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Some Applications of the Mellin Transform to Asymptotics of Series

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Abstract

Mellin transforms are used to find asymptotic approximations for functions defined by series. Such approximations were needed in the analysis of a water-wave problem, namely, the trapping of waves by submerged plates. The method seems to have wider applicability.

1 Introduction

It is well known that asymptotic approximations for functions defined by integrals can often be found using Mellin transforms [2], [8]. Analogous results can also be sought for functions defined by series,

$$f(x) = \sum_{n=1}^{\infty} c_n u(\mu_n x), \tag{1}$$

where c_n and μ_n are known constants and $u(y)$ is defined for all $y > 0$. We assume that the series is convergent for all $x > 0$, and seek the asymptotic behaviour of $f(x)$ as $x \to 0+$. Ramanujan [1, Ch. 15] considered some problems of this type, such as $\mu_n = n^p$ and $u(x) = \mathrm{e}^{-x}$, using the Euler-Maclaurin formula. In (1), $u(y)$ is sampled at points $y = \mu_n x$; we describe such points, and the series (1), as *separable*. Separable series can often be analysed using Mellin transforms; this method was used to confirm some of Ramanujan's results [1]. More generally, we study *non-separable* series

$$f(x) = \sum_{n=1}^{\infty} c_n u(\lambda_n(x)), \tag{2}$$

where $u(y)$ is sampled at non-separable points, $y = \lambda_n(x)$. We find asymptotic approximations to such series by first finding suitable separable approximations to $\lambda_n(x)$, and then use Mellin transforms.

Both separable and non-separable series arise in waveguide problems. In this context, the limit $1/h \to 0$ is of interest, where the walls are at $y = 0$ and $y = h$. A typical problem in acoustics concerns an infinite bifurcated waveguide, with a semi-infinite plate (the septum) along $y = d$, $x < 0$ (*closed geometry*). A waveguide mode is incident from $x = -\infty$ in the region $0 < y < d$; it is partially reflected at the end of the septum and partially transmitted into the rest of the guide. The same problem can be considered when $h = \infty$. This corresponds to an open-ended waveguide (*open geometry*). The connection between open-geometry and related closed-geometry problems is of interest because the latter are often easier to solve.

The same closed geometry has been used by Linton & Evans [4] in the context of linear water waves. The governing equation is the modified Helmholtz equation $(\nabla^2 - l^2)\phi = 0$, where l is the positive wavenumber in a direction perpendicular to the xy-plane. The septum and the bottom ($y = h$) are hard, whereas the boundary condition on the free surface $y = 0$ is $K\phi + \partial\phi/\partial y = 0$, where K is another positive wavenumber. Two more wavenumbers, k and k_0, are defined to be the unique positive real roots of

$$K = k \tanh kd \qquad \text{and} \qquad K = k_0 \tanh k_0 h, \tag{3}$$

respectively, and then l is chosen to satisfy $K < k_0 < l < k$. Hence, a surface wave incident from $x = -\infty$ will be totally reflected by the end of the plate. Linton & Evans [4] gave an explicit formula for the argument of the (complex) reflection coefficient, which they used to estimate the frequencies of waves trapped above a long horizontal submerged plate. We examine their formula below, and extract the limiting formula for deep water ($h \to \infty$). Indeed, it was a study of the limiting problem that originally motivated the present analysis.

2 Mellin transforms

To find asymptotic approximations for separable series (1), we use the Mellin transform. The Mellin transform of a function f, and its inverse, are

$$\widetilde{f}(z) = \int_0^\infty f(x) x^{z-1}\, dx \qquad \text{and} \qquad f(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \widetilde{f}(z) x^{-z}\, dz,$$

respectively. Typically, $\widetilde{f}(z)$ will be an analytic function of z within a strip, $a < \sigma < b$, say, where $z = \sigma + \mathrm{i}\tau$; within this strip, $|\widetilde{f}(z)| \to 0$ as $|\tau| \to \infty$; and $a < c < b$. We can obtain an asymptotic expansion of $f(x)$ for small

x by moving the inversion contour to the left; each term arises as a residue contribution from an appropriate pole in the analytic continuation of $\widetilde{f}(z)$ into $\sigma \le a$. Specifically, we have the following result.

THEOREM **1** ([6, p. 7]) *Suppose that $\widetilde{f}(z)$ is analytic in a left-hand plane, $\sigma \le a$, apart from poles at $z = -a_m$, $m = 0, 1, 2, \ldots$; let the principal part of the Laurent expansion of $\widetilde{f}(z)$ about $z = -a_m$ be given by*

$$\sum_{n=0}^{N(m)} A_{mn} \frac{(-1)^n n!}{(z + a_m)^{n+1}}.$$

Assume that $|\widetilde{f}(\sigma + \mathrm{i}\tau)| \to 0$ as $|\tau| \to \infty$ for $a' \le \sigma \le a$, and that $|\widetilde{f}(a' + \mathrm{i}\tau)|$ is integrable for $|\tau| < \infty$. Then, if a' can be chosen so that $-\mathrm{Re}\,(a_{M+1}) < a' < -\mathrm{Re}\,(a_M)$ for some M, $f(x)$ has the asymptotic expansion

$$f(x) \sim \sum_{m=0}^{M} \sum_{n=0}^{N(m)} A_{mn} x^{a_m} (\log x)^n \qquad \text{as } x \to 0+.$$

For more information on Mellin transforms, see [2, Ch. 4], [5], [8, Ch. 3].

3 Separable series: a problem of Ramanujan

EXAMPLE 1. Let ν be a real parameter. Find the behaviour of

$$f_\nu(x) = \sum_{n=1}^{\infty} n^{\nu-1} \mathrm{e}^{-nx} \qquad \text{as } x \to 0+.$$

We can take $c_n = n^{\nu-1}$, $\mu_n = n$ and $u(x) = \mathrm{e}^{-x}$. Hence

$$\widetilde{f_\nu}(z) = \zeta(z - \nu + 1)\Gamma(z), \tag{4}$$

where $\Gamma(z)$ is the the gamma function and $\zeta(z)$ is the Riemann zeta function. It is known that $\Gamma(z)$ is an analytic function of z, apart from simple poles at $z = -N$ with residue $(-1)^N/N!$, for $N = 0, 1, 2, \ldots$. It is also known that $\zeta(z)$ is analytic for all z, apart from a simple pole at $z = 1$; near $z = 1$, $\zeta(z) \simeq (z - 1)^{-1} + \gamma$, where $\gamma = 0.5772\ldots$ is Euler's constant.

Let us suppose that $0 < \nu < 1$. Then, $\widetilde{f_\nu}(z)$ is analytic for $\sigma > \nu$. We choose the inversion contour along $\sigma = c$, with $c > \nu$. Moving the contour to the left, we pick up a residue contribution from the simple pole at $z = \nu$: this gives the leading contribution as $f_\nu(x) \sim x^{-\nu}\Gamma(\nu)$ as $x \to 0+$. If we move the inversion contour further to the left, we formally obtain Ramanujan's expansion [1, p. 306],

$$f_\nu(x) \sim x^{-\nu}\Gamma(\nu) + \sum_{m=0}^{\infty} \frac{(-x)^m}{m!}\,\zeta(1-\nu-m) \quad \text{as } x \to 0. \tag{5}$$

The fact that this is an *asymptotic* expansion follows from Theorem 1 and the known properties of $\zeta(z)$ and $\Gamma(z)$ as $|\tau| \to \infty$. Actually, (5) is valid for all values of ν, apart from $\nu = -N$. In these cases, there is a double pole at $z = -N$, giving a term proportional to $x^N \log x$.

4 Non-separable series: a model problem

Let us consider some non-separable series involving the roots of the transcendental equation $(3)_2$. This has real roots $\pm k_0$ and an infinite number of pure imaginary roots, $\pm \mathrm{i}k_n$, $n = 1, 2, \ldots$; thus, k_n are the positive real roots of

$$K + k_n \tan k_n h = 0, \qquad n = 1, 2, \ldots; \tag{6}$$

they are ordered so that $(n - \frac{1}{2})\pi < k_n h < n\pi$. In the context of water-wave problems, h is the constant water depth and K is the positive real wavenumber. We are interested in the deep-water limit, $h \to \infty$. In dimensionless variables, we define $x = (Kh)^{-1}$ and $\lambda_n(x) = k_n h$, so that

$$\cos \lambda_n(x) + x\lambda_n(x) \sin \lambda_n(x) = 0 \qquad \text{with} \tag{7}$$

$$(n - \tfrac{1}{2})\pi < \lambda_n(x) < n\pi, \qquad n = 1, 2, \ldots. \tag{8}$$

It is straightforward to show that $\lambda_n(x)$ behaves as follows:

$$\lambda_n(x) \sim n\pi - (n\pi x)^{-1} - (x - \tfrac{1}{3})(n\pi x)^{-3} \tag{9}$$

as $n \to \infty$ for fixed x, and

$$\lambda_n(x) \sim (n - \tfrac{1}{2})\pi(1 + x + x^2) \tag{10}$$

as $x \to 0$ for fixed n. It is this non-uniform behaviour that causes difficulties. To find some uniform approximations, we rewrite the definition (7) as

$$\sin \nu_n(x) - x\{\mu_n + \nu_n(x)\} \cos \nu_n(x) = 0, \qquad \text{where} \tag{11}$$

$$\lambda_n(x) = \mu_n + \nu_n(x), \tag{12}$$

$\mu_n = (n - \frac{1}{2})\pi$ and $0 < \nu_n < \pi/2$. Discarding the second term inside the braces in (11) (this is certainly reasonable for large n), we obtain

$$\nu_n(x) \simeq \tan^{-1}(\mu_n x) = \nu_n^{(1)}(x), \tag{13}$$

say, which is a separable approximation to $\nu_n(x)$. The approximation $\lambda_n(x) \simeq \mu_n + \nu_n^{(1)}(x)$ agrees with the first two terms in (9) and with the first two terms in (10). This approximation can be improved by iteration: replace $\nu_n(x)$ by $\nu_n^{(1)}(x)$ inside the braces in (11) to give

$$\nu_n(x) \simeq \tan^{-1}\{\mu_n x + x\tan^{-1}(\mu_n x)\} = \nu_n^{(2)}(x), \tag{14}$$

say. Then, the approximation $\lambda_n(x) \simeq \mu_n + \nu_n^{(2)}(x)$ agrees with the three-term asymptotics in (9) and in (10).

Example 2. Find the behaviour of

$$f(x) = \pi \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n(x)} - \frac{1}{n\pi} \right) \qquad \text{as } x \to 0+.$$

The series converges for all $x \geq 0$; in fact, using the bounds (8), we have $0 < f(x) < 2\log 2$ for $x > 0$. As $\lambda_n(0) = (n - \frac{1}{2})\pi = \mu_n$, for all n, write

$$f(x) = 2\log 2 + S(x), \qquad \text{where} \tag{15}$$

$$S(x) = \pi \sum_{n=1}^{\infty} s_n(x) \qquad \text{and} \qquad s_n(x) = \frac{1}{\lambda_n(x)} - \frac{1}{\mu_n}.$$

We have $S(x) \to 0$ as $x \to 0$ and $S(x)$ is bounded as $x \to \infty$, whence $\widetilde{S}(z)$ is analytic in a strip $-\delta < \sigma < 0$, where $\delta > 0$. In fact, we note that $s_n(x) = O(x)$ as $x \to 0$ and is bounded as $x \to \infty$, whence $\widetilde{s}_n(z)$ is analytic for $-1 < \sigma < 0$; thus, we expect that $\delta = 1$.

We shall treat $S(x)$ using our separable approximations for $\nu_n(x)$. Since the latter may not be appropriate for small values of n, we split the sum:

$$S(x) = \pi \sum_{n=1}^{M} s_n(x) + \pi \sum_{n=M+1}^{\infty} s_n(x) = S_M(x) + S_M^{\infty}(x), \tag{16}$$

say, where M is fixed. For S_M, we can use (10) to give

$$S_M(x) \sim \pi \sum_{n=1}^{M} \mu_n^{-1}\{(1 + x + x^2)^{-1} - 1\} = -\pi x \sum_{n=1}^{M} \mu_n^{-1} + O(x^3) \tag{17}$$

as $x \to 0$. For S_M^{∞}, we start with $s_n(x) \simeq -\mu_n^{-2}(\nu_n - \nu_n^2/\mu_n)$, since $|\nu_n/\mu_n|$ is small. Next, we approximate ν_n by $\nu_n^{(2)}$ and ν_n^2 by $(\nu_n^{(1)})^2$. Finally, since $|\nu_n^{(1)}/\mu_n|$ is small, we can approximate $\nu_n^{(2)}$ using the Taylor approximation

$$\tan^{-1}(X + H) \simeq \tan^{-1} X + H(1 + X^2)^{-1} \tag{18}$$

for small H; the result is

$$s_n(x) \simeq -\mu_n^{-2}\{\nu_n^{(1)}(x) + x\nu_n^{(1)}(x)\,[1+(\mu_n x)^2]^{-1} - \mu_n^{-1}[\nu_n^{(1)}(x)]^{-2}\} = s_n^{(1)}(x),$$

say. This is our final separable approximation for $s_n(x)$. We find that the error, $|s_n - s_n^{(1)}|$ is $O(n^{-4})$ as $n \to \infty$ for fixed x, and is $O(x^3)$ as $x \to 0$ for fixed n. The Mellin transform of $s_n^{(1)}(x)$ is given by

$$\widetilde{s}_n^{(1)}(z) = -\mu_n^{-z-2}\widetilde{u}_1(z) + \mu_n^{-z-3}\widetilde{u}_2(z), \qquad \text{where} \tag{19}$$

$$\widetilde{u}_1(z) = \int_0^\infty x^{z-1}\tan^{-1}x\,dx = \frac{\pi}{2z\sin[\pi(z-1)/2]}, \tag{20}$$

$$\widetilde{u}_2(z) = \int_0^\infty x^{z-1}\{\tan^{-1}x - x(1+x^2)^{-1}\}\tan^{-1}x\,dx. \tag{21}$$

$\widetilde{u}_1(z)$ is analytic for $-1 < \sigma < 0$ and $\widetilde{u}_2(z)$ is analytic for $-4 < \sigma < 0$. Summing over n, using (16) and (19), gives

$$\widetilde{S}_M^\infty(z) \simeq -\psi_M(z+2)\widetilde{u}_1(z) + \psi_M(z+3)\widetilde{u}_2(z), \tag{22}$$

where, by definition,

$$\psi_M(z) = \pi\sum_{n=M+1}^{\infty}\mu_n^{-z} = \pi^{1-z}(2^z-1)\zeta(z) - \pi\sum_{n=1}^{M}\mu_n^{-z}. \tag{23}$$

$\psi_M(z)$ is analytic for all z, apart from a simple pole at $z=1$;

$$\psi_M(z) \simeq (z-1)^{-1} + \gamma + \log(4/\pi) - \pi\sum_{n=1}^{M}\mu_n^{-1} \quad \text{near } z=1. \tag{24}$$

To invert $\widetilde{S}_M^\infty(z)$, we start with the inversion contour to the left of $z=0$, and then move it further to the left; thus, we are interested in singularities in $\sigma < 0$. Consider the first term on the right-hand side of (22). From (20), we see that $\widetilde{u}_1(z)$ has simple poles at $z = -1, -3, \ldots$; near $z=-1$, we have $\widetilde{u}_1(z) \simeq (z+1)^{-1} + 1$. Hence, $\psi_M(z+2)\widetilde{u}_1(z)$ has a double pole at $z=-1$, giving terms proportional to $x\log x$ and x in $S_M^\infty(x)$. The next singularity at $z=-3$ gives a term in x^3, but we have already made errors of this order when we replaced $s_n(x)$ by $s_n^{(1)}(x)$. The second term on the right-hand side of (22) is analytic for $-4 < \sigma < 0$, apart from a simple pole at $z=-2$: this gives a term in x^2. Combining these results gives

$$S_M^\infty(x) = x\log x - x\left\{1 + \gamma + \log(4/\pi) - \pi\sum_{n=1}^{M}\mu_n^{-1}\right\} + O(x^2)$$

as $x \to 0$. Finally, using (16) and (17), we obtain

$$S(x) = x \log x - x\{1 + \gamma + \log(4/\pi)\} + O(x^2), \qquad \text{as } x \to 0; \tag{25}$$

$f(x)$ is given by (15). Note that this result does not depend on M; see (16).

5 A problem of Linton and Evans

In this section, we consider a water-wave problem described in §1 and solved by Linton & Evans [4]. They calculate a certain complex reflection coefficient; its argument is proportional to

$$E(h) = \tan^{-1}\left(\alpha^{-1}\sqrt{l^2 - k_0^2}\right) - \tan^{-1}(l/\alpha) - \tfrac{1}{2}\pi - (\alpha/\pi)L_0 + T, \tag{26}$$

where $L_0 = c\log(h/c) + d\log(h/d)$ and T is the sum

$$\sum_{n=1}^{\infty}\left\{\tan^{-1}\frac{\alpha}{\sqrt{l^2 + n^2\pi^2/c^2}} - \tan^{-1}\frac{\alpha}{\sqrt{l^2 + k_n^2}} + \tan^{-1}\frac{\alpha}{\sqrt{l^2 + \kappa_n^2}}\right\}.$$

The parameters d, l, K and κ_n $(n = 1, 2, \ldots)$ are fixed. k_0 is defined by $(3)_2$ and $\alpha = \sqrt{k^2 - l^2}$, where k is defined by $(3)_1$. We have $c = h - d > 0$ and $K < k_0 < l < k$.

Example 3. Find

$$\lim_{h\to\infty} E(h) = E_\infty, \tag{27}$$

say. This corresponds, physically, to solving the problem of Linton & Evans [4] when the water is infinitely deep.

Note that, as h varies, so too do k_0, k_n and c; all other parameters remain unchanged. To begin with, $(3)_2$ shows that $k_0 h \sim Kh(1 + 2\mathrm{e}^{-2Kh})$ as $Kh \to \infty$, so we can replace k_0 by K in the first term of $E(h)$, as $h \to \infty$. It is elementary to show that $L_0 = d(\log h + 1 - \log d) + o(1)$ as $h \to \infty$. For T, we note that the arguments of the three inverse tangents behave like $\alpha c/(n\pi)$, $\alpha h/(n\pi)$ and $\alpha d/(n\pi)$, respectively, as $n \to \infty$, and so we can write $T = T_1 - T_2 + T_3$, where

$$T_1 = \sum_{n=1}^{\infty}\left\{\tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + n^2\pi^2/c^2}}\right) - \frac{\alpha c}{n\pi}\right\}, \tag{28}$$

$$T_2 = \sum_{n=1}^{\infty}\left\{\tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + k_n^2}}\right) - \frac{\alpha h}{n\pi}\right\}, \tag{29}$$

$$T_3 = \sum_{n=1}^{\infty}\left\{\tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + \kappa_n^2}}\right) - \frac{\alpha d}{n\pi}\right\}, \tag{30}$$

using $c-h+d=0$. Note that T_1 is a separable series, T_2 is a non-separable series and T_3 is independent of h. So, at this stage, we have

$$\begin{aligned} E(h) &= -\tan^{-1}[\alpha(l^2-K^2)^{-1/2}] - \tan^{-1}(l/\alpha) \\ &\quad -(\alpha d/\pi)(\log h + 1 - \log d) + T_1 - T_2 + T_3 + o(1) \quad \text{as } h\to\infty. \end{aligned}$$

Deep-water behaviour of T_1. From (28), we have $T_1 = f(\pi/c)$, where $f(x)$ is defined by (1) with $c_n = 1$,

$$u(x) = \tan^{-1}\left(\frac{\alpha}{\sqrt{l^2+x^2}}\right) - \frac{\alpha}{x} \tag{31}$$

and $\mu_n = n$. Proceeding as in §3, we obtain $\widetilde{f}(z) = \zeta(z)\widetilde{u}(z)$, where $\widetilde{u}(z)$ is analytic for $1<\sigma<3$. We must find the singularities of $\widetilde{u}(z)$ in $0\le\sigma\le1$. For $1<\sigma<3$, we integrate by parts to give

$$\widetilde{f}(z) = (\alpha/z)\zeta(z)\widetilde{u}_1(z), \tag{32}$$

where

$$\widetilde{u}_1(z) = \int_0^\infty x^z \left\{ \frac{x}{\sqrt{x^2+l^2}\,(x^2+k^2)} - \frac{1}{x^2} \right\} dx. \tag{33}$$

is analytic for $1<\sigma<3$. It turns out that $\widetilde{u}_1$ can be continued analytically into $-2<\sigma<3$, apart from a simple pole at $z=1$; near $z=1$, we find that $\widetilde{u}_1(z) \simeq -(z-1)^{-1} + Q$, where

$$Q = \log(2/l) + (k/\alpha)\log[(k-\alpha)/l]. \tag{34}$$

Hence, (32), shows that $\widetilde{f}(z)$ has a double pole at $z=1$, a simple pole at $z=0$, and is otherwise analytic in $-2<\sigma<3$; near $z=1$,

$$\widetilde{f}(z) \simeq \alpha(1+w)^{-1}(w^{-1}+\gamma)(-w^{-1}+Q) \simeq \alpha\{-w^{-2}+w^{-1}(Q-\gamma+1)\},$$

where $w = z-1$, whereas near $z=0$,

$$\widetilde{f}(z) \simeq (\alpha/z)\zeta(0)\widetilde{u}_1(0) = -\tfrac{1}{2}z^{-1}\tan^{-1}(\alpha/l).$$

We can then move the inversion contour to the left of $z=0$, giving

$$f(x) = (\alpha/x)\log x + (\alpha/x)(Q-\gamma+1) - \tfrac{1}{2}\tan^{-1}(\alpha/l) + o(1)$$

as $x\to0$. Replacing x by π/c, and expanding for large h gives

$$\begin{aligned} T_1 &= (\alpha h/\pi)\{-\log h + \log\pi + Q - \gamma + 1\} - \tfrac{1}{2}\tan^{-1}(\alpha/l) \\ &\quad + (\alpha d/\pi)\{\log(h/\pi) - Q - \gamma\} + o(1) \quad \text{as } h\to\infty. \end{aligned} \tag{35}$$

Deep-water behaviour of T_2. As in Example 2, we expect the leading behaviour of the non-separable series T_2 to be given by (29) with $k_n h$ replaced by $\mu_n = (n - \frac{1}{2})\pi$. So, consider

$$T_\infty(1/h) = \sum_{n=1}^{\infty} \left\{ \tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + \mu_n^2/h^2}} \right) - \frac{\alpha h}{n\pi} \right\}. \tag{36}$$

We have

$$T_\infty(x) = \sum_{n=1}^{\infty} \left\{ \tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + \mu_n^2 x^2}} \right) - \frac{\alpha}{\mu_n x} \right\} + \frac{\alpha}{x} \sum_{n=1}^{\infty} \left(\frac{1}{\mu_n} - \frac{1}{n\pi} \right);$$

the second sum is $(2/\pi)\log 2$. Hence, $T_\infty(x) = (2\alpha/(\pi x))\log 2 + f(x)$, where $f(x)$ is the separable series (1), with $c_n = 1$, $\mu_n = (n - \frac{1}{2})\pi$ and $u(x)$ is again given by (31). We obtain

$$\widetilde{f}(z) = \pi^{-z}(2^z - 1)\zeta(z)\widetilde{u}(z) = (\alpha/z)\pi^{-z}(2^z - 1)\zeta(z)\widetilde{u}_1(z),$$

where $\widetilde{u}_1(z)$ is defined by (33). Note that, unlike the function defined by (32), here, $\widetilde{f}(z)$ does not have a pole at $z = 0$. However, it does have a double pole at $z = 1$; near $z = 1$,

$$\widetilde{f}(z) \simeq (\alpha/\pi)\{-(z-1)^{-2} + (z-1)^{-1}(Q - \gamma + 1 + \log\pi - 2\log 2)\}.$$

Hence $T_\infty(x) = (\alpha/\pi)\{x^{-1}\log x + x^{-1}(Q - \gamma + 1 + \log\pi)\} + o(1)$ as $x \to 0$, and so, as $h \to \infty$, we obtain

$$T_\infty(1/h) = -(\alpha/\pi)h\log h + (\alpha/\pi)h(Q - \gamma + 1 + \log\pi) + o(1). \tag{37}$$

We now examine the difference between T_2 and $T_\infty(1/h)$. Let

$$T_4 = T_2 - T_\infty(1/h) = \sum_{n=1}^{\infty} t_n, \qquad \text{where} \tag{38}$$

$$t_n = \tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + k_n^2}} \right) - \tan^{-1}\left(\frac{\alpha}{\sqrt{l^2 + \mu_n^2/h^2}} \right).$$

Clearly, $t_n = o(1)$ as $h \to \infty$, for fixed n, so we have

$$T_4 = \sum_{n=M+1}^{\infty} t_n + o(1) \qquad \text{as } h \to \infty,$$

where M is fixed (cf. (16)). Writing $k_n h = \mu_n + \nu_n$, as in (12), we have $l^2 + k_n^2 \simeq \Delta_n^2 + 2\nu_n\mu_n/h^2$ as $|\nu_n/\mu_n|$ is small, where $\Delta_n^2 = l^2 + \mu_n^2/h^2$. Hence, using the Taylor approximation (18), we find that

$$t_n \simeq \frac{-\alpha\nu_n\mu_n}{h^2\Delta_n(\Delta_n^2 + \alpha^2)}.$$

Finally, we use the approximation (13), $\nu_n \simeq \nu_n^{(1)} = \tan^{-1}(\mu_n/(Kh))$, giving $t_n \simeq -(\alpha/h)t_n^{(1)}(1/h)$, where

$$t_n^{(1)}(y) = \frac{\mu_n y \tan^{-1}(\mu_n y/K)}{\sqrt{l^2+\mu_n^2 y^2}\,(k^2+\mu_n^2 y^2)}, \tag{39}$$

using $k^2 = \alpha^2 + l^2$. So, we have approximated T_4 by a separable series:

$$T_4 = -(\alpha/h)T_M^\infty(1/h) + o(1) \qquad \text{as } h \to \infty, \text{ where} \tag{40}$$

$$T_M^\infty(x) = \sum_{n=M+1}^{\infty} t_n^{(1)}(x)$$

and $t_n^{(1)}(x)$ is defined by (39). As $t_n^{(1)}(x) \sim \frac{1}{2}\pi(\mu_n x)^{-2}$ as $x \to \infty$, we see that $\widetilde{T}_M^\infty(z)$ is analytic in a strip $\beta < \sigma < 2$, for some β, so we can take the inversion contour just to the left of $\sigma = 2$. We have $\widetilde{T}_M^\infty(z) = \pi^{-1}\psi_M(z)\widetilde{u}(z)$, where $\psi_M(z)$ is defined by (23) and

$$\widetilde{u}(z) = \int_0^\infty \frac{y^z \tan^{-1}(y/K)}{\sqrt{y^2+l^2}\,(y^2+k^2)}\,dy$$

is analytic for $-2 < \sigma < 2$. Hence, $\widetilde{T}_M^\infty(z)$ is analytic for $-2 < \sigma < 2$, apart from a simple pole at $z = 1$; using (24), we have $\widetilde{T}_M^\infty(z) \simeq L\pi^{-1}(z-1)^{-1}$ near $z = 1$, where

$$L = \widetilde{u}(1) = \int_0^\infty \frac{y \tan^{-1}(y/K)}{\sqrt{y^2+l^2}\,(y^2+k^2)}\,dy. \tag{41}$$

Hence, as the conditions of Theorem 1 are satisfied, we obtain

$$T_M^\infty(x) = L/(\pi x) + o(1)$$

as $x \to 0$, whence (40) gives $T_4 = -(\alpha/\pi)L + o(1)$ as $h \to \infty$. Finally, we combine this result with (37) and (38) to give

$$T_2 = (\alpha/\pi)\{-h\log h + h(Q - \gamma + 1 + \log\pi) - L\} + o(1) \quad \text{as } h \to \infty. \tag{42}$$

The integral defining L, (41), is not elementary. However, it can be expressed in terms of dilogarithms [3]; we find that

$$\alpha L = \tfrac{1}{4}\pi^2 - \tfrac{1}{2}A\log(k+K) + \tfrac{1}{2}A\log(k-K) - \delta\tan^{-1}(\psi/\alpha) - \mathcal{L} \tag{43}$$

where $\psi = \sqrt{l^2 - K^2}$, $A = -\log((k-\alpha)/l)$, $\delta = \tan^{-1}(\psi/K)$ and

$$\mathcal{L} = \mathrm{Li}_2(\mathrm{e}^{-A}, \delta) - \mathrm{Li}_2(\mathrm{e}^{-A}, \pi + \delta). \tag{44}$$

Here, the dilogarithm is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-w)\,\frac{dw}{w} \tag{45}$$

for complex z, and $\mathrm{Li}_2(r,\theta) = \mathrm{Re}\,\left\{\mathrm{Li}_2(r\mathrm{e}^{\mathrm{i}\theta})\right\}$.

Synthesis. From (35) and (42), we have

$$T_1 - T_2 = (\alpha/\pi)\{L + d(\log h - \log\pi - Q + \gamma)\} - \tfrac{1}{2}\tan^{-1}(\alpha/l) + o(1), \tag{46}$$

as $h \to \infty$, so that the terms involving h and $h\log h$ in (35) and (42) cancel. Moreover, when (46) is substituted into (31), we see that the terms in $\log h$ cancel, leaving only bounded terms as $h \to \infty$. Finally, substituting for Q and L, we obtain

$$E_\infty = \frac{\alpha d}{\pi}\left\{\log\frac{ld}{2\pi} + \gamma - 1\right\} - \frac{kd}{\pi}\log\frac{k-\alpha}{l} - \tan^{-1}\frac{\alpha}{\psi} - \frac{1}{2}\tan^{-1}\frac{l}{\alpha}$$
$$+ T_3 - \frac{1}{\pi}\mathcal{L} + \frac{1}{2\pi}\log\frac{k-\alpha}{l}\,\log\frac{k+K}{k-K} - \frac{1}{\pi}\tan^{-1}\frac{\psi}{K}\,\tan^{-1}\frac{\psi}{\alpha}. \tag{47}$$

This expression for E_∞ bears little resemblance to $E(h)$; indeed, it is perhaps surprising to see terms involving products of logarithms and products of inverse tangents. Nevertheless, the result can be checked by solving the deep-water problem directly. This has been done by Parsons [7], using the Wiener-Hopf technique; the two approaches yield the same result.

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