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# Some negative results on the use of Helmholtz integral equations for rough-surface scattering

## 1 Introduction

A plane time-harmonic sound wave is scattered by a bounded three-dimensional obstacle with surface $S$. The scattered field can be found by solving a Helmholtz integral equation. For example, if $S$ is sound-hard, the scattered field on $S$ satisfies

$$u(p) - \int_S u(q) \frac{\partial G}{\partial n_q}(p,q)\, dS_q = \int_S \frac{\partial u_{\mathrm{inc}}}{\partial n}\, G(p,q)\, dS_q, \tag{1}$$

where $p$ and $q$ are points on $S$, $G$ is the free-space Green's function, $\partial/\partial n_q$ denotes normal differentiation at $q$, and $u_{\mathrm{inc}}$ is the incident field. The Helmholtz integral equations are familiar boundary integral equations, for which there is a complete theoretical framework (Kleinman and Roach, 1974; Colton and Kress, 1983). Many examples of their numerical treatment, usually using boundary elements, can also be found.

Suppose now that $S$ is unbounded. A typical problem is the reflection of a plane wave by an infinite two-dimensional rough surface. This is a classical problem, going back to Lord Rayleigh. Standard texts usually treat the problem using approximate techniques, such as perturbation theory or Kirchhoff theory. More recently, there has been an effort to validate these approximate techniques by comparing them with 'exact' methods. In particular, comparisons have been made with numerical solutions of Helmholtz integral equations, such as (1).

However, the derivation of a Helmholtz integral equation for an infinite rough surface is not straightforward. Indeed, we are not aware of any correct derivations in the literature (even though such equations have been the subject of extensive computational studies). In fact, we can show that such an integral equation is definitely not valid in certain cases! These will be illustrated using some explicit examples.

## 2 Bounded obstacles

Consider a bounded three-dimensional obstacle with a smooth surface $S$, insonified by a plane wave. The problem is to calculate the scattered field $u$. In order to have a well-posed boundary-value problem, one imposes the Sommerfeld radiation condition,

$$r\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) \to 0 \qquad \text{as } r \to \infty, \tag{2}$$

uniformly in all directions. Here, $r$ is a spherical polar coordinate, $k$ is the wavenumber, and the time-dependence is $\mathrm{e}^{-\mathrm{i}\omega t}$.

Equation (1) is derived by applying Green's theorem to $u(Q)$ and $G(P, Q)$ in the region bounded internally by $S$ and externally by $C_r$, a large sphere of radius $r$; the point $P$ is in this region. The radiation condition implies that the integral

$$I(u; C_r) \equiv \int_{C_r} \left(u\,\frac{\partial G}{\partial r} - G\,\frac{\partial u}{\partial r}\right) dS \to 0 \qquad \text{as } r \to \infty, \tag{3}$$

and so only boundary integrals over $S$ remain (Colton and Kress, 1983). Equation (1) follows by letting $P \to p \in S$.

## 3 Unbounded obstacles

The simplest scattering problem for unbounded obstacles is reflection of a plane wave by an infinite flat plane, $S$. It is well known that the incident wave is reflected specularly as a single propagating plane wave. More generally, if $S$ is an infinite rough surface, an incident plane wave will be scattered into a spectrum of plane waves. For such problems, the Sommerfeld radiation condition is not appropriate as it is not satisfied by a plane wave. However, it is common to proceed, *assuming* that the scattered field can be represented in terms of plane waves, at least at some distance from $S$. Typically, this requires the discarding of an integral such as (3), but with the large sphere $C_r$ replaced by a large *hemisphere* $H_r$. Can this step be justified?

Assume that the two-dimensional rough surface, $S$, is sound-hard and that it is given by $z = s(x, y)$, $-\infty < x, y < \infty$, with $-h < s(x, y) \le 0$ for some constant $h \ge 0$. We can write the total field as $u_{\mathrm{tot}} = u_{\mathrm{inc}} + u$, where $u$ is the scattered field. The incident plane wave is

$$u_{\mathrm{inc}}(r, \theta, \phi) = \exp\{\mathrm{i}\mathbf{k}_{\mathrm{i}} \cdot \mathbf{x}\}, \qquad 0 \le \theta_{\mathrm{i}} \le \tfrac{1}{2}\pi, \tag{4}$$

where $\mathbf{k}_{\mathrm{i}} = k(\sin\theta_{\mathrm{i}}, 0, -\cos\theta_{\mathrm{i}})$, $\theta_{\mathrm{i}}$ is the angle of incidence, $(r, \theta, \phi)$ are spherical polar coordinates, and $\mathbf{x} = r\hat{\mathbf{x}} = r(\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta)$. All the fields $u_{\mathrm{tot}}$, $u_{\mathrm{inc}}$ and $u$ satisfy the Helmholtz equation, $(\nabla^2 + k^2)u = 0$, for $z > s$. The boundary condition is

$$\partial u_{\mathrm{tot}}/\partial n = 0 \qquad \text{on } S, \tag{5}$$

where $\partial/\partial n$ denotes normal differentiation *out* of the acoustic medium.

## 4 Reflection by a flat surface

Let us return to the simplest problem, reflection by a flat surface, so that $s = 0$. The text-book solution for $u$ is

$$u(r, \theta, \phi) = \exp\{\mathrm{i}\mathbf{k}_\mathrm{s} \cdot \mathbf{x}\} \qquad \text{for } 0 \le \theta_\mathrm{i} < \tfrac{1}{2}\pi, \tag{6}$$

where $\mathbf{k}_\mathrm{s} = k(\sin\theta_\mathrm{i}, 0, \cos\theta_\mathrm{i})$. When $\theta_\mathrm{i} = \frac{1}{2}\pi$ ('grazing incidence'), we have $u \equiv 0$. So, for $0 \le \theta_\mathrm{i} < \frac{1}{2}\pi$, $u_\mathrm{tot} = 2\,\mathrm{e}^{\mathrm{i}kx\sin\theta_\mathrm{i}}\cos(kz\cos\theta_\mathrm{i})$ solves the problem. But consider

$$u'_\mathrm{tot} \equiv u_\mathrm{tot} + u_\mathrm{g} \tag{7}$$

with $u_\mathrm{g} = V(\beta)\,\mathrm{e}^{\mathrm{i}k(x\cos\beta + y\sin\beta)}$, where $\beta$ and $V(\beta)$ are arbitrary, with $-\pi < \beta \le \pi$. $u'_\mathrm{tot}$ also 'solves' the problem, in that it satisfies the Helmholtz equation and the boundary condition. Of course, we disallow this second solution, unless $V \equiv 0$: but why? The answer is: because of the radiation condition (which we have yet to specify). For example, take $\beta = 0$ and $V(0) = 1$, so that $u_\mathrm{g} = \mathrm{e}^{\mathrm{i}kx}$; this gives an 'outgoing' grazing wave at $x = +\infty$ but it is an 'incoming' grazing wave at $x = -\infty$ — we must therefore exclude it. Indeed, we must exclude *all* contributions $u_\mathrm{g}$, for any $\beta$ and $V$.

A similar condition is imposed on the two-dimensional problem (DeSanto and Martin, 1997). However, the three-dimensional problem has an extra feature: we could consider replacing $u_\mathrm{g}$ in (7) by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} V(\beta)\,\mathrm{e}^{\mathrm{i}k(x\cos\beta + y\sin\beta)}\,d\beta$$

where $V$ is a continuous function; but, as $u_\mathrm{g}$ has been excluded, we must also exclude all linear combinations of such plane grazing waves. In particular, by taking $V(\beta) = (-\mathrm{i})^n\,\mathrm{e}^{\mathrm{i}n\beta}$, we see that we must exclude the cylindrical standing waves

$$J_n(kR)\,\mathrm{e}^{\mathrm{i}n\phi}, \tag{8}$$

where $R = r\sin\theta$ and $(R, \phi, z)$ are cylindrical polar coordinates of the point at $\mathbf{x}$. On the other hand, the exact scattered field, given by (6), when evaluated on any plane $z =$ constant, has an azimuthal Fourier component proportional to

$$J_n(k_\mathrm{i}R)\,\mathrm{e}^{\mathrm{i}n\phi}, \tag{9}$$

where $k_\mathrm{i} = k\sin\theta_\mathrm{i} \le k$. Thus, if one wants to formulate a radiation condition, mathematically, it must be such that fields (8) are excluded but fields (9) are permitted. All this suggests that the specification of a *mathematical* radiation condition for the present class of problems (plane-wave scattering by an infinite two-dimensional rough surface) will not be straightforward. However, the *physical* purpose of a radiation condition is clear: it is to exclude all 'incoming' waves apart from the incident wave.

## 5 A radiation condition

The scattered field in the half-space $z > 0$ may be written using an angular-spectrum representation, as a superposition (integral) of propagating and evanescent plane waves (DeSanto and Martin, 1996). A typical propagating plane wave is

$$v(r,\theta,\phi;\alpha,\beta) = \exp\{\mathrm{i}\mathbf{k}\cdot\mathbf{x}\}, \qquad 0 \le \alpha \le \tfrac{1}{2}\pi, \quad |\beta| \le \pi, \tag{10}$$

where $\mathbf{k} = k\hat{\mathbf{k}} = k(\sin\alpha\,\cos\beta, \sin\alpha\,\sin\beta, \cos\alpha)$. It turns out that the propagating waves give the dominant contribution to $I(u;H_r)$, where $H_r$ is a large hemisphere of radius $r$ and centre $O$.

We can now give our radiation condition. We require that all propagating plane-wave components $v(r,\theta,\phi;\alpha,\beta)$ in $u$ propagate *outwards* through $H_r$, away from $O$. When imposing this, we have to be careful with grazing waves ($\alpha = \frac{1}{2}\pi$; see the discussion following (7)). One convenient way is to partition the half-space $z > 0$ and the hemisphere $H_r$ into four parts. Thus, with

$$H_r^m = \{(r,\theta,\phi) : 0 \le \theta \le \tfrac{1}{2}\pi,\ \tfrac{1}{2}(m-3)\pi \le \phi < \tfrac{1}{2}(m-2)\pi\}, \quad m = 1,2,3,4,$$

being the surfaces of four octants of a sphere, we require the following conditions for the regions specified:

$$\begin{array}{lll} m=1: & \text{in } x<0,\ y\le 0, & \text{use } -\pi \le \beta < -\tfrac{1}{2}\pi; \\ m=2: & \text{in } x\ge 0,\ y<0, & \text{use } -\tfrac{1}{2}\pi \le \beta < 0. \\ m=3: & \text{in } x>0,\ y\ge 0, & \text{use } 0 \le \beta < \tfrac{1}{2}\pi; \\ m=4: & \text{in } x\le 0,\ y>0, & \text{use } \tfrac{1}{2}\pi \le \beta < \pi; \end{array} \tag{11}$$

This partitioning makes it easy to ensure that only plane waves propagating out through $H_r^m$ are included. We have used this form of radiation condition to derive boundary integral equations.

# 6 Boundary integral equations

Apply Green's theorem to $u$ and $G$ in the region $D_r$ with boundary $\partial D_r = H_r \cup S_r \cup T_r$, where $S_r = \{(x,y,z) : z = s(x,y),\ 0 \le x^2+y^2 < r^2\}$ is a truncated rough surface, and

$$T_r = \{(x,y,z) : x^2+y^2 = r^2,\ s(x,y) \le z \le 0\} \tag{12}$$

is the surface of a truncated circular cylinder joining $H_r$ and $S_r$. After using the boundary condition (5), the result is

$$2u(P) = \int_{S_r} \left\{ u(q)\,\frac{\partial G}{\partial n_q}(P,q) + \frac{\partial u_{\mathrm{inc}}}{\partial n}\,G(P,q) \right\} dS_q + I(u;H_r) + I(u;T_r), \tag{13}$$

where $P \in D_r$,

$$I(u;\mathcal{S}) = \int_{\mathcal{S}} \left\{ u(q)\,\frac{\partial G}{\partial n_q}(P,q) - \frac{\partial u}{\partial n}\,G(P,q) \right\} dS_q$$

and $\partial/\partial n_q$ denotes normal differentiation at $q$ in a direction away from the origin (so that $\partial/\partial n = \partial/\partial r$ on $H_r$, consistent with (3)).

The next step is to estimate $I(u;H_r)$ and $I(u;T_r)$ for large $r$. This is described in detail by DeSanto and Martin (1998). Here, we consider a single propagating plane-wave component in the angular-spectrum representation of $u$, and focus on $I(v;H_r)$ as $r\to\infty$, where $v$ is defined by (10). In fact, for this calculation, it is enough to set $\beta=0$, and so we write

$$v(r,\theta,\phi;\alpha,0)=v(r,\theta,\phi;\alpha)=v(\mathbf{x};\alpha), \qquad 0\le\alpha\le\tfrac{1}{2}\pi.$$

The simplest way to evaluate the large-$r$ limit is to use the method of stationary phase. We then consider the contribution from $I(u;T_r)$ to (13). The results are surprising, and motivate some explicit examples.

# 7 The method of stationary phase

We use the method of stationary phase to estimate $I(v;H_r)$. We are interested in large values of $r=|\mathbf{x}|$, for fixed $\mathbf{y}$ and $k$. We have

$$G(P,q)\simeq(B/r)\,\exp\{\mathrm{i}k(r-\mathbf{y}\cdot\hat{\mathbf{x}})\} \tag{14}$$

where $P$ and $q$ have position vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, and $B=-\tfrac{1}{2}/\pi$. Hence, for large $r$, $I(v;H_r)\simeq\mathrm{i}B\,\mathrm{e}^{\mathrm{i}kr}\,L(kr)$, where

$$L(\lambda)=\lambda\int_{\mathcal{D}}g(\theta,\phi)\,\mathrm{e}^{\mathrm{i}\lambda F(\theta,\phi)}\,d\theta\,d\phi, \tag{15}$$

$$g(\theta,\phi)=(1-\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})\,\exp\{-\mathrm{i}k\mathbf{y}\cdot\hat{\mathbf{x}}\}\,\sin\theta, \tag{16}$$

$F(\theta,\phi)=\hat{\mathbf{k}}\cdot\hat{\mathbf{x}}$ and $\mathcal{D}=\{(\theta,\phi):0\le\theta\le\tfrac{1}{2}\pi,-\pi\le\phi\le\pi\}$ is the rectangular domain of integration.

It turns out that there are three cases, depending on the value of $\alpha$: $\alpha=0$, $0<\alpha<\tfrac{1}{2}\pi$ and $\alpha=\tfrac{1}{2}\pi$.

*Case I*: $\alpha=0$. This corresponds to a plane wave $v$ propagating along the $z$-axis. We have

$$L(\lambda)=\lambda\int_0^{\pi/2}(1-\cos\theta)\,b(\theta;\mathbf{y})\,\mathrm{e}^{\mathrm{i}\lambda\cos\theta}\,\sin\theta\,d\theta \tag{17}$$

where

$$b(\theta;\mathbf{y})=\int_{-\pi}^{\pi}\exp\{-\mathrm{i}k\mathbf{y}\cdot\hat{\mathbf{x}}\}\,d\phi=2\pi\,\mathrm{e}^{-\mathrm{i}k\rho\cos\theta\,\cos\Theta}\,J_0(k\rho\sin\theta\,\sin\Theta)$$

and

$$\mathbf{y}=\rho\hat{\mathbf{y}}=\rho(\sin\Theta\,\cos\Phi,\sin\Theta\,\sin\Phi,\cos\Theta). \tag{18}$$

The integral (17) can be estimated for large $\lambda$ using the (one-dimensional) method of stationary phase. The only stationary-phase point is at $\theta=0$; as the integrand vanishes at $\theta=0$, we deduce that $L(\lambda)=O(1)$ as $\lambda\to\infty$. In fact, an integration by parts shows that $L(\lambda)\sim\mathrm{i}\,b(\tfrac{1}{2}\pi;\mathbf{y})$ as $\lambda\to\infty$, whence

$$I(v;H_r)=\mathrm{e}^{\mathrm{i}kr}\,J_0(k\rho\sin\Theta)+O((kr)^{-1/2}) \qquad \text{as } kr\to\infty\text{, for } \alpha=0. \tag{19}$$

*Case II*: $0 < \alpha < \frac{1}{2}\pi$. We can estimate $L(\lambda)$, defined by (15), for large $\lambda$, using the method of stationary phase for two-dimensional integrals. Thus, we look for stationary-phase points $\mathbf{c} = (\theta, \phi) \in \mathcal{D}$ at which $\operatorname{grad} F = \mathbf{0}$; such points may be in the interior of $\mathcal{D}$ or on the boundary, $\partial\mathcal{D}$. Each $\mathbf{c}$ contributes a term to $L(\lambda)$ proportional to $g(\mathbf{c}) \exp\{\mathrm{i}\lambda F(\mathbf{c})\}$, the next term being $O(\lambda^{-1})$. The relevant stationary-phase points are at $\mathbf{c} = (0, \pm\frac{1}{2}\pi)$ and $(\alpha, 0)$. At these points, $g(\mathbf{c}) = 0$, whence $L(\lambda) = o(1)$ as $\lambda \to \infty$. In fact, we find that

$$I(v; H_r) = O((kr)^{-1/2}) \qquad \text{as } kr \to \infty, \text{ for } 0 < \alpha < \tfrac{1}{2}\pi.$$

*Case III*: $\alpha = \frac{1}{2}\pi$. Now, two more stationary-phase points appear, at $\mathbf{c} = (\frac{1}{2}\pi, \pm\pi)$. These give a non-trivial contribution; the result is

$$I(v; H_r) = 2\,\exp\{\mathrm{i}\mathbf{k}\cdot\mathbf{y}\} + O((kr)^{-1/2}) \qquad \text{as } kr \to \infty, \text{ for } \alpha = \tfrac{1}{2}\pi.$$

Let us summarise these results. The symmetry axis of the hemisphere $H_r$ is the $z$-axis. We considered plane waves $v$ propagating out of the hemisphere, at an angle $\alpha$ to the $z$-axis. We saw that $I(v; H_r) \to 0$ as $r \to \infty$, for $0 < \alpha < \frac{1}{2}\pi$. For $\alpha = \frac{1}{2}\pi$ ('grazing waves', with respect to the plane $z = 0$), $I(v; H_r) \to 2\,\exp\{\mathrm{i}\mathbf{k}\cdot\mathbf{y}\}$, a finite quantity, as $r \to \infty$. For $\alpha = 0$ ('normal waves', with respect to $z = 0$), $I(v; H_r) \sim \mathrm{e}^{\mathrm{i}kr} J_0(k\rho \sin\Theta)$, which means that $I(v; H_r)$ does not have a limit (in this case) as $r \to \infty$. This is an unpleasant result!

# 8 Asymptotic behaviour of $I(u; T_r)$

The truncated cylindrical surface $T_r$ is defined by (12). A point $q \in T_r$, with position vector $\mathbf{x}$, has cylindrical polar coordinates $(r, \phi, z)$. Then, for large $r$, $|\mathbf{x} - \mathbf{y}| \simeq r - \rho \sin\Theta\, \cos(\phi - \Phi)$, where the fixed point $P$ has position vector $\mathbf{y}$ and spherical polar coordinates $(\rho, \Theta, \Phi)$ defined by (18). It follows that

$$I(u; T_r) \sim \frac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}kr} \int_{-\pi}^{\pi} \mathcal{E}(r, \phi)\, \exp\{-\mathrm{i}k\rho \sin\Theta\, \cos(\phi - \Phi)\}\, d\phi \qquad \text{as } r \to \infty,$$

where

$$\mathcal{E}(r, \phi) = \int_s^0 \left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) dz$$

and the lower limit is $s(r\cos\phi, r\sin\phi)$.

When is it true that $I(u; T_r) \to 0$ as $r \to \infty$? To answer this question, we need additional assumptions. For example, a sufficient condition is that $s \to 0$ as $r \to \infty$, for all $\phi$, which means that the rough surface approaches the flat plane $z = 0$ at large distances, in all directions. We could also put some restriction on the behaviour of $u$ near $S$.

However, there are situations in which $I(u; T_r)$ does not vanish as $r \to \infty$. An explicit example of this is given next.

# 9 An example

Recall the integral representation (13) for $u(P)$ when $P \in D_r$, the region bounded by the hemisphere $H_r$, the truncated rough surface $S_r$ and the truncated circular cylinder $T_r$:

$$2u(P) = I(u; S_r) + I(u; H_r) + I(u; T_r). \tag{20}$$

We note that the left-hand side of (20) does not depend on $r$, so that the right-hand side must have a limit as $r \to \infty$. Nevertheless, it is instructive to see how the different surface pieces contribute. We do this by examining a very simple example, which is for a plane wave *normally* incident upon a *flat* surface at $z = -h$. Thus

$$u_{\text{inc}} = A_0 \, \mathrm{e}^{-\mathrm{i}kz} \text{ and } u = \mathrm{e}^{\mathrm{i}kz} \text{ with } A_0 = \mathrm{e}^{-2\mathrm{i}kh}.$$

This is the exact solution. Let us see how this solution is reconstructed by the representation (20). For simplicity, we take $P$ at the origin; this will permit all the integrals to be evaluated exactly (without any asymptotic approximations).

On $H_r$, we have $u = \mathrm{e}^{\mathrm{i}kr\cos\theta}$, $\partial u/\partial n = \mathrm{i}ku\cos\theta$, $G = Br^{-1}\,\mathrm{e}^{\mathrm{i}kr}$ and $\partial G/\partial n = (\mathrm{i}k - r^{-1})G$. Integrating over $\theta$ gives

$$I(u; H_r) = \mathrm{e}^{\mathrm{i}kr}. \tag{21}$$

On $S_r$, we have $u = \mathrm{e}^{-\mathrm{i}kh}$, $\partial u/\partial n = -\mathrm{i}ku$,

$$G = BR_\sigma^{-1}\mathrm{e}^{\mathrm{i}kR_\sigma}, \qquad \partial G/\partial n = hR_\sigma^{-2}\left(\mathrm{i}k - R_\sigma^{-1}\right)G$$

and $R_\sigma = \sqrt{\sigma^2 + h^2}$, whence the integral over $S_r$ is

$$\begin{aligned} I(u; S_r) &= 2\pi B\,\mathrm{e}^{-\mathrm{i}kh}\int_0^r \mathrm{e}^{\mathrm{i}kR_\sigma}\left\{\mathrm{i}k + hR_\sigma^{-1}\left(\mathrm{i}k - R_\sigma^{-1}\right)\right\}\frac{\sigma\,d\sigma}{R_\sigma} \\ &= -\mathrm{e}^{-\mathrm{i}kh}\int_h^R \mathrm{e}^{\mathrm{i}kt}\left\{\mathrm{i}k + ht^{-1}\left(\mathrm{i}k - t^{-1}\right)\right\}dt \\ &= -\mathrm{e}^{-\mathrm{i}kh}\left[\mathrm{e}^{\mathrm{i}kt} + ht^{-1}\,\mathrm{e}^{\mathrm{i}kt}\right]_{t=h}^R \\ &= 2 - \left(1 + \frac{h}{R}\right)\mathrm{e}^{\mathrm{i}k(R-h)} \end{aligned} \tag{22}$$

where $R = \sqrt{r^2 + h^2}$.

On $T_r$, we have $u = \mathrm{e}^{\mathrm{i}kz}$ and $\partial u/\partial n = \partial u/\partial r = 0$, whence

$$\begin{aligned} I(u; T_r) &= 2\pi Br\int_{-h}^0 \mathrm{e}^{\mathrm{i}kz}\frac{\partial}{\partial r}\left(\frac{\mathrm{e}^{\mathrm{i}kR_z}}{R_z}\right)dz \\ &= -\left[(1 - zR_z^{-1})\,\mathrm{e}^{\mathrm{i}k(R_z+z)}\right]_{z=-h}^0 \\ &= -\mathrm{e}^{\mathrm{i}kr} + \left(1 + \frac{h}{R}\right)\mathrm{e}^{\mathrm{i}k(R-h)} \end{aligned} \tag{23}$$

where $R_z = \sqrt{r^2 + z^2}$.

Adding equations (21), (22) and (23), we see that their sum is exactly 2, which is $2u(P)$ evaluated at the origin. Note that, as $r \to \infty$,

$$\begin{aligned} I(u; S_r) &\sim 2 - \mathrm{e}^{\mathrm{i}k(r-h)}, \\ I(u; T_r) &\sim -\mathrm{e}^{\mathrm{i}kr} + \mathrm{e}^{\mathrm{i}k(r-h)}, \end{aligned}$$

and $I(u; H_r)$ does not simplify further. Thus, the boundary integral over the truncated rough surface does not have a limit as $r \to \infty$. Moreover, the integral over the truncated cylinder $T_r$ does not have a limit as $r \to \infty$, and it is not negligible. This is a genuine three-dimensional effect, which is not seen in the two-dimensional case (DeSanto and Martin, 1997).

# 10 Discussion

The results described above show that if the scattered field includes a plane wave propagating along the $z$-axis away from the rough surface ('normal waves'), then the usual Helmholtz integral equation is not valid: the boundary integral diverges. It is possible to devise a modified integral equation which reduces to the standard Helmholtz integral equation when normal waves are absent (DeSanto and Martin, 1998).

This is unsatisfactory, even though the mathematical difficulty may be overcome. Indeed, this difficulty is due entirely to the unphysical problem posed at the outset: *plane-wave* reflection by an *infinite* rough surface. Clearly, we can realise neither a plane wave nor an infinite rough surface. Moreover, the mathematical difficulty disappears if we consider either point-source insonification or a finite patch of roughness on an otherwise flat surface.

At the Metsovo conference, Ralph Kleinman asked if the results in section 9 could be extended to point-source insonification: they can. Thus, suppose that the incident field is generated by a point source at $(x, y, z) = (0, 0, z_0)$ with $z_0 > 0$:

$$u_{\rm inc} = AR^{-1}\,{\rm e}^{{\rm i}kR} \quad \text{with} \quad R = \sqrt{x^2 + y^2 + (z - z_0)^2}.$$

If we take $A = A_0 z_0\,{\rm e}^{-{\rm i}kz_0}$ with $A_0 = {\rm e}^{-2{\rm i}kh}$, then $u_{\rm inc} \sim A_0\,{\rm e}^{-{\rm i}kz}$ as $z_0 \to \infty$, which means that the point-source incident field reduces to the incident plane wave considered in section 9.

We know the exact solution for the field scattered by an infinite plane surface at $z = -h$, using an image singularity. It is

$$u = AR_1^{-1}\,{\rm e}^{{\rm i}kR_1} \quad \text{with} \quad R = \sqrt{x^2 + y^2 + (z + z_0 + 2h)^2}.$$

If we take $P$ at $O$, we can evaluate the integrals discussed in section 9. For example, we find that

$$I(u; H_r) = \frac{A}{z_0 + 2h}\left(1 - \frac{r}{R_0}\right){\rm e}^{{\rm i}k(R_0 + r)},$$

where $R_0 = \sqrt{r^2 + (z_0 + 2h)^2}$. Substituting for $A$ gives, exactly,

$$I(u; H_r) = \frac{z_0}{z_0 + 2h}\left(1 - \frac{r}{R_0}\right){\rm e}^{{\rm i}k(R_0 + r - z_0 - 2h)}.$$

If we fix $z_0$ and $h$, we see that $I(u; H_r) \to 0$ as $r \to \infty$, because $R_0 \sim r$. On the other hand, if we fix $r$ and $h$ and let $z_0 \to \infty$, we recover (21).

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