WAVES IN WOOD

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Abstract Elastic waves in materials with cylindrical orthotropy are considered, this being a plausible model for waves in wood. For time-harmonic motions, the problem is reduced to some coupled ordinary differential equations. Methods for solving these equations are discussed. These include the method of Frobenius (power-series exapansions) and the use of Neumann series (expansions in series of Bessel functions of various orders).

1. INTRODUCTION

As children, we learn that we can determine the age of a sawn log or tree by counting the annual rings visible at a sawn end. The presence of these rings influences the mechanical properties of the wood, of course. This observation leads to a constitutive model in which the wood is assumed to be an elastic solid with *cylindrical orthotropy.* Thus, we assume that the elastic stiffnesses are constants when referred to cylindrical polar coordinates, r, θ, z. (In general, the material will be inhomogeneous when described in Cartesian coordinates.)

We are interested in the propagation of elastic waves in wooden poles, using the cylindrical-orthotropy model. One application is to understand the use of ultrasonic devices for determining whether wooden telegraph poles have decayed internally [1]. Application of these techniques to live trees has also been made [2]. We are also interested in methods for inspecting wire ropes and overhead transmission lines; wire rope, consisting of several helically wound strands, may be 'homogenised' [3], giving a similar mathematical model for the material.

Further information on waves in wood can be found in the book by Bucur [4].

In this paper, we begin by formulating the problem of wave propagation in a material with cylindrical anisotropy, using matrix notation where possible. We look for time-harmonic solutions, with a prescribed dependence on θ and z, leading to a 3×3 system of coupled ordinary differential equations in the radial direction. To simplify further, we suppose that the material is cylindrically orthotropic. We review the known exact solutions, corresponding to isotropy, axisymmetry (no dependence on θ) or no dependence on z. We then discuss two methods for solving the remaining situations. These are the method of Frobenius (expansions in powers of r) and a generalization using Neumann series (expansions in Bessel functions of various orders). The two methods are compared and contrasted. The method based on Neumann series seems to be better suited to problems involving wave propagation, and should find further applications.

2. GOVERNING EQUATIONS

In the absence of body forces, the governing equations of motion are

$$\frac{\partial}{\partial r}\left(r\mathbf{t}_r\right) + \frac{\partial}{\partial \theta}\mathbf{t}_\theta + \mathsf{K}\mathbf{t}_\theta + r\frac{\partial}{\partial z}\mathbf{t}_z = \rho r \frac{\partial^2}{\partial t^2}\widetilde{\mathbf{u}}, \tag{1.1}$$

where ρ is the density,

$$\mathbf{t}_r = \begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \\ \tau_{rz} \end{pmatrix}, \quad \mathbf{t}_\theta = \begin{pmatrix} \tau_{\theta r} \\ \tau_{\theta\theta} \\ \tau_{\theta z} \end{pmatrix}, \quad \mathbf{t}_z = \begin{pmatrix} \tau_{zr} \\ \tau_{z\theta} \\ \tau_{zz} \end{pmatrix},$$

$$\mathsf{K} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{\mathbf{u}} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}$$

is the displacement vector and τ_{ij} are the stress components. In what follows, we generalise the matrix formulation of Ting [5] for static problems; he gives expressions for the traction vectors $\mathbf{t}_i$ in terms of $\widetilde{\mathbf{u}}$.

We look for time-harmonic solutions of (1.1) in the form

$$\widetilde{\mathbf{u}}(r,\theta,z,t) = \mathrm{Re_i}\left\{\mathbf{u}_m(r)\,\mathrm{e}^{\mathrm{j}m\theta}\,\mathrm{e}^{\mathrm{i}\xi z}\mathrm{e}^{-\mathrm{i}\omega t}\right\},$$

where i and j are two non-interacting complex units, m is an integer, ξ is the axial wavenumber, ω is the radian frequency, and Re_i denotes the real part with respect to i. Use of $\mathrm{e}^{\mathrm{j}m\theta}$ rather than $\cos m\theta$ and $\sin m\theta$ allows us to retain the nice matrix notation in what follows. Thus, we find that $\mathbf{u}_m(r)$ solves

$$r\mathbf{Q}\left(r\mathbf{u}_m'\right)' + r\mathbf{A}\mathbf{u}_m' + \mathbf{B}\mathbf{u}_m = \mathbf{0}, \tag{1.2}$$

where

$$\begin{aligned}\mathbf{A} &= \mathbf{R}\mathbf{K}_m + \mathbf{K}_m\mathbf{R}^T + \mathrm{i}\xi r(\mathbf{P} + \mathbf{P}^T),\\ \mathbf{B} &= \rho\omega^2 r^2\mathbf{I} - \xi^2 r^2\mathbf{M} + \mathbf{K}_m\mathbf{T}\mathbf{K}_m + \mathrm{i}\xi r(\mathbf{P} + \mathbf{K}_m\mathbf{S} + \mathbf{S}^T\mathbf{K}_m),\end{aligned}$$

$$\mathbf{Q} = \begin{pmatrix} C_{11} & C_{16} & C_{15}\\ C_{16} & C_{66} & C_{56}\\ C_{15} & C_{56} & C_{55}\end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} C_{16} & C_{12} & C_{14}\\ C_{66} & C_{26} & C_{46}\\ C_{56} & C_{25} & C_{45}\end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} C_{66} & C_{26} & C_{46}\\ C_{26} & C_{22} & C_{24}\\ C_{46} & C_{24} & C_{44}\end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} C_{15} & C_{14} & C_{13}\\ C_{56} & C_{46} & C_{36}\\ C_{55} & C_{45} & C_{35}\end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} C_{55} & C_{45} & C_{35}\\ C_{45} & C_{44} & C_{34}\\ C_{35} & C_{34} & C_{33}\end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} C_{56} & C_{46} & C_{36}\\ C_{25} & C_{24} & C_{23}\\ C_{45} & C_{44} & C_{34}\end{pmatrix},$$

$\mathbf{I}$ is the identity, $\mathbf{K}_m = \mathbf{K} + \mathrm{j}m\mathbf{I}$, $\mathbf{R}^T$ is the transpose of $\mathbf{R}$, and we have used the contracted notation $C_{\alpha\beta}$ for the elastic stiffnesses. Note that $\mathbf{Q}$, $\mathbf{T}$ and $\mathbf{M}$ are symmetric matrices.

For two-dimensional motions independent of z ($\xi = 0$), we recover the equations studied in [1]. If we also put $m = 0$ (axisymmetry) and $\omega = 0$ (static), we obtain the equations solved by Ting [5].

Setting $\mathbf{u}_m = (u_m, v_m, w_m)$, (1.2) gives three coupled ordinary differential equations for the three components of $\mathbf{u}_m$. In general, these equations do not decouple.

3. CYLINDRICAL ORTHOTROPY

For materials with cylindrical orthotropy, there are nine non-trivial stiffnesses, namely C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}, C_{55} and C_{66}. The matrices $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$, $\mathbf{P}$, $\mathbf{M}$ and $\mathbf{S}$ simplify to

$$\mathbf{Q} = \begin{pmatrix} C_{11} & 0 & 0\\ 0 & C_{66} & 0\\ 0 & 0 & C_{55}\end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} 0 & C_{12} & 0\\ C_{66} & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$\mathsf{T} = \begin{pmatrix} C_{66} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{44} \end{pmatrix}, \qquad \mathsf{P} = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & 0 \\ C_{55} & 0 & 0 \end{pmatrix},$$

$$\mathsf{M} = \begin{pmatrix} C_{55} & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}, \qquad \mathsf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C_{23} \\ 0 & C_{44} & 0 \end{pmatrix},$$

and the system (1.2) simplifies accordingly.

Note that isotropy is a special case of cylindrical orthotropy. For isotropic materials, $C_{11} = C_{22} = C_{33} = \lambda + 2\mu$, $C_{12} = C_{13} = C_{23} = \lambda$ and $C_{44} = C_{55} = C_{66} = \mu$, where λ and μ are the Lamé moduli. Exact solutions of (1.2) are well known for isotropic solids. They are given in textbooks on elastic waves; see, for example, Graff [6, §8.2]. Some of these solutions will be mentioned below.

4. AXISYMMETRIC MOTIONS

For axisymmetric motions ($m = 0$) of a cylindrically orthotropic solid, we have $\mathsf{K}_0 = \mathsf{K}$, $\mathsf{RK} = -\mathsf{KR}^T$,

$$\mathsf{A} = \mathrm{i}\xi r \begin{pmatrix} 0 & 0 & C_{13} + C_{55} \\ 0 & 0 & 0 \\ C_{13} + C_{55} & 0 & 0 \end{pmatrix}$$

and

$$\mathsf{B} = \begin{pmatrix} B_5 r^2 - C_{22} & 0 & \mathrm{i}\xi r (C_{13} - C_{23}) \\ 0 & B_4 r^2 - C_{66} & 0 \\ \mathrm{i}\xi r (C_{55} + C_{23}) & 0 & B_3 r^2 \end{pmatrix},$$

where

$$B_i = \rho\omega^2 - \xi^2 C_{ii} \qquad \text{(no sum)}.$$

So, the (axisymmetric) torsional component v_0 decouples from the radial and axial components, u_0 and w_0, respectively. We find that v_0 satisfies

$$r C_{66} \left(r v_0' \right)' + (B_4 r^2 - C_{66}) v_0 = 0.$$

This is Bessel's equation; solutions are

$$J_1(\ell r) \quad \text{and} \quad Y_1(\ell r),$$

where $\ell = \sqrt{B_4 / C_{66}}$.

The remaining pair of equations for u_0 and w_0 is as follows:

$$r C_{11} \left(r u_0' \right)' + \mathrm{i}\xi r^2 (C_{13} + C_{55}) w_0'$$

$$+\,(B_5 r^2 - C_{22})u_0 + \mathrm{i}\xi r(C_{13} - C_{23})w_0 = 0, \tag{1.3}$$

$$rC_{55}\,(rw_0')' + \mathrm{i}\xi r^2(C_{13} + C_{55})u_0' + \mathrm{i}\xi r(C_{55} + C_{23})u_0 + B_3 r^2 w_0 = 0. \tag{1.4}$$

Note that axisymmetric motions in a cylindrically orthotropic material do not depend on the stiffness C_{12}.

Equations (1.3) and (1.4) decouple when $\xi = 0$ [1]. Typical solution pairs are found to be

$$u_0 = 0 \quad \text{and} \quad w_0 = J_0(\kappa_a r),$$

and

$$u_0 = J_\gamma(\kappa_1 r) \quad \text{and} \quad w_0 = 0,$$

where $\gamma = \sqrt{C_{22}/C_{11}}$, $\kappa_1 = \omega\sqrt{\rho/C_{11}}$ and $\kappa_a = \omega\sqrt{\rho/C_{55}}$.

For isotropic materials, one solution pair for (1.3) and (1.4) is

$$u_0 = \xi J_1(Kr) \quad \text{and} \quad w_0 = \mathrm{i}K J_0(Kr),$$

where $K^2 = \rho\omega^2/\mu - \xi^2$. Another is

$$u_0 = k J_1(kr) \quad \text{and} \quad w_0 = -\mathrm{i}\xi J_0(kr),$$

where $k^2 = \rho\omega^2/(\lambda + 2\mu) - \xi^2$. See, for example, [6, p. 471].

5. TWO-DIMENSIONAL MOTIONS

For motions of a cylindrically orthotropic material that are independent of z ($\xi = 0$), we again find that the 3×3 system (1.2) partially decouples. However, now we obtain a pair of equations for u_m and v_m, and a single equation for w_m. The latter can be solved exactly: two independent solutions are

$$J_\beta(\kappa_a r) \quad \text{and} \quad Y_\beta(\kappa_a r),$$

where $\beta = m\sqrt{C_{44}/C_{55}}$ and $\kappa_a = \omega\sqrt{\rho/C_{55}}$.

The pair of equations for $u_m(r)$ and $v_m(r)$ is as follows [1]:

$$rC_{11}\,(ru_m')' + \mathrm{j}mr(C_{66} + C_{12})v_m' + (\rho\omega^2 r^2 - m^2 C_{66} - C_{22})u_m - \mathrm{j}m(C_{66} + C_{22})v_m = 0, \tag{1.5}$$

$$rC_{66}\,(rv_m')' + \mathrm{j}mr(C_{66} + C_{12})u_m' + (\rho\omega^2 r^2 - C_{66} - m^2 C_{22})v_m + \mathrm{j}m(C_{66} + C_{22})u_m = 0. \tag{1.6}$$

Note that motions independent of z in a cylindrically orthotropic medium depend on only six of the nine stiffnesses, namely C_{11}, C_{12}, C_{22}, C_{44}, C_{55} and C_{66}.

For a homogeneous isotropic elastic solid, solutions of (1.5) and (1.6) are known. Let (u_m, v_m) be a solution pair. There are four independent solution pairs. The first two pairs are

$$u_m = J_m'(k_P r) \quad \text{and} \quad v_m = \mathrm{j}m(k_P r)^{-1} J_m(k_P r) \tag{1.7}$$

and

$$u_m = m(k_S r)^{-1} J_m(k_S r) \quad \text{and} \quad v_m = \mathrm{j}\, J_m'(k_S r), \tag{1.8}$$

where $k_P = \omega\sqrt{\rho/(\lambda + 2\mu)}$ is the compressional wavenumber and $k_S = \omega\sqrt{\rho/\mu}$ is the shear wavenumber. The second two pairs are obtained by replacing J_n in these expressions by Y_n.

6. COUPLED SYSTEMS

In special cases, we have seen that we can find explicit solutions of (1.2). However, in general, we are left with a coupled system of ordinary differential equations to solve. This will be a 2×2 system if $m = 0$ ((1.3) and (1.4)) or if $\xi = 0$ ((1.5) and (1.6)), but it will be a 3×3 system in the general case.

How can we solve such systems? A standard technique for solving ordinary differential equations is the method of Frobenius, in which one looks for solutions in the form of power series. The method proceeds by writing

$$\begin{aligned} u_m(r) &= \sum_{n=0}^{\infty} \hat{a}_n \, (\kappa r)^{2n+\alpha}, \\ v_m(r) &= \mathrm{j} \sum_{n=0}^{\infty} \hat{b}_n \, (\kappa r)^{2n+\alpha}, \\ w_m(r) &= \sum_{n=0}^{\infty} \hat{c}_n \, (\kappa r)^{2n+\alpha}, \end{aligned}$$

where the coefficients $\hat{a}_n$, $\hat{b}_n$ and $\hat{c}_n$, and the exponent α are to be determined. It turns out that there is no loss of generality in using $(\kappa r)^{2n}$ rather than $(\kappa r)^n$.

Substituting these expansions in (1.2) and collecting terms leads to an indicial equation for α and coupled recursion relations for the coefficients. This yields an efficient method for computing the coefficients. In the present context, it has been used by many authors, including Ohnabe and Nowinski [7], Chou and Achenbach [8], Markuš and Mead [9] and Yuan and Hsieh [10].

The main drawback of the method of Frobenius is that it is essentially a *static* method: power series in κr are only expected to be good for small values of κr.

If we examine the known exact solutions for isotropic solids, such as (1.7) and (1.8), we see that they can be written as linear combinations of two Bessel functions, with orders differing by two. For example, (1.7) can be written as

$$\begin{aligned} u_m(r) &= \tfrac{1}{2}\left\{J_{m-1}(k_P r) - J_{m+1}(k_P r)\right\}, \\ v_m(r) &= \tfrac{1}{2}\,\mathrm{j}\left\{J_{m-1}(k_P r) + J_{m+1}(k_P r)\right\}. \end{aligned}$$

This suggests that we should use a generalization of the method of Frobenius, in which u_m, v_m and w_m are expanded as *Neumann series*,

$$\begin{aligned} u_m(r) &= \sum_{n=0}^{\infty} a_n\, J_{2n+\alpha}(kr), \\ v_m(r) &= \mathrm{j}\sum_{n=0}^{\infty} b_n\, J_{2n+\alpha}(kr), \\ w_m(r) &= \sum_{n=0}^{\infty} c_n\, J_{2n+\alpha}(kr), \end{aligned}$$

where the coefficients a_n, b_n, c_n and α are to be determined. Note that the parameter k is to some extent at our disposal; its choice is discussed in [1], making use of the asymptotic behaviour of solutions of the governing differential equations for *large* r.

In [1], we have used Neumann series to solve (1.5) and (1.6), corresponding to $\xi = 0$. We investigated two methods for finding a_n and b_n, which we call *direct* and *indirect*. In the direct method, we substitute the Neumann-series expansions directly into (1.5) and (1.6) and then group terms. This requires manipulating series of Bessel functions, and so is more complicated than at the analogous stage of the method of Frobenius. It turns out that α solves the same indicial equation as before. Eventually, we obtain some recurrence relations for a_n and b_n; they are fairly complicated but they are well behaved numerically [1].

For the indirect method, we begin with the standard method of Frobenius, leading to the computation of the coefficients $\hat{a}_n$ and $\hat{b}_n$. From these, we then compute the coefficients a_n and b_n, using the known expansion of an arbitrary power in terms of Bessel functions:

$$(\tfrac{1}{2}kr)^{\nu} = \sum_{n=0}^{\infty} \frac{(2n+\nu)\,\Gamma(n+\nu)}{n!}\, J_{2n+\nu}(kr).$$

(Compare this with the definition of a Bessel function,

$$J_{\nu}(kr) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}(\tfrac{1}{2}kr)^{2n+\nu},$$

which can itself be obtained by the method of Frobenius.)

We have found that the use of Neumann series is more efficient for the problems of wave propagation in wood that we have described above. We think that the method will find application to other problems involving wave propagation in materials with cylindrical anisotropy.

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