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ON THE RADIATION OF WATER WAVES BY OSCILLATING SUBMERGED PLATES

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We consider the radiation of small-amplitude surface water waves by an oscillating thin rigid plate, in three dimensions. The problem is to calculate the hydrodynamic forces on the plate. It is simplest if the plate lies *in* the free surface. Such *dock problems* have been studied by MacCamy (1961), Miles (1987) and others. They are relatively simple because they can be reduced to a Fredholm integral equation of the second kind with a weakly-singular kernel. In fact, the equation is a (direct) boundary integral equation for the velocity potential ϕ on the plate.

Suppose now that the plate is submerged. Then, we can reduce the problem to a hypersingular integral equations over the plate; the unknown is $[\phi]$, the discontinuity in the potential across the plate. Similar reductions have been made for two-dimensional problems (Parsons & Martin 1992, 1994, 1995). Previous work on submerged plates in three dimensions is limited: for water of finite depth, matched eigenfunction expansions have been used by Yu & Chwang (1993).

In this paper, we describe our work on the simplest situation, namely, the heave (vertical) oscillations of a horizontal circular plate. This is an axisymmetric problem. We start from the governing hypersingular integral equation for $[\phi]$. We then explicitly invert the hypersingular part to obtain a Fredholm integral equation of the second kind. Finally, we introduce a new unknown function, leading to a very simple Fredholm integral equation of the second kind. This equation is easy to solve numerically (except when the disc is close to the free surface). It also has an elegant structure: for example, in the absence of waves, it reduces to Love's integral equation (Love 1949) for the electrostatic field of a parallel-plate capacitor.

The motivation for the present work is twofold: first, we want some benchmark results which can be used for comparison with numerical solutions of the original hypersingular integral equation; and, second, we are interested in the problem of small submergence, when (quasi-) resonant motions of the fluid above the disc are expected, by analogy with those found above a truncated vertical circular cylinder by Longuet-Higgins (1967), Miles (1986) and others.

Boundary integral equations

For any smooth submerged plate, the governing hypersingular integral equation for $[\phi]$ is

$$\frac{1}{4\pi} \oint_{\Omega} [\phi(q)] \frac{\partial^2}{\partial n_p \partial n_q} G(p,q)\, ds_q = V(p), \qquad p \in \Omega, \tag{1}$$

which is to be solved subject to $[\phi] = 0$ around the edge of Ω. Here, V is the prescribed normal velocity on the plate Ω and G is a fundamental solution, corresponding to a 3-D wave source: it is given by

$$G(P,Q) \equiv G(x,y,z;\xi,\eta,\zeta) = \frac{1}{\sqrt{R^2 + (z-\zeta)^2}} + \int_0^\infty \mathrm{e}^{-k(z+\zeta)}\, J_0(kR) \frac{k+K}{k-K}\, dk \tag{2}$$

where $R = \sqrt{(x-\xi)^2 + (y-\eta)^2}$ and J_0 is a Bessel function. G is harmonic and satisfies the linearized free-surface condition

$$K\phi + \partial\phi/\partial z = 0 \qquad \text{on } z = 0, \tag{3}$$

where $K = \omega^2/g$, ω is the frequency and g is the acceleration due to gravity; moreover, the integration path in (2) is indented below the pole of the integrand at $k = K$ so that G also satisfies the radiation condition.

The first term in (2) gives rise to the hypersingularity in (1). For a flat circular disc, the corresponding hypersingular operator has an explicit inverse, which we apply. Moreover, if the circular disc is also horizontal, the non-singular part is especially simple. Finally, if the forcing is very simple, $V = V_0$, a constant, the solution is axisymmetric. Explicitly, suppose that the disc's centre is at

$$z = \tfrac{1}{2}b - (V_0/\omega)\cos\omega t.$$

Then, the vertical hydrodynamic force on the horizontal disc, F, can be expressed as

$$F = \rho a^3 V_0 \omega \left\{ \mathcal{A} \cos\omega t + \mathcal{B} \sin\omega t \right\},$$

where a is the radius of the disc, ρ is the fluid's density, and $\mathcal{A}$ and $\mathcal{B}$ are the dimensionless added-mass and damping coefficients, respectively. These coefficients can be expressed in terms of an integral of $[\phi]$ over the disc. For a disc $0 \le r < 1$, we write

$$[\phi](r) = -\frac{4}{\pi} \int_r^1 \frac{\psi(t)\, dt}{\sqrt{t^2 - r^2}},$$

and find that ψ solves

$$\psi(x) - \frac{b}{\pi} \int_{-1}^{1} \frac{\psi(y)}{b^2 + (x-y)^2}\, dy - \frac{2K}{\pi} \int_{-1}^{1} \psi(y)\, \Phi_0(x-y,b)\, dy = x, \qquad -1 \le x \le 1, \tag{4}$$

where Φ_0 is a *two-dimensional* wave-source potential:

$$\Phi_0(X, Y) = \fint_0^\infty \mathrm{e}^{-kY} \cos kX \, \frac{dk}{k - K}.$$

In terms of ψ, we find that

$$\mathcal{A}(K, b) + \mathrm{i}\mathcal{B}(K, b) = 8 \int_0^1 \psi(x) \, x \, dx.$$

The integral equation (4) has a nice structure. For example, when $b \to \infty$ (deeply submerged plate), it gives $\psi(x) = x$, which gives the classical added mass for a plate oscillating in an infinite fluid. When $K = 0$, the second integral term in (4) vanishes, leaving an equation known as Love's equation (Love 1949); this corresponds to two identical coaxial circular plates in an infinite fluid (in this limit, the free surface is replaced by a rigid wall; see (3)).

Numerical solutions of (4) will be presented. These show the occurrence of negative added mass for a range of frequencies and for sufficiently small submergence b. Current work on the small-b limit will also be described.

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