothing process. If L is an integral operator defined by a Cauchy principal-value integral (a *singular* integral operator), Lf will have the same smoothness as f: in some sense, L is similar to the identity operator, I. Hypersingular operators coarsen. If L is such an operator, Lf will have less smoothness than f: in some sense, L is similar to a differential operator, and this operator will be of first order in our applications.

We usually identify hypersingular operators in one of two ways. One is as in (8): the integral operator is defined in terms of a finite-part integral (and does not exist as an improper integral or as a Cauchy principal-value integral). The second way involves Fourier transforms. Roughly, we can write (locally) $Lf = \mathcal{F}^{-1}\{\sigma\mathcal{F}\{f\}\}$, where σ is called the *symbol* and $\mathcal{F}$ denotes Fourier transformation. Then, if σ is a linear function of the transform variable(s), L is a first-order differential operator; if σ is linear for large values of the transform variable(s), then L will be identified as one of our hypersingular operators — we will see examples in Section 7.

Returning to (8), let us project onto D (as in Section 2), giving

$$\frac{1}{4\pi}\,{\large\times}\!\!\!\!\!\!\int_D K(x_0, y_0;\, x, y)\, w(x, y)\,\mathrm{d}A = b(x_0, y_0), \quad (x_0, y_0) \in D, \tag{10}$$

where

$$\begin{aligned} K = {} & R_1^{-3}(1 - \mathrm{i}kR_1)\,\mathrm{e}^{\mathrm{i}kR_1}\{\mathbf{N}(p)\cdot\mathbf{N}(q)\} \\ & - R_1^{-5}(3 - 3\mathrm{i}kR_1 - k^2R_1^2)\,\mathrm{e}^{\mathrm{i}kR_1}(\mathbf{N}(p)\cdot\mathbf{R}_1)(\mathbf{N}(q)\cdot\mathbf{R}_1), \end{aligned}$$

$\mathbf{R}_1 = (x - x_0,\, y - y_0,\, S(x, y) - S(x_0, y_0))$, $R_1 = |\mathbf{R}_1|$ and

$$b(x, y) = -\partial u_{\mathrm{in}}/\partial N = -\mathrm{i}k\mathbf{N}\cdot\hat{\boldsymbol{\alpha}}\exp\left(\mathrm{i}k\mathbf{q}\cdot\hat{\boldsymbol{\alpha}}\right) \tag{11}$$

when u_{in} is given by (5). Notice that $K(x_0, y_0;\, x, y) = K(x, y;\, x_0, y_0)$.

Equation (10) is to be solved subject to the edge condition

$$w(x, y) = 0 \text{ for all points } (x, y) \text{ on } \partial D. \tag{12}$$

Let

$$S_1 = \partial S/\partial x \quad \text{and} \quad S_2 = \partial S/\partial y \quad \text{evaluated at } (x, y), \tag{13}$$

with S_1^0 and S_2^0 being the corresponding quantities at (x_0, y_0). Then $\mathbf{N}(q) = (-S_1, -S_2, 1)$ and $\mathbf{N}(p) = (-S_1^0, -S_2^0, 1)$. Let

$$R = \{(x - x_0)^2 + (y - y_0)^2\}^{1/2} \tag{14}$$

and $\Lambda = \{S(x, y) - S(x_0, y_0)\}/R$. Also, define the angle Θ by

$$x - x_0 = R\cos\Theta \quad \text{and} \quad y - y_0 = R\sin\Theta,$$

whence $\mathbf{R}_1 = R(\cos\Theta,\, \sin\Theta,\, \Lambda)$. Hence

$$K = \frac{\mathrm{e}^{\mathrm{i}kRX}}{R^3}\left\{\frac{1 - \mathrm{i}kRX}{X^3}(1 + S_1S_1^0 + S_2S_2^0) - \frac{Y}{X^5}(3 - 3\mathrm{i}kRX - (kRX)^2)\right\}, \tag{15}$$

where $X = \sqrt{1 + \Lambda^2}$ and

$$Y = (S_1\cos\Theta + S_2\sin\Theta - \Lambda)(S_1^0\cos\Theta + S_2^0\sin\Theta - \Lambda).$$

This formula for K is exact. If we expand K for small R about p, we find that

$$K \sim R^{-3}\sigma(p; \Theta) \tag{16}$$

where

$$\sigma(p; \Theta) = \frac{1 + (S_1^0)^2 + (S_2^0)^2}{1 + (S_1^0\cos\Theta + S_2^0\sin\Theta)^2}.$$

In particular, $\sigma \equiv 1$ when S is constant. Equation (16) exhibits the strong singularity in the kernel K, and is typical of hypersingular operators defined over surfaces.

4 Low-frequency scattering

Before the advent of computers, it was traditional to obtain approximate solutions of scattering problems, assuming that the frequency is low, so that the crack is assumed to be small compared to the wavelength of the incident field. These approximations are still useful today. In three dimensions, it is known that the scattered field is an analytic function of k: it has a Maclaurin expansion with respect to k. Thus, associated static problems (where $k = 0$) feature.

The basic static problem is as follows. Let ϕ_a denote the velocity potential of a given ambient flow. Then, we seek another potential ϕ, where ϕ is a bounded solution of $\nabla^2\phi = 0$ in the fluid, with

$$\frac{\partial\phi}{\partial n} + \frac{\partial\phi_\mathrm{a}}{\partial n} = 0 \quad \text{on } \Omega$$

and $\phi = O(r^{-1})$ as $r \to \infty$. For a uniform ambient flow, we have

$$\phi_\mathrm{a}(x, y, z) = x\alpha_1 + y\alpha_2 + z\alpha_3. \tag{17}$$

In general, we can find ϕ by solving a hypersingular integral equation analogous to (8), namely

$$\frac{1}{4\pi} \oint_{\Omega} [\phi(q)] \frac{\partial^2}{\partial n_p \partial n_q} G_0(p,q) \, \mathrm{d}S_q = -\frac{\partial \phi_{\mathrm{a}}}{\partial n_p}, \quad p \in \Omega, \tag{18}$$

with $[\phi(q)] = 0$ for all $q \in \partial\Omega$. Here, $G_0 = \mathcal{R}^{-1}$ is the static free-space Green's function; see (4). The right-hand side of (18) reduces to $-\hat{\boldsymbol{\alpha}} \cdot \mathbf{n}(p)$ when ϕ_{a} is given by (17).

Returning to (8), we seek solutions in powers of k. We can write

$$G = G_0 + kG_1 + \cdots \quad \text{and} \quad u_{\mathrm{in}} = u_{\mathrm{in}}^0 + ku_{\mathrm{in}}^1 + \cdots ,$$

and then (8) implies that $[u]$ has a similar expansion,

$$[u] = u_0 + ku_1 + \cdots .$$

Substituting and collecting up powers of k, we obtain a sequence of equations from which u_j can be determined. The first two are

$$\mathcal{L}_0 u_0 = b_0 \quad \text{and} \quad \mathcal{L}_0 u_1 = b_1,$$

where

$$\mathcal{L}_0 u = \frac{1}{4\pi} \oint_{\Omega} u(q) \, \frac{\partial^2}{\partial n_p \partial n_q} G_0(p,q) \, \mathrm{d}S_q, \quad b_0(p) = -\frac{\partial u_{\mathrm{in}}^0}{\partial n_p},$$
$$b_1(p) = -\frac{\partial u_{\mathrm{in}}^1}{\partial n_p} - \frac{1}{4\pi} \oint_{\Omega} u_0(q) \, \frac{\partial^2}{\partial n_p \partial n_q} G_1(p,q) \, \mathrm{d}S_q$$

and $u_j = 0$ on $\partial\Omega$ for $j = 0, 1, 2, \ldots$. Thus, each u_j is obtained by solving a certain static problem. The problem itself could be solved by any convenient method, not necessarily via the equation $\mathcal{L}_0 u_j = b_j$.

For an incident plane wave, (5) gives

$$u_{\mathrm{in}}(x,y,z) = 1 + \mathrm{i}k(x\alpha_1 + y\alpha_2 + z\alpha_3) + \cdots .$$

Thus, $u_{\mathrm{in}}^0 \equiv 1$ and $b_0 \equiv 0$. As the equation $\mathcal{L}_0 u_j = b_j$ is uniquely solvable (subject to $u_j = 0$ on $\partial\Omega$), we obtain $u_0 \equiv 0$. Then, the equation for u_1 reduces to

$$\mathcal{L}_0 u_1 = -\mathrm{i}\hat{\boldsymbol{\alpha}} \cdot \mathbf{n}(p).$$

Hence, comparison with (18) gives $u_1 = \mathrm{i}[\phi]$ and

$$[u] \simeq \mathrm{i}k[\phi] \quad \text{for small } k.$$

This approximation can then be inserted in (6) to give a low-frequency approximation for the far-field pattern.

5 Some strategies

In general, we encounter equations, such as (10), that we can write in operator form as

$$\mathcal{H}w = b, \tag{19}$$

where $\mathcal{H}$ is a linear operator, b is known and w is to be found. We write (19) as

$$(\mathcal{L}_0 + \mathcal{L}_1)w = b \quad \text{with} \quad \mathcal{L}_1 = \mathcal{H} - \mathcal{L}_0, \tag{20}$$

where $\mathcal{L}_0$ is another operator. If $\mathcal{L}_0$ is invertible, we obtain

$$(I + \mathcal{M})w = g \quad \text{with } \mathcal{M} = \mathcal{L}_0^{-1}\mathcal{L}_1 \text{ and } g = \mathcal{L}_0^{-1}b. \tag{21}$$

Several stategies for solving (19) follow this general pattern. To be effective, $\mathcal{L}_0^{-1}$ should be available explicitly or it should be easier to compute than $\mathcal{H}^{-1}$.

For example, if $\mathcal{L}_0$ is $\mathcal{H}$ evaluated at $k = 0$, we obtain a method with two virtues. First, the operator $\mathcal{M}$ will not be hypersingular: the operator $\mathcal{H}$ has been *regularized.* Second, we have access to analytical approximations for low-frequency scattering, as discussed in Section 4. This is because $\mathcal{M}$ is small in some norm, so that $(I + \mathcal{M})w = g$ can be solved by iteration.

For another example, if $\mathcal{L}_0$ is $\mathcal{H}$ evaluated for a simpler geometry, we obtain a method for perturbed screens (such as wrinkled discs).

We shall see examples of these strategies later.

6 Flat cracks as a special case

Almost all the literature on scattering by three-dimensional screens (and cracks) assumes that the screen is flat. Thus, we assume that $S(x, y) \equiv 0$ whence $D \equiv \Omega$. From (10) and (15), the governing hypersingular integral equation reduces to

$$\frac{1}{4\pi} ⨎_D (1 - \mathrm{i}kR)\frac{\mathrm{e}^{\mathrm{i}kR}}{R^3}\,[u(x, y)]\,\mathrm{d}A = b(x_0, y_0), \quad (x_0, y_0) \in D, \tag{22}$$

where R is defined by (14) and

$$b(x, y) = -\partial u_{\mathrm{in}}/\partial z \quad \text{evaluated on } z = 0. \tag{23}$$

Noticing that

$$\begin{aligned}\frac{1}{R^3}(1 - \mathrm{i}kR)\,\mathrm{e}^{\mathrm{i}kR} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\frac{\mathrm{e}^{\mathrm{i}kR}}{R} \\ &= \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + k^2\right)\frac{\mathrm{e}^{\mathrm{i}kR}}{R},\end{aligned}$$

we can rewrite (22) as

$$\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + k^2\right)\int_D [u(x, y)]\,\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\,\mathrm{d}A = 4\pi b(x_0, y_0), \quad (x_0, y_0) \in D. \tag{24}$$

This can be regarded as a kind of regularization, because the finite-part integral has gone, although it has been replaced by a differential operator.

For an incident plane wave, (23) gives

$$b(x,y) = -\mathrm{i}k\alpha_3 \mathrm{e}^{\mathrm{i}k(x\alpha_1+y\alpha_2)} = \frac{1}{\mathrm{i}k\alpha_3}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\mathrm{e}^{\mathrm{i}k(x\alpha_1+y\alpha_2)}$$

(for $\alpha_3 \neq 0$). Then, (24) can be integrated to give

$$\int_D [u(x,y)]\,\frac{\mathrm{e}^{\mathrm{i}kR}}{R}\,\mathrm{d}A = \frac{4\pi}{\mathrm{i}k\alpha_3}\,\mathrm{e}^{\mathrm{i}k(x_0\alpha_1+y_0\alpha_2)} + \Psi_0(x_0,y_0), \quad (x_0,y_0)\in D, \tag{25}$$

where $\Psi_0(x,y)$ solves

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\Psi_0(x,y) = 0, \quad (x,y)\in D.$$

Denote the left-hand side of (25) by $\mathcal{S}[u]$; $\mathcal{S}$ is a single-layer operator. The equation $\mathcal{S}w = g$ arises when the analogous scattering problem for a sound-soft screen (Dirichlet condition) is solved. It is known that, in general, the solution of $\mathcal{S}w = g$ is infinite around ∂D, whereas we want w to satisfy the edge condition (12). This condition can only be satisfied by making an appropriate choice for Ψ_0; it is not clear how to make this choice in practice.

It is known that $\mathcal{S}$ smooths by one order. Thus, the operator on the left-hand side of (24) coarsens by one order.

7 Flat cracks: direct approach

Here, we assume from the outset that the crack or screen is flat and lying in the xy-plane. To proceed, we use two-dimensional Fourier transforms. Thus, define

$$U(\xi,\eta,z) = \mathcal{F}\{u_{\mathrm{sc}}\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{\mathrm{sc}}(x,y,z)\,\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}\,\mathrm{d}x\,\mathrm{d}y$$

with inverse

$$u_{\mathrm{sc}}(x,y,z) = \mathcal{F}^{-1}\{U\} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\xi,\eta,z)\,\mathrm{e}^{-\mathrm{i}(\xi x+\eta y)}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Transforming (1) gives $(k^2-\xi^2-\eta^2+\partial^2/\partial z^2)U = 0$ with solution

$$U(\xi,\eta,z) = A_\pm(\xi,\eta)\,\mathrm{e}^{\pm\gamma z} \quad \text{for } \pm z > 0.$$

Here, A_+ and A_- are arbitrary functions, and γ is defined as follows:

$$\gamma = \begin{cases} \sqrt{s^2-k^2}, & s > k, \\ -\mathrm{i}\sqrt{k^2-s^2}, & 0 \le s < k, \end{cases} \qquad \text{with } s = \sqrt{\xi^2+\eta^2}.$$

This definition ensures that the radiation condition is satisfied.

As $\partial u_{\rm sc}/\partial z$ is continuous across $z = 0$ for all (x, y), we infer that $A_+ = -A_-$. This implies that $u_{\rm sc}(x, y, z)$ is an odd function of z, $u_{\rm sc}(x, y, -z) = -u_{\rm sc}(x, y, z)$, so we can now assume that $z \geq 0$ and write

$$u_{\rm sc}(x, y, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\xi, \eta)\, \mathrm{e}^{-\gamma z}\, \mathrm{e}^{-\mathrm{i}(\xi x+\eta y)}\, \mathrm{d}\xi\, \mathrm{d}\eta, \quad z > 0. \tag{26}$$

Let us identify A in terms of $[u]$. From (26), we have

$$[u(x, y)] = \frac{2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\xi, \eta)\, \mathrm{e}^{-\mathrm{i}(\xi x+\eta y)}\, \mathrm{d}\xi\, \mathrm{d}\eta = 2\mathcal{F}^{-1}\{A\},$$

so that $2A = \mathcal{F}\{[u]\}$. Explicitly, as $[u(x, y)] = 0$ for $(x, y) \notin D$, we have

$$2A(\xi, \eta) = \mathcal{F}\{[u]\} = \int_D [u(x, y)]\, \mathrm{e}^{\mathrm{i}(\xi x+\eta y)}\, \mathrm{d}x\, \mathrm{d}y. \tag{27}$$

Then, application of the boundary condition (2) on $z = 0+$ yields

$$\frac{\partial u_{\rm in}}{\partial z} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma A(\xi, \eta)\, \mathrm{e}^{-\mathrm{i}(\xi x+\eta y)}\, \mathrm{d}\xi\, \mathrm{d}\eta, \quad (x, y) \in D,$$

or, more concisely,

$$-\tfrac{1}{2}\mathcal{F}^{-1}\left\{\gamma \mathcal{F}\{[u]\}\right\} = b(x, y), \quad (x, y) \in D, \tag{28}$$

where b is defined by (23). This is another equation for $[u]$; it should be compared with (22). Once solved, $u_{\rm sc}$ is given by (26) with (27).

Equation (28) can be regarded as a hypersingular equation. This can be seen by noticing that $\gamma \sim s$ as $s \to \infty$, so that the right-hand side of (28) is similar to a first derivative of $[u]$. For an extensive review of the use of (28) for scattering computations, see [2].

Write $\gamma = s + \beta(s)$ where $\beta = \gamma - s$. Then, we can write (28) as (20) with

$$\mathcal{L}_0 w = -\tfrac{1}{2}\mathcal{F}^{-1}\{s\mathcal{F}\{w\}\}, \quad \mathcal{L}_1 w = -\tfrac{1}{2}\mathcal{F}^{-1}\{\beta \mathcal{F}\{w\}\}$$

and $w = [u]$. The operator $\mathcal{L}_1$ is similar to a single-layer operator: its symbol $\beta \sim -\frac{1}{2}k^2/s$ as $s \to \infty$. The operator $\mathcal{L}_0$ is hypersingular but it does not depend on k: it is the corresponding static operator. If D has a simple shape (such as a circular disc), $\mathcal{L}_0$ has an explicit inverse (subject to the edge condition on ∂D) and this can be used in order to obtain the regularized equation (21).

8 Flat cracks: how to compute $[u]$

For a flat screen D, we found two equations for $[u]$, the discontinuity in u across D, namely (22) and (28); write these formally as $\mathcal{H}[u] = b$. A familiar way to solve such equations is to expand $[u]$ with a set of basis functions, writing

$$[u(x, y)] = \sum_n a_n u_n(x, y);$$

evidently, the functions u_n have to be selected and then the coefficients a_n have to be found. Substitution in $\mathcal{H}[u] = b$ gives

$$\sum_n a_n\,(\mathcal{H}u_n)(x,y) = b(x,y), \quad (x,y) \in D,$$

and then various methods (such as collocation) immediately suggest themselves for the numerical determination of a_n.

For the expansion functions, u_n, one option is *radial basis functions.* Thus, choose N points $(x_n, y_n) \in D$, $n = 1, 2, \ldots, N$, put

$$u_n(x,y) = \chi\left(\sqrt{(x-x_n)^2 + (y-y_n)^2}\right)$$

and then choose the function χ. Examples are $\chi(r) = \mathrm{e}^{-r^2/c^2}$ [15] and $\chi(r) = (1 - r^2/c^2)^\alpha H(c-r)$ [4], where c and α are positive constants and H is the Heaviside unit function. Other choices could be made but, to be effective, one should be able to compute $\mathcal{H}u_n$ efficiently if not analytically.

One virtue of radial basis functions is that they provide flexibility: the shape of D is relatively unimportant. On the other hand, we know that $[u] = 0$ around the edge ∂D; in fact, we know that $[u]$ must vanish as the square-root of the distance from ∂D. This knowledge could be incorporated by using special approximations near ∂D. However, if D is simple in shape, such as circular, elliptical or rectangular, it is possible to construct functions u_n with the correct square-root edge behaviour. As an example, we consider a circular screen D of radius a. For this geometry, we have functions u_n satisfying $\mathcal{L}_0 u_n = b_n$, where $\mathcal{L}_0$ is $\mathcal{H}$ at $k = 0$ and b_n is known explicitly.

Introduce plane polar coordinates (r, θ) so that the crack occupies $0 \le r < a$. For simplicity, suppose that $u_n(x,y)$ is an even function of $x = r\cos\theta$ and put

$$u_n(x,y) = a w_n(r/a)\,H(a-r)\,\cos n\theta.$$

Then, a standard calculation (using the Jacobi expansion [9, p. 37] twice) gives

$$\mathcal{F}^{-1}\left\{\gamma\mathcal{F}\{u_n\}\right\} = \cos n\theta \int_0^\infty a^3\gamma J_n(rs) \int_0^1 w_n(\rho)\,J_n(as\rho)\,\rho\,\mathrm{d}\rho\,s\,\mathrm{d}s,$$

where J_n is a Bessel function. This formula simplifies if we expand $w_n(\rho)$ in a series of functions $w_m^n(\rho)$, defined by

$$w_m^n(r) = r^n C_{2m+1}^{n+1/2}(\sqrt{1-r^2}),$$

where C_m^λ is a Gegenbauer polynomial. Each function $w_m^n(r)$ is equal to $\sqrt{1-r^2}$ multiplied by a polynomial in r; in particular, $w_0^n(r) = (2n+1)r^n\sqrt{1-r^2}$. Hence [5]

$$\int_0^1 w_m^n(r)\,J_n(asr)\,r\,\mathrm{d}r = \frac{2\,\Gamma(n+m+3/2)}{\Gamma(n+1/2)\,m!}\,\frac{j_{n+2m+1}(as)}{as},$$

where j_n is a spherical Bessel function. Then, writing

$$[u(x,y)] = aH(a-r)\sum_{n=0}\sum_{m=0} W_m^n w_m^n(r/a)\,\cos n\theta, \tag{29}$$

we find that

$$(\mathcal{H}[u])(r,\theta) = \sum_{n=0} \sum_{m=0} W_m^n t_m^n(r/a) \cos n\theta, \quad r < a,$$

where

$$t_m^n(r/a) = -\frac{\Gamma(n+m+3/2)}{\Gamma(n+1/2)\, m!} \int_0^\infty a^2 \gamma J_n(rs)\, j_{n+2m+1}(as)\, \mathrm{d}s.$$

The remaining integral must be evaluated numerically. However, in the static limit (replace γ by s), we have [1]

$$\int_0^\infty a^2 s J_n(rs)\, j_{n+2m+1}(as)\, \mathrm{d}s = \frac{\Gamma(m+3/2)\,\Gamma(n+1/2)}{(n+m)!} \frac{w_m^n(r/a)}{\sqrt{1-r^2/a^2}},$$

a polynomial in r/a, and this gives the explicit evaluation of $\mathcal{L}_0[u]$. Also, if

$$b(r,\theta) = \sum_{n=0} \sum_{m=0} B_m^n \frac{w_m^n(r/a)}{\sqrt{1-r^2/a^2}} \cos n\theta,$$

then the solution of $\mathcal{L}_0[u] = b$ is given by (29) with

$$W_m^n = -\frac{(n+m)!\, B_m^n}{\Gamma(n+m+3/2)\,\Gamma(m+3/2)};$$

the coefficients B_m^n can be found using the orthogonality of the Gegenbauer polynomials, which gives

$$\int_0^1 w_l^n(r)\, w_m^n(r)\, \frac{r\, \mathrm{d}r}{\sqrt{1-r^2}} = h_m^n \delta_{lm},$$

where δ_{ij} is the Kronecker delta and h_m^n is a known constant. Thus, in effect, $\mathcal{L}_0^{-1}$ is available, and it can be used as in Section 5.

If D is flat but not circular, a possible strategy is the following. Find a conformal mapping that maps the interior of D onto the interior of a disc. Use this mapping in the integral equation (22); as it is a *conformal* mapping, it does not change the basic hypersingularity in the kernel, and so the dominant operator is the static operator $\mathcal{L}_0$ for the disc. The operator $\mathcal{L}_1$ will include some effects due to the mapping and some due to the dynamics. The details have not been worked out, except for static problems [6].

9 Curved cracks

If the crack or screen is not flat, one may have to resort to solving (numerically) the hypersingular integral equation over the screen Ω, (8), or the version of this equation obtained by projection onto the flat region D, (10). Obviously, care must be taken in handling the hypersingularity and with the edge condition, (9) or (12).

The surface Ω is defined by $z = S(x,y)$ for $(x,y) \in D$. For non-constant S, the singularity in the kernel of the integral equation (10) is essentially different from that occurring in the integral equation for constant S; this is revealed by the presence of σ in (16). This difference means that the equation over D, (10), cannot be regularized using known results for flat screens. However, we can make analytical progress when Ω is *almost* flat.

Suppose that

$$S(x,y) = \varepsilon f(x,y)$$

where ε is a small dimensionless parameter and f is independent of ε. Setting

$$\Lambda = \varepsilon\lambda \quad \text{with} \quad \lambda = \{f(x,y) - f(x_0,y_0)\}/R, \tag{30}$$

we find from (15) that

$$K = R^{-3}\mathrm{e}^{\mathrm{i}kR}\{1 - \mathrm{i}kR + \varepsilon^2 K_2 + O(\varepsilon^4)\} \quad \text{as } \varepsilon \to 0,$$

where

$$\begin{aligned} K_2 = {} & (1-\mathrm{i}kR)(f_1 f_1^0 + f_2 f_2^0 - \tfrac{3}{2}\lambda^2) + \tfrac{1}{2}\lambda^2(kR)^2 \\ & - 3(1 - \mathrm{i}kR - \tfrac{1}{3}(kR)^2)(f_1\cos\Theta + f_2\sin\Theta - \lambda)(f_1^0\cos\Theta + f_2^0\sin\Theta - \lambda) \end{aligned}$$

and f_j, f_j^0 are defined similarly to S_j; see (13).

We expand b similarly. For an incident plane wave, (11) gives

$$b(x,y) = \mathrm{i}k\{b_0(x,y) + \varepsilon b_1(x,y) + \cdots\},$$

where, for example, $b_0(x,y) = -\alpha_3 \mathrm{e}^{\mathrm{i}k(x\alpha_1 + y\alpha_2)}$.

Then, if we expand w in (10) as

$$w(x,y) = \mathrm{i}k(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots),$$

we find that

$$H_k w_0 = b_0, \quad H_k w_1 = b_1 \quad \text{and} \quad H_k w_2 = b_2 - \mathcal{K}_2 w_0,$$

where

$$(H_k w)(x_0,y_0) = \frac{1}{4\pi} ⨎_D (1-\mathrm{i}kR)\,\frac{\mathrm{e}^{\mathrm{i}kR}}{R^3}\,w(x,y)\,\mathrm{d}A$$

is the basic hypersingular operator for acoustic scattering by a flat sound-hard screen D (see (22)) and

$$(\mathcal{K}_2 w)(x_0,y_0) = \frac{1}{4\pi} ⨎_D K_2(x,y;\,x_0,y_0)\,\frac{\mathrm{e}^{\mathrm{i}kR}}{R^3}\,w(x,y)\,\mathrm{d}A.$$

Thus, we have a sequence of hypersingular integral equations, $H_k w_n = f_n$, to solve.

When $k=0$, w_0, w_1 and w_2 have been found explicitly for particular geometries, namely inclined elliptical discs and spherical caps [7]. The results obtained agree with known exact results. Similar results for perturbed penny-shaped cracks have also been obtained [8].

For $k > 0$, we can see that w_0 is simply the solution for a flat screen; however, the far-field pattern will be different, it being given by (7) with w replaced by w_0. It should be possible to obtain w_1 without too much difficulty, as $f_1 = b_1$ is simple. For higher-order terms, one would have to evaluate $\mathcal{K}_2 w$.

If it is assumed further that the incident waves are long compared to the diameter of the scatterer, $2a$, low-frequency approximations may be made. Then, each w_n can be expanded in powers of ka. This approach has been pursued for a shallow crack in the shape of a spheroidal cap [10].

One difficulty with these approximation methods is that there are few results to compare with. For acoustic scattering by spherical caps, see, for example, the papers by Thomas [14] and Miles [11]. Numerical results for elastic-wave scattering by cracks in the shape of spherical and spheroidal caps have been given by Boström and Olsson [3].

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