ON THE NULL-FIELD EQUATIONS FOR WATER-WAVE RADIATION PROBLEMS
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Abstract
Consider a rigid body which is performing simple-harmonic oscillations in the free surface of deep water under gravity. Under certain geometrical conditions on D, the wetted surface of the body, it is known that the linear boundary-value problem \( \mathbf{V} \) for the corresponding velocity potential \( \phi \) is uniquely solvable at all frequencies. The usual method for solving \( \mathbf{V} \) is to derive a Fredholm integral equation of the second kind over \( D \); it is also well-known that the usual boundary integral equations are uniquely solvable, except at an infinite discrete set of frequencies (the irregular frequencies). In this paper, we shall describe an alternative method for solving \( \mathbf{V} \), called the 'null-field' method; this method was originally devised by Waterman for acoustic and electromagnetic scattering problems. We derive an infinite system of non-linear equations (the 'null-field' equations), which are to be solved for the boundary values of \( \phi \). These equations are uniquely solvable at all frequencies the physical irregular frequencies do not occur. For the special case of a half-immersed circular cylinder, we find a simple connection between the null-field method and the method of multipoles. We discuss the numerical solution of the null-field equations and present some results for the heating, half-immersed, elliptic cylinder. Finally, we sketch how the null-field method can be extended to three-dimensional problems and to problems where the water is of constant finite depth.

1. Introduction
Consider a rigid body which is floating in the free surface of a fluid. We suppose that the fluid is incompressible and inviscid, and assume that the effects of surface tension are negligible. We denote the fluid domain by \( D \), the free surface by \( T \) and the wetted surface of the body by \( \Gamma \), which we assume has properties A (John, 1950). Let \( \partial D \) denote the curve of the surface \( \partial T \) and its mirror image in the plane of the free surface. We shall say that \( \partial D \) has properties A if \( \partial T \) is a convex, curve-differentiable surface. (In particular, \( \partial D \) must intersect the free surface perpendicularly.)

Let us assume, for simplicity, that the fluid is of infinite depth and that the body is a horizontal cylinder of infinite length. We take Cartesian coordinates \((x, y, z)\) with the \( z \)-axis parallel to the generators of the cylinder and the \( y \)-axis vertical (\( y \) increasing with depth), such that \( D \) occupies a region of the plane \( \gamma = 0 \).

Suppose that the cylinder performs simple-harmonic oscillations of small amplitude and radian frequency \( \omega \). The motion is assumed to be independent of \( z \) and \( y \), for irrotational motion, we can formulate the following well-known, linear, two-dimensional boundary-value problem for a velocity potential \( \phi(x, \gamma) = \exp(-i\omega t) \).

Boundary-value problem \( \mathbf{V} \)
Determine a function \( \phi \) satisfying Laplace's equation,

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial \gamma^2} = 0 \quad \text{in } D,
\]

the free-surface condition

\[
\frac{\partial \phi}{\partial \gamma} = 0 \quad \text{on } T,
\]

and the boundary condition

\[
\frac{\partial \phi}{\partial \gamma} = \psi(y) \quad \text{on } \partial D,
\]

where \( K = \sqrt{\gamma/g} \) is the acceleration due to gravity, and the function \( \psi(y) \) is prescribed on \( \partial D \) (see Figure 1). In addition, there is the radiation condition that waves travel outwards to infinity, and the condition that the fluid motion vanishes as \( \gamma = \pm \infty \).

The notation is as follows: capital letters \( F, G \) denote points of \( D \); small letters \( p, q \) denote points of \( \partial D \); the origin \( 0 \) is assumed to lie in \( F \); the portion of the water which is inside the cylinder \( D \); denotes the interior of the body, \( \partial D \) denotes the region with boundary \( \partial D \); \( F, G \) denote points of \( \partial D \), \( p, q \) is the length \( \delta \); \( \delta \phi \) denotes normal differentiation at the point \( p \) in the direction from \( p \) into \( D \).

The following theorem on the solvability of \( \phi \) has been proved by John (1950):
Figure 1. The floating cylinder.

Theorem 1. Suppose that \( D \) has properties 1 and 2 and that \( V_k \) is continuous on \( D \). Then, there exists a unique solution of the boundary-value problem \( \mathbf{F} \) for all real values of \( k \).

We shall henceforth assume that the conditions of Theorem 1 are always satisfied.

The usual approach for solving the boundary-value problem \( \mathbf{F} \) is to derive an integral equation of the second kind, over the boundary \( \partial D \). One way of doing this is to assume that \( \phi(\mathbf{r}) \) can be represented as a distribution of sources over \( D \); the source strength is then found to be the solution of a Fredholm integral equation of the second kind.

Alternatively, an integral equation of the second kind can be derived directly from Green's theorem. It is well known that both of these methods (which will be described in section 2) lead to boundary integral equations of the second kind which are singular at a certain infinite discrete set of frequencies, corresponding to the eigenvalues of a related interior problem. This phenomenon is clearly a consequence of the method of solution, for we have already remarked that the original boundary-value problem \( \mathbf{F} \) is known to have a unique solution at all frequencies, provided that \( \partial D \) has properties 1 and 2 (Theorem 1).

A different approach to the related problem in acoustics has been employed by Wettstein (1955). His method is based on solving the Helmholtz formula in the interior of the body and leads to an infinite system of equations, rather than a single (integral) equation. Martino (1950) has studied these equations (called the "null-field equations of acoustics"), and proved that they always have a unique solution, i.e., difficulties at interior eigenvalues do not occur with this method.

In this paper, we shall derive the corresponding equations for water-wave radiation problems. After a review of integral-equation methods in section 2, we derive the null-field equations for water waves in section 3. In section 4, we consider the special case of an oscillating half-submerged circular cylinder. For this geometry, we show that the null-field equations may be obtained by suitably modifying Wettstein's method of multipoles. In section 5, we discuss the numerical solution of the null-field equations. We propose a simple numerical scheme which we use, in section 6, to solve the equations corresponding to a heaving, half-submerged, elliptic cylinder. We obtain good agreement with other published results (for the virtual-mass coefficient). Finally, we briefly describe how the null-field method may be extended to solve problems in three dimensions and problems where the fluid is of constant finite depth.

2. Boundary Integral Equations

Let \( G_2(\mathbf{r}, \mathbf{r}') \) be the potential at \( \mathbf{r} \) due to a single water-source at \( \mathbf{r}' \), i.e., (Theorems, 1953)

\[
G_2(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{1}{(\mathbf{r} - \mathbf{r}')^2} - \frac{2\pi}{k} \frac{1}{(\mathbf{r} - \mathbf{r}')^2} \cos k \cdot \mathbf{r} - k \cdot \mathbf{r}'
\]

where, in order to satisfy the radiation condition, the path of integration passes below the pole of the integrand at \( k = k_0 \). \( G_2(\mathbf{r}, \mathbf{r}') \) also satisfies Laplace's equation in \( D \) (except at \( \mathbf{r} = \mathbf{r}' \), where it has a logarithmic singularity) and the free-surface condition (1.1).

If we apply Green's theorem to \( \partial D \), we obtain the following three equations, depending on the location of \( \mathbf{r} \):

\[
2\pi \delta(\mathbf{r}) = \int_D \left[ \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n} \right] \mathbf{f}(\mathbf{r}') d\mathbf{r}'
\]

(2.1)

\[
\mathbf{v}(\mathbf{r}) = \int_D \left[ \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n} \right] \mathbf{v}(\mathbf{r}') d\mathbf{r}'
\]

(2.2)

\[
\phi(\mathbf{r}) = \int_D \left( \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n} \right) \phi(\mathbf{r}') d\mathbf{r}'
\]

(2.3)

(here, the radiation condition on \( \mathbf{r} \) and \( \phi \) ensures that there is no contribution from infinity.) These equations are called the Helmholtz formulae in acoustics (see, e.g., Baker and Giesing, 1950); we shall use the same terminology here.

If we use the boundary condition (1.2) in (2.1), we obtain

\[
\mathbf{v}(\mathbf{r}) = \int_D \left[ \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n} \right] \mathbf{v}(\mathbf{r}') d\mathbf{r}'
\]

(2.5)

which is a Fredholm integral equation of the second kind for the unknown boundary values of \( \mathbf{v}_b \). This integral equation (Green's integral equation) possesses a unique solution unless the corresponding homogeneous integral equation,

\[
\mathbf{v}_b(\mathbf{r}) = \int_D \left[ \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n} \right] \mathbf{v}_b(\mathbf{r}') d\mathbf{r}' = 0
\]

(2.6)

has a non-trivial solution. It was shown by John (1950) that (2.6) does have non-trivial solutions whenever \( k \) coincides with an eigenvalue of the 'interior water-bubblelet problem', where the Dirichlet condition \( \phi = 0 \) is satisfied on \( \Gamma \) and the free-surface condition (1.1) is satisfied on \( \Gamma_w \). At such values of \( k \) (called spectral by John), Green's integral equation (2.5) does not have a unique solution for general \( \mathbf{v}(\mathbf{r}) \).

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A different approach for solving \( \mathbf{V} \) is to represent it by a distribution of simple wave-sources over \( D \),

\[
\Omega = \frac{1}{\mu} \int \delta (x - x_0) \mu (x) \, d\mathbf{x}.
\]  

(2.7)

On applying the boundary condition (1.2), we find that the unknown source strength \( \mu (x) \) satisfies

\[
\nu (x) = \frac{1}{\mu} \int \frac{1}{4} \psi (y) \delta (x - y) \, d\mathbf{y} = \nu (y),
\]  

(2.9)

This integral equation (the source integral equation) is of the same form as (2.5), except that the kernel of (2.9) is the transpose of the kernel appearing in (2.5). (Here, we have used the symmetry of (2.1).) Hence, (2.9) has the same irregular values as (2.5).

If \( K \) is not an irregular value, we can construct the solution of (2.9) by substituting the unique solution of (2.9) into (2.7). For the representation (2.9) satisfies the radiation condition and the free-surface condition (for any continuous \( \phi (x) \)), whilst it automatically satisfies the boundary condition (1.2) on \( D \) (if \( \nu (x) \) satisfies (2.9)).

The situation is not so straightforward with Green's integral equation (2.5). Nevertheless, if we substitute the unique solution of (2.9) into (2.7) and then use the boundary condition (1.2), we can define a function \( \Omega (x) \), say, by

\[
\Omega (x) = \frac{1}{\mu} \int \frac{1}{4} \psi (y) \delta (x - y) \, d\mathbf{y} = \Omega (x),
\]  

(2.9)

which does solve \( \mathbf{V} \). (A proof is given by Martin (1981); it is necessary to show that \( \psi \) satisfies the boundary condition on \( D \).

When \( K \) is an irregular value, the integral equations (2.5) and (2.9) are not uniquely solvable for general \( \Omega (x) \) (actually, \( \Omega (x) \) has none). This difficulty was first overcome by John (1959), who was able to prove that \( \Omega \) is uniquely solvable at the irregular values by giving an appropriately complex argument involving the non-trivial solutions of the homogeneous forms of (1.2). Another way of overcoming the difficulty at the irregular values is to use a different fundamental solution in place of the wave-sources \( \Omega (x) \); see, e.g., Ursell (1953), 1954, and Fay (1960).

Numerical solutions of the integral equations (2.5) and (2.9) have been obtained by many authors, for various \( D \) and \( \mu (x) \). It is known that the discretized versions of these integral equations become ill-conditioned within a narrow band of frequencies around each irregular frequency. Although several computational devices have been used to alleviate this difficulty, it is not pertinent to discuss all these here. (For a recent review, see Nel (1970); for a comparison between several methods, see Between (1980).)

Let us now examine (2.1), the third of the Helmholtz formulas. This is an integral relation which asserts that the potential induced in \( D \) by the sources on \( D \) is exactly cancelled by the potential induced by the dipoles on \( D \). In other words, although the combination of the actual potential in \( D \) (i.e., the solution of the boundary-value problem \( \mathbf{V} \)), across \( D \), does not vanish in \( b \) (otherwise, it would vanish everywhere), the potential generated by the source and dipole distributions over \( D \) (which are used to represent the actual potential in \( D \)) does not vanish throughout \( D \). Waterman (1960) calls this the "extended boundary condition," and (2.4) the "extended integral equation." According to Nel (1970), p. 102, (2.4) has not been used for wave-wave calculations. In acoustics, however, Ramsden (1949) has replaced the integral relation (2.4) by an infinite system of equations, called the multi-field equations. More recently, Javors (1980) has obtained the corresponding equations for wave-wave problems. These will be derived in the next section.

2. The Multi-Field Equations for Wave Waves

Recently, Ursell (1981) has shown that the simple wave-source (2.1) has a bilinear expansion, i.e.,

\[
\psi (x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n}(x) \phi_{m,n}(y),
\]  

(3.1)

where \( \phi_{m,n} \) and \( c_{m,n} \) are regular and satisfy the free-surface condition (2.1), whilst \( \phi_{m,n} \) satisfy the free-surface and radiation conditions, and are singular at \( 0 \); \( \phi_{m,n} \) are usually known as "multipoles" potentials; \( \phi_{10} \) and \( \phi_{20} \) correspond to a source and a horizontal dipole, respectively, whilst \( \phi_{11} \), \( \phi_{12} \), etc. are wave-free potentials.

Let \( C \) be the inscribed circle to \( \partial D \), which is centered at \( 0 \). Similarly, let \( C' \) be the circumscribed circle to \( \partial D \), and let \( C' \) be the semi-circular region which is bounded by \( C' \) and the lower half of \( C \). (See Figure D.) When both \( C \) and \( C' \) are such that \( \partial D \), and the lower half of \( C \), we may substitute the bilinear expansion (3.1) into the third of the Helmholtz formulas, (2.1), to give

\[\text{Figure D. The inscribed and circumscribed semi-circles.}\]
where we have used the boundary condition (1.12), (2.2) holds for all \( P \) in \( \mathbb{R}^3 \). Since the functions \( \Phi_i(\mathbf{r}) \) are regular solutions of Laplace's equation in \( \mathbb{R}^3 \), it follows that each term in (3.2) must vanish, so we obtain the following set of equations:

\[
\begin{align*}
C_i(r) &= \frac{\partial^2 \Phi_i(r)}{\partial r^2} + \frac{\partial \Phi_i(r)}{\partial r} + \frac{\partial^2 \Phi_i(r)}{\partial \theta^2} + \frac{\partial \Phi_i(r)}{\partial \theta} + \frac{\partial^2 \Phi_i(r)}{\partial \phi^2} = 0, \\
\text{(3.3)}
\end{align*}
\]

for \( i = 1, 2 \) and \( m = 0, 1, \ldots \). We call these the null-field equations for water waves. Equations of this type were first obtained by Korteweg (1980), for electromagnetic scattering problems. Since then, the null-field method (also called the "B' matrix method") has been used to solve many such problems, as well as problems in acoustics and elastodynamics; for a collection of papers on these topics, see the conference proceedings edited by Varadan and Varadan (1986). However, the null-field equations for water waves appear to be new.

Having obtained the null-field equations, we would like to know whether they are solvable. The following theorem has been proved by Martin (1991):

**Theorem 2.** Suppose that 2D has properties J and that \( \Phi(r) \) is continuous on \( 2D \). Then, the null-field equations for water waves, (3.3), possess a unique solution for all values of \( \Phi \).

The proof of this theorem involves showing that \( \Phi \) satisfies (3.3) if and only if \( \Phi \) satisfies a Fredholm integral equation of the second kind, which is itself known to possess a unique solution. Such an integral equation has been obtained by Gressell (1981). We replace \( \Phi \) by a set of fundamental solutions, \( \Phi_i \), defined by

\[
\Phi_i(r) = C_i(r) + \int_{\mathbb{R}^3} \frac{\Phi(r')}{r} G_i(r, r') \, dr',
\]

where \( C_i \) are constants. Applying Green's theorem in \( 2D \), to \( \Phi(r) \) and \( \Phi_i(r) \) (as in section 2), then gives

\[
\Phi(r) = \int_{\mathbb{R}^3} \frac{\Phi(r')}{r} \, dr' = \int_{\mathbb{R}^3} \frac{\Phi(r')}{r} \, dr' - \int_{\mathbb{R}^3} \frac{\Phi_i(r')}{r} \, dr',
\]

where \( \Phi_i \) is another null-field solution of the second kind for \( \Phi(r) \). Gressell (1981) has proved

\[
\text{(3.4)}
\]

**Theorem 3.** Let \( \Phi_i \) have properties J and let \( \Phi_i \) be constants \( \Phi_i \) in (3.3) be chosen such that \( \text{Im}(\Phi_i) > 0 \) for \( i = 1, 2 \) and \( m = 0, 1, \ldots \). Then, the integral equation (3.4) is uniquely solvable for any given value of \( \Phi \), provided that \( \Phi \) is sufficiently large.

We now know that the null-field equations are uniquely solvable, but what is the solution of the original boundary-value problem? It can be shown that if \( \Phi(r) \) satisfies the null-field equations (3.4), then the solution of \( \Phi \) is given by

\[
\Phi(r) = \int_{\text{S}} \frac{C_i(r')}{R(r, r')} \, dS - \int_{\text{S}} \frac{C_i(r')}{R(r, r')} \, dS,
\]

where the coefficients \( C_i \) are given by

\[
C_i(r) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\Phi_i(r')}{r} \, dr',
\]

for \( i = 1, 2 \) and \( m = 0, 1, \ldots \).

(3.5) implies that, exterior to \( \text{S} \), \( \Phi(r) \) has an expansion in terms of a set of functions \( \Phi_i(r) \) of which \( \Phi_i(r) \) satisfies Laplace's equation in \( \mathbb{R}^3 \), the free-surface condition and the radiation condition. Equations of this type were first used by Ursell (1959) to solve the problem of the bending of a semi-infinite circular cylinder (see section 4). Later (1960), he proved that the set \( \Phi_i(r) \) is complete. Therefore, we can assume that (3.4) holds exterior to \( \text{S} \) and proceed to give an alternative derivation of the null-field equations: Apply Green's theorem to \( \Phi(r) \) and \( \Phi_i(r) \) in the region bounded by \( P \), \( S \) and \( F \), where \( S \) is a large semi-circle, of radius \( R \), enclosing \( P \) and centred on \( O \). There is no contribution from integrating over \( P \), since \( \Phi \) and \( \Phi_i \) both satisfy (1.1). We can now show that the contribution from integrating over \( S \) varies as \( R^{-1} \), by using (3.4), and then using asymptotic properties of \( \Phi_i(r) \) to prove that

\[
\lim_{R \to \infty} \left( \frac{\Phi(r)}{R} \right) = 0.
\]

(3.7)

Thus, we see that the null-field equations do not depend, essentially, on the bilateral expansion of the wave-source (3.1), or on the interior integral relation (2.4), but on the asymptotic properties which satisfy the free-surface and radiation conditions, as (3.8).
6. The Half-Immersed Circular Cylinder

Consider the special case of a half-immersed circular cylinder, with wetted surface \( C \), floating in the free surface of deep water. We define circular polar coordinates \((r, \theta)\) such that points on \( C \) have coordinates \((a, \theta)\), with \(-\pi \leq \theta \leq \pi\). The symmetric boundary-value problem \( P^* \) corresponding to vertical oscillations of the cylinder was first solved by Orrin (1930a). We shall now briefly describe his method (the "method of multipoles").

For the particular geometry considered here, \( C \) coincides with \( C \). Thus, we can represent \( \psi \) as an infinite series of multiple potentials, throughout the entire fluid domain \( D \), i.e. we can write

\[
\psi(r, \theta) = \sum_{n=0}^{\infty} \psi_n(r, \theta) \quad \text{for} \quad r \geq a, \quad -\pi \leq \theta \leq \pi,
\]

where \( \psi_n \) is the potential due to a simple wave source at \( C \). These are symmetric wave-free potentials; see the appendix.

(L.1) satisfies Laplace's equation in \( D \), the free-surface condition (L.1) and the radiation condition. (L.1) also satisfies the boundary condition (L.2) on \( C \). If the coefficients \( c_n \) can be chosen such that

\[
\int_V \psi_n(r_0, \theta_0) \rho(r, \theta) \sin \theta \, dV = 0 \quad \text{for} \quad n \neq 0,
\]

the boundary condition is satisfied. However, the coefficients must be chosen such that the potentials \( \psi_n \) are orthogonal to each other.

To find the unknown coefficients \( c_n \), we suggest two methods. In one of these, we multiply (L.2) by the complete set \( \cos \alpha \sin \theta \), \( \alpha = 0, 1, \ldots \), and integrate over \( C \) to get

\[
\int_C \psi(r, \theta) \cos \alpha \sin \theta \, ds = 0 \quad \text{for} \quad \alpha = 0, 1, \ldots.
\]

This is an infinite system of linear algebraic equations for \( c_n \). These equations may be obtained by numerically solving a truncated system of equations.

Instead of multiplying (L.2) by \( \cos \alpha \sin \theta \), let us multiply by the complete set \( \psi_n(r, \theta) \), \( \alpha = 0, 1, \ldots \), and integrate over \( C \). We obtain

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad \text{for} \quad n \neq 0, \ldots
\]

We now apply Green's theorem to \( \psi \) and \( \psi_n \) in the region bounded by \( C \), \( F \) and \( S \), where \( S \) is a large semi-circle of radius \( R \). There is no contribution from integrating over \( F \) and the contribution from integrating over \( S \) also vanishes as \( R \to \infty \). Hence, (L.3) becomes

\[
\int_C \frac{1}{4\pi} \left( \frac{1}{|r - r'|} + \frac{1}{|r - r'|} \right) \psi(r, \theta) \psi_n(r', \theta) \, ds = 0 \quad (L.4)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.5)
\]

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad (L.6)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.7)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.8)
\]

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad (L.9)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.10)
\]

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad (L.11)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.12)
\]

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad (L.13)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.14)
\]

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad (L.15)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.16)
\]

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad (L.17)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.18)
\]

\[
\int_C \psi(r, \theta) \psi_n(r, \theta) \, ds = 0 \quad (L.19)
\]

\[
\int_0^{2\pi} \psi(r, \theta) \psi_n(r, \theta) \, d\theta = 0 \quad (L.20)
\]
\( n_m \) is the Frobenius delta, for then (5.3) yields \( n = n_m \) and (5,2) becomes

\[
\sum_{n} \beta_n \mathcal{F}_n = 0.
\]

However, we do not know a priori which functions \( \mathcal{F}_n \) satisfy the orthogonality relation, (5.3), for a given boundary \( \beta \); for each value of \( n \), the determination of \( \mathcal{F}_n \) is equivalent to solving the null-field equations (with \( \mathcal{F}_n \) replaced by \( \beta_n \)). Computationally, it is not worth while to determine \( \mathcal{F}_n \) (see Bates and Wall [1937, p. 57] for some considerations of this point, in acoustics). However, it may be possible to choose \( \beta_n \) so that (5.3) is nearly satisfied, i.e., so that the infinite system of equations (5.3) is diagonally dominant. Assuming that such a choice can be made, we truncate the infinite system and obtain

\[
\sum_{n} \beta_n \mathcal{F}_n = 0, \quad \mathcal{F}_n = 0, 1, \ldots, N. \tag{5.4}
\]

This is a system of \( N + 1 \) simultaneous, linear, algebraic equaions for the \( N + 1 \) coefficients \( \beta_n \), \( n = 0, 1, \ldots, N \). Note that we have now made too approximations: we have only used the first \( N + 1 \) of the null-field equations and we have only used \( N + 1 \) of the complete set \( \mathcal{F}_n \) to represent \( \mathcal{F} \) on \( S \).

To obtain a practical numerical method, we must now choose the set of functions \( \beta_n \). Simple choices, such as trigonometric functions or orthogonal polynomials, will often be satisfactory, but sometimes, other choices (perhaps functions which depend on \( X \)) will be more appropriate. Also, from computational experience gained by solving the analogous exterior problem of acoustics, we expect that the system (5.4) will be ill-conditioned for elongated bodies. Several methods for alleviating this difficulty have been devised (see, e.g., Naim, 1930), but, even in acoustics, no satisfactory algorithm exists for choosing \( \beta_n \). Nevertheless, if we have made a reasonable choice for \( \beta_n \), we can solve (5.3) and obtain a good approximation to \( \mathcal{F} \); in the next section, we shall describe the use of this procedure to solve a particular problem \( \mathcal{P} \).

5. The Half-Illuminated Elliptic Cylinder

We consider the vertical oscillations of a half-illuminated elliptic cylinder. The curve-pousing boundary-value problem \( \mathcal{P} \) has been treated by several authors. For example, Foster (1940) (cf. Truesdell, 1948b) has used conformal mapping and the method of multipole, while McK (1953) has solved the source integral equations (6.3).

Let an arbitrary point \( \xi \) on the wetted surface of the cylinder have coordinates

\[
x = x_0, \quad y = \rho \cos \theta, \quad -\pi \leq \theta \leq \pi,
\]

where \( x_0 \) and \( \rho \) are the base and height, respectively, of the cylinder. In circular polar coordinates \((r, \theta)\), we have

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad -\pi \leq \theta \leq \pi,
\]

where \( r = \rho \cos \theta + \rho \sin \theta \), \( \rho = \rho_0 \), and \( \theta = \theta_0 \). Since the motion is symmetric about \( \theta = 0 \), we only require the even \((s = 1)\) null-field equations, and only need to integrate over half of \( 2\theta \). Moreover, we have \( \mathcal{F}(s, \theta) = \mathcal{F} \cos \theta \), where the cylinder is oscillating with vertical velocity \( \mathcal{F}(s, \theta) \).

Thus, the null-field equations become

\[
\int_0^{\pi} \beta(s, \theta) \mathcal{F}(s, \theta) \cos \theta \, \mathrm{d} \theta = 0, \quad n = 0, 1, 2, \ldots \tag{6.1}
\]

where the constants \( \beta_n \) are given by

\[
\beta_n = \int_0^{\pi} \mathcal{F}(s, \theta) \cos \theta \, \mathrm{d} \theta. \tag{6.2}
\]

The source potential, \( \Phi_0 \), may be evaluated using the expression given by Yu and Truesdell (1955), namely

\[
\Phi_0(x, y) = \frac{1}{2} \left[ \frac{1}{2} \left( \log \rho - \frac{1}{2} \right) \right] \rho \cos \theta,
\]

where \( \gamma = 0.5772 \ldots \) is Euler's constant. The wave-front potentials, \( \phi_n \), are real and are given by

\[
\phi_n(x, y) = \cos \left( \frac{\pi n x}{2 a} \right) \cos \left( \frac{\pi n y}{2 b} \right).
\]

The normal derivative of \( \phi_n \) at \( \theta = \pi \), may be evaluated using

\[
\frac{\partial \phi_n}{\partial \theta} = \sin \left( \frac{\pi n x}{2 a} \right) \cos \left( \frac{\pi n y}{2 b} \right) \sin \left( \frac{\pi n y}{2 b} \right) \cos \left( \frac{\pi n x}{2 a} \right),
\]

where \( \theta = \pi \), and \( \tan \theta = \tan \theta_0 \).

To solve the null-field equations, we use the method outlined in section 5. Write

\[
\mathcal{F}(n, \theta) = \sum_{n} \beta_n \mathcal{F}_n(n, \theta),
\]

and substitute into the first \( N + 1 \) of (6.1), and write the system (5,1), with \( \mathcal{F}_n(n, \theta) \) given by (6.2) and \( \beta_n \) given by

\[
\beta_n = \int_0^{\pi} \mathcal{F}_n(n, \theta) \cos \theta \, \mathrm{d} \theta.
\]

\( \beta_n \) and \( \mathcal{F}_n \) may be evaluated numerically using any suitable quadrature formula (6.6).
integrands are non-singular). In our numerical work, we tried $f_n(x) = \cos \varphi_n$ and $g_n(x) = \sin \varphi_n$ in a Chebyshev polynomial expansion. Although other choices could have been made, we found the Chebyshev polynomials to be quite satisfactory for our problem.

In Table 1, we give values of the virtual-mass coefficient for various values of $K_0$ and $\pi$, where

$$\text{virtual-mass coefficient} = \frac{\pi^2}{8} \left( 1 + 2 \sum_{n=1}^{\infty} \frac{\cos \varphi_n}{2^n} \right)$$

(Here, the virtual mass has been normalized by the mass of the field displaced by a half-inverted circular cylinder of radius $a$.)

The results shown were obtained using Chebyshev polynomials and $N = 7$. Comparing our numerical results with the graphical results of Letter (1940, Fig. 13) and Linn (1965, Fig. 15), we see that the agreement is good (Linn's results must be multiplied by $\pi / 2$). We believe that these are correct to 5 significant figures.

As expected, our simple numerical scheme for solving the null-field equations does not converge for very thin ellipses ($\pi < 0.4$ and $K > 3$, approximately), even though the complete set of equations is guaranteed to have a unique solution. It is hoped that this difficulty can be overcome by using a more sophisticated numerical technique; see, e.g., Varadan and Taranin (1955), where various techniques for solving the null-field equations of anisotropy are described. However, when the equations are not too ill-conditioned, our simple scheme is very efficient; the machine-time required is rather less than that required by integral-equation methods. In addition, the null-field method does not exhibit irregular frequencies that is important, computationally, because the location of these frequencies in the spectrum is unknown a priori for an arbitrary curve $30$. Thus, the null-field method may be computationally attractive, especially for problems which are fully three-dimensional (see Section 7).

### Table 1

<table>
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<tr>
<th>$K_0$</th>
<th>$0.5$</th>
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<th>$1.5$</th>
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</tr>
<tr>
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<td>0.6443</td>
<td>0.6496</td>
<td>0.7493</td>
<td>0.8458</td>
</tr>
</tbody>
</table>

The second set is obtained by multiplying the first by $\pi / 2$.

### Conclusion

The best known method for treating water-wave radiation problems is to solve an integral equation of the second kind over the (mean) wetted surface. However, it is well-known that the usual boundary integral equations are not uniquely solvable at the irregular values of $K$. In this paper, we have described an alternative method, which is to solve the infinite system of null-field equations. These equations (which appear to be new in the context of wave-motion problems) always have a unique solution - the mathematical irregular values do not occur. Moreover, this solution may be used to solve the original boundary-value problem $\Psi$.

For simplicity, we have only presented the null-field equations for two-dimensional motions in deep water. However, the method may be extended to three-dimensional motions and to water of constant finite depth. For the two-dimensional, finite-depth case, we simply replace $\pi^2$ by the corresponding multiple potentials; see Thome (1953) or Triell (1951). The three-dimensional case is similar and is treated by Taranin (1952); the corresponding multiple potentials were also obtained by Thome (1953), and the bilinear expansion of the point wave-source, corresponding to (2.3), has been given by Linn (1965). In each case, it can be shown that the infinite set of null-field equations has precisely one solution, for all values of $K$.

In Section 3, we examined several aspects of the numerical solution of the null-field equations. We described a simple exact method for reducing the null-field equations to an infinite system of linear algebraic equations. Transposing this system leads to a practical method for solving the null-field equations. In Section 5, we described a successful application of this method to the two-dimensional problem involving the vertical oscillation of a half-inverted, elliptic cylinder.

It is clear that much work remains to be done on the development of more sophisticated techniques for solving the null-field equations. Nevertheless, the null-field method does provide an alternative approach which, it is hoped, will be as computationally useful as it is already known to be in other branches of mathematical physics.

### Appendix

The following theorem has been proved by Triell (1951):

**Theorem.** When $\rho < \rho_1$, the two-dimensional source potential, defined by (2.1), can be expanded as

$$\phi_0(\rho, \theta) = \sum_{n=1}^{\infty} \frac{1}{n \cos \phi_n} \frac{\rho_1}{\rho} \sum_{m=-\infty}^{\infty} \frac{e^{i \beta m \theta}}{r^2}$$

where

<table>
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<th>$\rho$</th>
<th>$0.5$</th>
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<th>$1.5$</th>
<th>$2.0$</th>
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<td>0.6496</td>
<td>0.7493</td>
<td>0.8458</td>
</tr>
</tbody>
</table>
\[ \psi_n^{(2)} = \sqrt{\frac{2y}{2}} \cos \frac{m \pi y}{2a} \quad \text{and} \quad \psi_n^{(1)} = \frac{1}{2} \sin \frac{m \pi y}{2a}, \]

\[ \psi_n^{(2)} = \cos \frac{m \pi y}{2a} \quad \text{and} \quad \psi_n^{(1)} = \frac{1}{2} \sin \frac{m \pi y}{2a}, \]

\[ \psi_n^{(2)} = -e^{-i \omega t} \cos \frac{m \pi y}{2a} \quad \text{and} \quad \psi_n^{(1)} = \frac{1}{2} e^{-i \omega t} \sin \frac{m \pi y}{2a}, \]

\[ \psi_n^{(2)} = (\frac{m \pi}{2a})^{\frac{1}{2}} \cos \frac{m \pi y}{2a} \quad \text{and} \quad \psi_n^{(1)} = \frac{1}{2} (\frac{m \pi}{2a})^{\frac{1}{2}} \sin \frac{m \pi y}{2a}, \]

where \( m = 1, 2, \ldots \), the point \( \mathbf{P} (x, y) \) has circular polar coordinates given by \( x = r \cos \theta \), \( y = r \sin \theta \) (with \( r = 1 \)), and \( \psi \) is a solution to the Laplace equation.

References


