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SCATTERING OF ELASTIC WAVES BY INCLUSIONS WITH THIN INTERFACE LAYERS

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ABSTRACT

Elastic waves are scattered by an elastic inclusion. The interface between the inclusion and the surrounding material is imperfect: the discontinuity in the displacement (traction) vector across the interface is taken to be proportional to the average of the traction (displacement) vectors on the two sides of the interface. Uniqueness theorems are obtained, and boundary integral equations over the interface are derived.

1. INTRODUCTION

Consider a bounded obstacle embedded in an unbounded solid. Both the obstacle (the 'inclusion') and the surrounding solid (the 'matrix') are composed of homogeneous, isotropic elastic materials. We consider the scattering of elastic waves by the inclusion. For small time-harmonic oscillations, this leads to a vector transmission problem, which we call the *inclusion problem*, in which conditions are specified on the smooth interface, S, between the matrix and the inclusion.

Usually, the matrix and the inclusion are assumed to be welded together, i.e. the displacement and traction vectors are both continuous across S, which is then called a *perfect interface.* The opposite extreme is when there is no interaction ('complete debonding'). Intermediate situations arise when the two solids can slip or separate, or when there is a thin layer of a different material (such as glue or lubricant) between the solids. In this paper, we are especially interested in those intermediate situations that can be modelled by simple linear modifications to the perfect-interface continuity conditions. In fact, we suppose that the discontinuity in the displacement (traction) vector across the interface depends linearly on the average of the traction (displacement) vectors on the two sides of the interface (see (2.4) below). These transmission conditions include many of the phenomenological models of imperfect interfaces in the literature.

New results are obtained, extending the work in [1]. Specifically, we obtain a general uniqueness theorem, giving sufficient conditions on the coefficients in the interface conditions. We also derive some new quasi-Fredholm systems of coupled boundary integral equations over S. It remains to analyse the solvability of these systems.

2. INCLUSION PROBLEMS

Let B_i denote a bounded domain, with a smooth closed boundary S and simply-connected exterior, B_e. We seek displacements $\mathbf{u}_e(P)$ and $\mathbf{u}_i(P)$ so that

$$L_e\mathbf{u}_e(P) = \mathbf{0}, \quad P \in B_e \qquad \text{and} \qquad L_i\mathbf{u}_i(P) = \mathbf{0}, \quad P \in B_i,$$

where $\mathbf{u}(P) = \mathbf{u}_e(P) + \mathbf{u}_{inc}(P)$ for $P \in B_e$, $\mathbf{u}_{inc}$ is the given incident wave and $\mathbf{u}_e$ satisfies a radiation condition at infinity. In addition, we shall impose certain continuity conditions across S; these are specified below. The operator L_a is defined by

$$L_a\mathbf{u} = k_a^{-2} \text{ grad div } \mathbf{u} - K_a^{-2} \text{ curl curl } \mathbf{u} + \mathbf{u}$$

where $\rho_a\omega^2 = (\lambda_a + 2\mu_a)k_a^2 = \mu_a K_a^2$ and $a = e$ or i. ρ_a is the density of the solid in B_a, λ_a and μ_a are the Lamé moduli, and the time-dependence $e^{-i\omega t}$ is suppressed throughout. The traction operator T_a is defined on S by

$$(T_a\mathbf{u})_m(p) = \lambda_a n_m \text{ div } \mathbf{u} + \mu_a n_\ell(\partial u_m/\partial x_\ell + \partial u_\ell/\partial x_m)$$

where $\mathbf{n}(p)$ is the unit normal at $p \in S$, pointing into B_e.

If S is a perfect interface, we impose

$$[\mathbf{t}] = \mathbf{0} \qquad \text{and} \qquad [\mathbf{u}] = \mathbf{0}, \tag{2.1}$$

where $\mathbf{t} = T_e\mathbf{u}$ and $\mathbf{t}_i = T_i\mathbf{u}_i$ are traction vectors and square brackets denote discontinuities across the interface:

$$[\mathbf{u}] = \mathbf{u} - \mathbf{u}_i, \qquad \text{evaluated on } S.$$

The corresponding inclusion problem has been studied extensively; see, e.g., Kupradze *et al.* [2] and Mura [3]. For a simplified treatment of the two-dimensional problem, and further references, see [4].

The perfect-interface conditions (2.1) were first modified by Newmark in 1943 [5]. He was concerned with the transmission of static loads between straight beams, and explicitly allowed slipping to occur. Similar modifications were used by Mal and Bose [6] for scattering by spherical inclusions. These modifications, and those of many other authors, are special cases of the following interface conditions,

$$[\mathbf{t}] = \mathbf{0} \qquad \text{and} \qquad [\mathbf{u}] = F.\mathbf{t}, \tag{2.2}$$

where the matrix F is given. We shall allow F to be a full matrix, with elements that vary with position p on S; in the engineering literature, F is usually taken as a constant diagonal matrix.

An alternative modification to (2.1) is

$$[\mathbf{t}] = G.\mathbf{u} \qquad \text{and} \qquad [\mathbf{u}] = \mathbf{0}, \tag{2.3}$$

where the elements of the matrix G could vary with position p on S. A special case of (2.3) was used by Olsson *et al.* [7].

Finally, we consider a model that includes both (2.2) and (2.3), namely

$$[\mathbf{t}] = G.\langle\mathbf{u}\rangle \qquad \text{and} \qquad [\mathbf{u}] = F.\langle\mathbf{t}\rangle, \tag{2.4}$$

where $\langle\mathbf{u}\rangle = \frac{1}{2}(\mathbf{u} + \mathbf{u}_i)$ is the average of $\mathbf{u}$ and $\mathbf{u}_i$ on S. A special case of (2.4) was used by Baik and Thompson [8, 9].

A review of the literature on imperfect interfaces is given in [1].

3. UNIQUENESS THEOREMS

Consider the problem of scattering by an inclusion with an imperfect interface. We can prove uniqueness theorems for interfaces characterized by (2.2), (2.3) or (2.4); for its generality, we use (2.4) here, and always assume that all the elements of the matrices F and G are finite. We adapt standard arguments from [2]. Thus, surround S with a large sphere S_R of radius R. For $P \in B_e$, write

$$\mathbf{u}(P) = \mathbf{u}^{(p)} + \mathbf{u}^{(s)},$$

where

$$\mathbf{u}^{(p)} = -k_e^{-2} \text{ grad div } \mathbf{u}, \qquad \mathbf{u}^{(s)} = \mathbf{u} - \mathbf{u}^{(p)},$$

and $\mathbf{u}_{inc} \equiv \mathbf{0}$. Then, an application of Betti's reciprocal theorem to $\mathbf{u}$ and its complex conjugate, $\overline{\mathbf{u}}$, in the region between S and S_R gives

$$k_e(\lambda_e + 2\mu_e) \lim_{R\to\infty} \int_{S_R} |\mathbf{u}^{(p)}|^2 \, ds + K_e\mu_e \lim_{R\to\infty} \int_{S_R} |\mathbf{u}^{(s)}|^2 \, ds + I = 0, \quad (3.1)$$

where

$$I = \frac{1}{2i} \int_S (\mathbf{u}.\overline{\mathbf{t}} - \overline{\mathbf{u}}.\mathbf{t}) \, ds = \mathcal{I}m \int_S \mathbf{u}.\overline{\mathbf{t}} \, ds, \quad (3.2)$$

$\mathcal{I}m$ denotes imaginary part and the radiation condition has been used (see [2], Chpt. 3, §2). If we can show that $I \geq 0$, we can deduce from (3.1) that $\mathbf{u}^{(p)} \equiv \mathbf{0}$ and $\mathbf{u}^{(s)} \equiv \mathbf{0}$, whence $\mathbf{u} \equiv \mathbf{0}$ in B_e. Then, (2.4) imply that $\mathbf{u}_i = \mathbf{0}$ and $\mathbf{t}_i = \mathbf{0}$ on S, whence $\mathbf{u}_i \equiv \mathbf{0}$ in B_i.

Now, applying Betti's theorem in B_i to $\mathbf{u}_i$ and $\overline{\mathbf{u}}_i$ gives

$$0 = \frac{1}{2i} \int_S (\mathbf{u}_i.\overline{\mathbf{t}}_i - \overline{\mathbf{u}}_i.\mathbf{t}_i) \, ds = \mathcal{I}m \int_S \mathbf{u}_i.\overline{\mathbf{t}}_i \, ds. \quad (3.3)$$

But, since

$$\mathbf{u}.\overline{\mathbf{t}} - \mathbf{u}_i.\overline{\mathbf{t}}_i = \frac{1}{2} \left\{ (\overline{\mathbf{t}} + \overline{\mathbf{t}}_i).(\mathbf{u} - \mathbf{u}_i) + (\mathbf{u} + \mathbf{u}_i).(\overline{\mathbf{t}} - \overline{\mathbf{t}}_i) \right\},$$

subtracting (3.3) from (3.2) gives

$$I = \mathcal{I}m \int_S \left\{ \langle \overline{\mathbf{t}} \rangle . F . \langle \mathbf{t} \rangle + \langle \mathbf{u} \rangle . \overline{G} . \langle \overline{\mathbf{u}} \rangle \right\} \, ds$$

after using (2.4). Thus, $I \geq 0$, provided that

$$F_{k\ell} = \overline{F}_{\ell k} \quad \text{for} \quad k \neq \ell \qquad \text{and} \qquad \mathcal{I}m(F_{kk}) \geq 0 \quad \text{(no sum)} \quad (3.4)$$

and

$$G_{k\ell} = \overline{G}_{\ell k} \quad \text{for} \quad k \neq \ell \qquad \text{and} \qquad \mathcal{I}m(G_{kk}) \leq 0 \quad \text{(no sum)}. \quad (3.5)$$

So, if the elements of F and G are finite and satisfy (3.4) and (3.5), respectively (for all $p \in S$ if F and G vary with p), we have proved that the corresponding inclusion problem has at most one solution.

4. BOUNDARY INTEGRAL EQUATIONS

In this section, we derive (direct) boundary integral equations over S for inclusions with imperfect interfaces characterized by (2.4), in the plane case. In fact, our aim is to derive *quasi-Fredholm* systems of singular integral equations, for all the usual Fredholm theorems hold for such systems [10]. In particular, we can analyse solvability by showing that the corresponding homogeneous system has only the trivial solution.

First, we introduce two fundamental Green's tensors, $\mathbf{G}_a(P;Q)$ $(a = e, i)$:

$$(\mathbf{G}_a(P;Q))_{ij} = \frac{1}{\mu_a}\left\{\Psi_a\delta_{ij} + \frac{1}{K_a^2}\frac{\partial^2}{\partial x_i \partial x_j}(\Psi_a - \Phi_a)\right\}$$

where $\Phi_a = -(i/2)H_0^{(1)}(k_a R)$, $\Psi_a = -(i/2)H_0^{(1)}(K_a R)$ and $R = |P - Q|$. Next, we define elastic single-layer and double-layer potentials by

$$(S_a\mathbf{u})(P) = \int_S \mathbf{u}(q).\mathbf{G}_a(q;P)\,ds_q$$

and

$$(D_a\mathbf{u})(P) = \int_S \mathbf{u}(q).T_a^q\mathbf{G}_a(q;P)\,ds_q,$$

respectively, where T_a^q means T_a applied at $q \in S$. Then, three applications of Betti's theorem (one in B_e to $\mathbf{u}_e$ and $\mathbf{G}_e$, one in B_i to $\mathbf{u}_{inc}$ and $\mathbf{G}_e$, and one in B_i to $\mathbf{u}_i$ and $\mathbf{G}_i$) yield the familiar representations

$$2\mathbf{u}_e(P) = (S_e\mathbf{t})(P) - (D_e\mathbf{u})(P), \qquad P \in B_e, \tag{4.1}$$

and

$$-2\mathbf{u}_i(P) = (S_i\mathbf{t}_i)(P) - (D_i\mathbf{u}_i)(P), \qquad P \in B_i. \tag{4.2}$$

Letting $P \to p \in S$, (4.1) and (4.2) give

$$(I + \overline{K_e^*})\mathbf{u} - S_e\mathbf{t} = 2\mathbf{u}_{inc} \tag{4.3}$$

and

$$(I - \overline{K_i^*})\mathbf{u}_i + S_i\mathbf{t}_i = \mathbf{0}, \tag{4.4}$$

respectively, where

$$\overline{K_a^*}\mathbf{u} = \int_S \mathbf{u}(q).T_a^q \mathbf{G}_a(q;p)\, ds_q$$

is a singular integral operator. We can obtain two further relations by calculating the tractions on S corresponding to (4.1) and (4.2). However, we shall forego this possibility here. This self-imposed restriction prevents us from obtaining quasi-Fredholm systems for some choices of F and G, including the case of a perfect interface ($F = G = 0$); see [4] for such a system, involving a regularization of the operator TD.

When $G = 0$, we have (2.2), which we can use in (4.4) to give

$$(I - \overline{K_i^*})\mathbf{u} + \{S_i - (I - \overline{K_i^*})F\}\mathbf{t} = \mathbf{0}. \tag{4.5}$$

The pair (4.3) and (4.5) is a system of four coupled singular integral equations for the four components of the two vectors $\mathbf{u}(p)$ and $\mathbf{t}(p)$, $p \in S$. It can be shown that this system is a quasi-Fredholm system, provided that F is a non-singular matrix (for all $p \in S$ if F varies with p). This system was derived in [1].

When $F = 0$ in (2.4), we have (2.3). It is easily seen that the use of (2.3) in (4.4), as before, does *not* lead to a quasi-Fredholm system for any G (including $G = 0$).

Suppose that, in (2.4), F is non-singular. Then, we have

$$\mathbf{t} = A\mathbf{u} - B\mathbf{u}_i \qquad \text{and} \qquad \mathbf{t}_i = B\mathbf{u} - A\mathbf{u}_i$$

on S, where the matrices A and B are given by

$$A = F^{-1} + \frac{1}{4}G \qquad \text{and} \qquad B = F^{-1} - \frac{1}{4}G.$$

Substituting into (4.3) and (4.4) gives

$$\left.\begin{aligned} (I + \overline{K_e^*} - S_e A)\mathbf{u} + S_e B\mathbf{u}_i &= 2\mathbf{u}_{inc} \\ S_i B\mathbf{u} + (I - \overline{K_i^*} - S_i A)\mathbf{u}_i &= \mathbf{0} \end{aligned}\right\} \tag{4.6}$$

This is a quasi-Fredholm system for $\mathbf{u}(p)$ and $\mathbf{u}_i(p)$. In particular, if $G = 0$, we have $A = B = F^{-1}$, whence (4.6) reduces to the pair (4.3) and (4.5), since $\mathbf{t} = F^{-1}[\mathbf{u}]$.

Finally, suppose instead that G is non-singular in (2.4), whence

$$\mathbf{u} = C\mathbf{t} - D\mathbf{t}_i \qquad \text{and} \qquad \mathbf{u}_i = D\mathbf{t} - C\mathbf{t}_i$$

on S, where the matrices C and D are given by

$$C = G^{-1} + \frac{1}{4}F \qquad \text{and} \qquad D = G^{-1} - \frac{1}{4}F.$$

Substituting into (4.3) and (4.4) gives

$$\left.\begin{array}{l} \{(I + \overline{K_e^*})C - S_e\}\mathbf{t} - (I + \overline{K_e^*})D\mathbf{t}_i = 2\mathbf{u}_{inc} \\ (I - \overline{K_i^*})D\mathbf{t} - \{(I - \overline{K_i^*})C - S_i\}\mathbf{t}_i = \mathbf{0} \end{array}\right\} \tag{4.7}$$

This is a system of singular integral equations for $\mathbf{t}(p)$ and $\mathbf{t}_i(p)$. To analyse it, let

$$M = \sigma(I + \overline{K_e^*}) \qquad \text{and} \qquad N = \sigma(I - \overline{K_i^*}),$$

where $\sigma(L)$ is the symbol matrix of the singular integral operator L; see [10] or [4]. Then, we have to examine the determinant, Δ, of

$$\begin{pmatrix} MC & -MD \\ ND & -NC \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} C & -D \\ D & -C \end{pmatrix}.$$

Thus,

$$\Delta = \det(M)\det(N)\Delta_0$$

where

$$\Delta_0 = \det \begin{pmatrix} C & -D \\ D & -C \end{pmatrix}.$$

It follows that (4.7) is a quasi-Fredholm system provided that Δ_0 does not vanish. This is readily shown to be the case if $F = 0$ or if F and G are real and symmetric; cf. the sufficient conditions for uniqueness obtained in § 3.

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REFERENCES

[1] P.A. Martin, 'Thin interface layers: adhesives, approximations and analysis', in *Elastic Waves and Ultrasonic Nondestructive Evaluation* (ed. S.K. Datta, J.D. Achenbach and Y.S. Rajapakse), North-Holland, Amsterdam, 1990, pp. 217–222.

[2] V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili, & T.V. Burchuladze, *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam, 1979.

[3] T. Mura, *Micromechanics of Defects in Solids*, 2nd revised ed., Martinus Nijhoff, Dordrecht, 1987.

[4] P.A. Martin, 'On the scattering of elastic waves by an elastic inclusion in two dimensions', *Quart. J. Mech. Appl. Math.* **43** (1990) 275–291.

[5] N.M. Newmark, C.P. Siess & I.M. Viest, 'Tests and analysis of composite beams with incomplete interaction', *Proc. Soc. for Experimental Stress Analysis* **9**, No.1 (1951) 75–92.

[6] A.K. Mal & S.K. Bose, 'Dynamic elastic moduli of a suspension of imperfectly bonded spheres', *Proc. Camb. Phil. Soc.* **76** (1974) 587–600.

[7] P. Olsson, S.K. Datta & A. Boström, 'Elastodynamic scattering from inclusions surrounded by thin interface layers', *J. Appl. Mech.* to appear.

[8] J. Baik & R.B. Thompson, 'Long wavelength elastic scattering from a planar distribution of inclusions', *J. Appl. Mech.* **52** (1985) 974–976.

[9] J.M. Baik & R.B. Thompson, 'Ultrasonic scattering from imperfect interfaces: a quasi-static model', *J. Nondestructive Evaluation*, **4** (1984) 177–196.

[10] N.I. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen, 1953.