ON THE NULL-FIELD EQUATIONS FOR WATER-WAVE RADIATION PROBLEMS

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Abstract

Consider a rigid body which is performing simple-harmonic oscillations in the free surface of deep water under gravity. Under certain geometrical conditions on ∂D, the wetted surface of the body, it is known that the linear boundary-value problem $\mathcal{P}$ for the corresponding velocity potential ϕ is uniquely solvable at all frequencies. The usual method for solving $\mathcal{P}$ is to derive a Fredholm integral equation of the second kind over ∂D. It is also well-known that the usual boundary integral equations are uniquely solvable, except at an infinite discrete set of frequencies (the irregular frequencies). In this paper, we shall describe an alternative method for solving $\mathcal{P}$, called the 'null-field' method; this method was originally devised by Waterman for acoustic and electromagnetic scattering problems. We derive an infinite system of moment-like equations (the 'null-field' equations), which are to be solved for the boundary values of ϕ. These equations are uniquely solvable at all frequencies - the unphysical irregular frequencies do not occur. For the special case of a half-immersed circular cylinder, we find a simple connection between the null-field method and the method of multipoles. We discuss the numerical solution of the null-field equations and present some results for the heaving, half-immersed, elliptic cylinder. Finally, we sketch how the null-field method can be extended to three-dimensional problems and to problems where the water is of constant finite depth.

1. Introduction

Consider a rigid body which is floating in the free surface of a fluid. We suppose that the fluid is incompressible and inviscid, and assume that the effects of surface tension are negligible. We denote the fluid domain by D, the free surface by F and the wetted surface of the body by ∂D, which we assume has

Properties J (John, 1950). Let ∂D^* denote the union of the surface ∂D and its mirror image in the plane of the free surface. We shall say that ∂D has properties J if ∂D^* is a convex, twice-differentiable surface. (In particular, ∂D must intersect the free surface perpendicularly.)

Let us assume, for simplicity, that the fluid is of infinite depth and that the body is a horizontal cylinder of infinte length. We take Cartesian coordinates (x, y, z) with the z-axis parallel to the generators of the cylinder and the y-axis vertical (y increasing with depth), such that F occupies a region of the plane $y = 0$.

Suppose that the cylinder performs simple-harmonic oscillations of small amplitude and radian frequency ω. The motion is assumed to be independent of z and so, for irrotational motion, we can formulate the following well-known, linear, two-dimensional boundary-value problem for a velocity potential $\mathrm{Re}\{\phi(x, y)\exp(-i\omega t)\}$:

Boundary-value problem $\mathcal{P}$

Determine a function $\phi(P)$ satisfying Laplace's equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \phi(P) = 0 \quad \text{in } D,$$

the free-surface condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on } F \tag{1.1}$$

and the boundary condition

$$\frac{\partial\phi(p)}{\partial n_p} = V(p) \quad \text{on } \partial D, \tag{1.2}$$

where $K = \omega^2/g$, g is the acceleration due to gravity, and the function V(p) is prescribed on ∂D (see Figure 1). In addition, there is the radiation condition that waves travel outwards to infinity, and the condition that the fluid motion vanishes as $y \to \infty$.

The notation is as follows: capital letters P, Q denote points of D; small letters p, q denote points of ∂D; the origin O is assumed to lie in F_-, the portion of the x-axis which is inside the cylinder; D_- denotes the interior of the body, i.e. the region with boundary $\partial D \cup F_-$; P_-, Q_- denote points of D_-; r_P is the length OP; $\partial/\partial n_p$ denotes normal differentiation at the point p, in the direction from ∂D into D.

The following theorem on the solvability of $\mathcal{P}$ has been proved by John (1950):

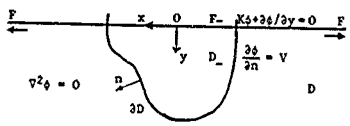

Figure 1. The floating cylinder.

Theorem 1. Suppose that ∂D has properties J and that V(q) is continuous on ∂D. Then, there exists a unique solution of the boundary-value problem $\mathcal{P}$, for all real values of K.

We shall henceforth assume that the conditions of Theorem 1 are always satisfied.

The usual approach for solving the boundary-value problem $\mathcal{P}$ is to derive an integral equation of the second kind, over the boundary ∂D. One way of doing this is to assume that $\phi(P)$ can be represented as a distribution of sources over ∂D; the source strength is then found to be the solution of a Fredholm integral equation of the second kind. Alternatively, an integral equation of the second kind can be derived directly from Green's theorem. It is well known that both of these methods (which will be described in section 2) lead to boundary integral equations of the second kind which are singular at a certain infinite discrete set of frequencies, corresponding to the eigenvalues of a related interior problem. This phenomenon is clearly a consequence of the method of solution, for we have already remarked that the original boundary-value problem $\mathcal{P}$ is known to have a unique solution at all frequencies, provided that ∂D has properties J and V(q) is continuous (Theorem 1).

A different approach to the related problems in acoustics has been employed by Waterman (1969). His method is based on solving the Helmholtz formula in the interior of the body and leads to an infinite system of equations, rather than a single (integral) equation. Martin (1980) has studied these equations (called the 'null-field equations of acoustics'), and proved that they always have a unique solution, i.e. difficulties at interior eigenvalues do not occur with this method.

In this paper, we shall derive the corresponding equations for water-wave radiation problems. After a review of integral-equation methods in section 2, we derive the null-field equations for water waves in section 3. In section 4, we consider the special case of an oscillating half-immersed circular cylinder. For this geometry, we show that the null-field equations may be obtained by suitably modifying Ursell's method of multipoles. In section 5, we discuss the numerical solution of the null-field equations. We propose a simple numerical scheme which we use, in section 6, to solve the equations corresponding to a heaving, half-immersed, elliptic cylinder. We obtain good agreement with other published results (for the virtual-mass coefficient). Finally, we briefly describe how the null-field method may be extended to solve problems in three dimensions and problems where the fluid is of constant finite depth.

2. Boundary Integral Equations

Let $G_0(P, Q)$ be the potential at P due to a simple wave-source at Q, i.e. (Thorne, 1953)

$$G_0(P,Q) \equiv G_0(x,y;\xi,\eta) = \tag{2.1}$$
$$= \frac{1}{2}\log\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2} - 2\int_0^\infty e^{-k(y+\eta)}\cos k(x-\xi)\frac{dk}{k-K},$$

where, in order to satisfy the radiation condition, the path of integration passes below the pole of the integrand at $k = K$. $G_0(P, Q)$ also satisfies Laplace's equation in D (except at $P = Q$, where it has a logarithmic singularity) and the free-surface condition (1.1).

If we apply Green's theorem in D, to $\phi(P)$ and $G_0(P, Q)$, we obtain the following three equations, depending on the location of P:

$$2\pi\phi(P) = \int_{\partial D}\{G_0(P,q)\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\frac{\partial}{\partial n_q}G_0(P,q)\}ds_q, \tag{2.2}$$

$$\pi\phi(p) = \int_{\partial D}\{G_0(p,q)\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\frac{\partial}{\partial n_q}G_0(p,q)\}ds_q, \tag{2.3}$$

$$0 = \int_{\partial D}\{G_0(P_-,q)\frac{\partial}{\partial n_q}\phi(q) - \phi(q)\frac{\partial}{\partial n_q}G_0(P_-,q)\}ds_q. \tag{2.4}$$

(Here, the radiation condition on ϕ and G_0 ensures that there is no contribution from infinity.) These equations are called the Helmholtz formulae in acoustics (see, e.g. Baker and Copson, 1950); we shall use the same terminology here.

If we use the boundary condition (1.2) in (2.3), we obtain

$$\pi\phi(p) + \int_{\partial D}\phi(q)\frac{\partial}{\partial n_q}G_0(p,q)ds_q = \int_{\partial D}G_0(p,q)V(q)ds_q, \tag{2.5}$$

which is a Fredholm integral equation of the second kind for the unknown boundary values of ϕ. This integral equation (Green's integral equation) possesses a unique solution unless the corresponding homogeneous integral equation,

$$\pi\phi(p) + \int_{\partial D}\phi(q)\frac{\partial}{\partial n_q}G_0(p,q)ds_q = 0, \tag{2.6}$$

has a non-trivial solution. It was shown by John (1950) that (2.6) does have non-trivial solutions whenever K coincides with an eigenvalue of the 'interior wave-Dirichlet problem', where the Dirichlet condition $\phi = 0$ is satisfied on ∂D and the free-surface condition (1.1) is satisfied on F_-. At such values of K (called irregular by John), Green's integral equation (2.5) does not have a unique solution for general V(p).

A different approach for solving $\mathcal{P}$ is to represent $\phi(P)$ by a distribution of simple wave-sources over ∂D,

$$\phi(P) = \int_{\partial D} \mu(q) G_0(P,q) ds_q. \qquad (2.7)$$

On applying the boundary condition (1.2), we find that the unknown source strength $\mu(q)$ satisfies

$$\pi\mu(p) + \int_{\partial D} \mu(q) \frac{\partial}{\partial n_p} G_0(p,q) ds_q = V(p). \qquad (2.8)$$

This integral equation (the source integral equation) is of the same form as (2.5), except that the kernel of (2.8) is the transpose of the kernel appearing in (2.5). (Here, we have used the symmetry of (2.1).) Hence, (2.8) has the same irregular values as (2.5).

When K is not an irregular value, we can construct the solution of $\mathcal{P}$ by substituting the unique solution of (2.8) into (2.7). For the representation (2.7) satisfies Laplace's equation in D, the radiation condition and the free-surface condition (for any continuous $\mu(q)$), whilst it automatically satisfies the boundary condition (1.2) on ∂D if $\mu(q)$ satisfies (2.8). The situation is not so straight-forward with Green's integral equation (2.5). Nevertheless, if we substitute the unique solution of (2.5) into (2.2), and then use the boundary condition (1.2), we can define a function $U(P)$, say, by

$$U(P) = \frac{1}{2\pi} \int_{\partial D} \{G_0(P,q)V(q) - \phi(q) \frac{\partial}{\partial n_q} G_0(P,q)\} ds_q$$

which does solve $\mathcal{P}$. (A proof is given by Martin (1981); it is necessary to show that U satisfies the boundary condition on ∂D.)

When K is an irregular value, the integral equations (2.5) and (2.8) are not uniquely solvable for general $V(p)$ (actually, (2.5) has more than one solution whilst (2.8) has none). This difficulty was first overcome by John (1950). He was able to prove that $\mathcal{P}$ is uniquely solvable at the irregular values by giving a rather complicated argument involving the non-trivial solutions of the homogeneous form of equation (2.8). Another way of overcoming the difficulty at the irregular values is to use a different fundamental solution in place of the simple wave-source $G_0(P, Q)$, see, e.g., Ursell (1953, 1981) and Sayer (1980).

Numerical solutions of the integral equations (2.5) and (2.8) have been obtained by many authors, for various ∂D and $V(q)$. It is known that the discretised versions of these integral equations become ill-conditioned within a narrow band of frequencies around each irregular frequency. Although several computational devices have been used to alleviate this difficulty, it is not pertinent to describe them all here; for a recent review, see Mei (1978); for a comparison between several methods, see Børresen (1980).

Let us now examine (2.4), the third of the Helmholtz formulae. This is an integral relation which asserts that the potential induced in D_- by the sources on ∂D is exactly cancelled by the potential induced by the dipoles on ∂D. In other words, although the continuation of the actual potential in D (i.e. the solution of the boundary-value problem $\mathcal{P}$), across ∂D, does not vanish in D_- (otherwise, it would vanish everywhere), the potential generated by the source and dipole distributions over ∂D (which are used to represent the actual potential in D) does vanish throughout D_-. Waterman (1969) calls this the 'extended boundary condition', and (2.4) the 'extended integral equation'. According to Mei (1978, p.402), (2.4) has not been used for water-wave calculations. In acoustics, however, Waterman (1969) has replaced the interior integral relation (2.4) by an infinite system of equations, called the null-field equations. More recently, Martin (1981) has obtained the corresponding equations for water-wave problems. These will be derived in the next section.

3. The Null-Field Equations for Water Waves

Recently, Ursell (1981) has shown that the simple wave-source (2.1) has a bilinear expansion, i.e.

$$G_0(P,Q) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^\sigma(P) \phi_m^\sigma(Q). \qquad (3.1)$$

for $r_Q > r_P$, where the harmonic functions α_m^σ and ϕ_m^σ are defined in an appendix below; α_m^σ are regular and satisfy the free-surface condition (1.1), whilst ϕ_m^σ satisfy the free-surface and radiations conditions, and are singular at O (ϕ_m^σ are usually known as 'multipole' potentials: ϕ_0^1 and ϕ_0^2 correspond to a source and a horizontal dipole, at O, respectively, whilst ϕ_m^σ, for $m > 0$, are wave-free potentials).

Let C_- be the inscribed circle to ∂D^*, which is centred on O. Similarly, let C_+ be the escribed circle to ∂D^*. Let D_N be the semi-circular region which is bounded by F_- and the lower half of C_- (see Figure 2). When P_- lies inside D_N (where $r_p < r_q$), we may substitute the bilinear expansion (3.1) into the third of the Helmholtz formulae, (2.4), to give

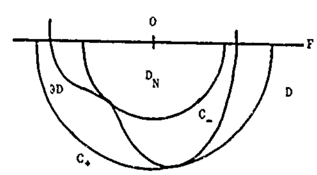

Figure 2. The inscribed and escribed semi-circles.

$$\sum_{m=0}^{\infty}\sum_{\sigma=1}^{2}\alpha_m^\sigma(P_-)\int_{\partial D}\{\phi(q)\frac{\partial}{\partial n_q}\phi_m^\sigma(q) - V(q)\phi_m^\sigma(q)\}ds_q = 0, \tag{3.2}$$

where we have used the boundary condition (1.2). (3.2) holds for all P_- in D_N. Since the functions $\alpha_m^\sigma(P_-)$ are regular solutions of Laplace's equation in D_N, it follows that each term in (3.2) must vanish, and so we obtain the following set of equations:

$$\int_{\partial D}\{\phi(q)\frac{\partial}{\partial n_q}\phi_m^\sigma(q) - V(q)\phi_m^\sigma(q)\}ds_q = 0, \quad \text{(N.F)}$$

$\sigma = 1, 2$ and $m = 0, 1, \ldots$. We call these the <u>null-field equations for water waves.</u> Equations of this type were first obtained by Waterman (1965), for electromagnetic scattering problems. Since then, the null-field method (also called the 'T-matrix' method) has been used to solve many such problems, as well as problems in acoustics and elastodynamics; for a collection of papers on these topics, see the conference proceedings edited by Varadan and Varadan (1980). However, the null-field equations for water waves appear to be new.

Having obtained the null-field equations, we would like to know whether they are solvable. The following theorem has been proved by Martin (1981):

<u>Theorem 2</u>. Suppose that ∂D has properties J and that $V(q)$ is continuous on ∂D. Then, the null-field equations for water waves, (N.F), possess a unique solution for all values of K.

The proof of this theorem involves showing that ϕ satisfies (N.F) if and only if ϕ satisfies a Fredholm integral equation of the second kind, which is itself known to possess a unique solution. Such an integral equation has been obtained by Ursell (1981). He replaced G_0 by a different fundamental solution, G_1, defined by

$$G_1(P,Q) = G_0(P,Q) + \sum_{m=0}^{N}\sum_{\sigma=1}^{2} a_m^\sigma\phi_m^\sigma(P)\phi_m^\sigma(Q), \tag{3.3}$$

where a_m^σ are constants. Applying Green's theorem in D, to $\phi(P)$ and $G_1(P,Q)$ (as in section 2), then gives

$$\pi\phi(p) + \int_{\partial D}\phi(q)\frac{\partial}{\partial n_q}G_1(p,q)ds_q = \int_{\partial D}G_1(p,q)V(q)ds_q. \tag{3.4}$$

This is another Fredholm integral equation of the second kind for $\phi(q)$. Ursell (1981) has proved

<u>Theorem 3</u>. Let ∂D have properties J and let the constants a_m^σ in (3.3) be chosen such that $\mathrm{Im}(a_m^\sigma) > 0$ for $\sigma = 1, 2$ and $m = 0, 1, \ldots N$. Then, the integral equation (3.4) is uniquely solvable at any given value of K, provided that N is sufficiently large.

We now know that the null-field equations are uniquely solvable, but what is the solution of the original boundary-value problem? It can be shown that if $\phi(q)$ satisfies the null-field equations (or, equivalently, the integral equation (3.4)), then the solution of P is given by

$$\phi(P) = \frac{1}{2\pi}\int_{\partial D}\{G_1(P,q)V(q) - \phi(q)\frac{\partial}{\partial n_q}G_1(P,q)\}ds_q. \tag{3.5}$$

Suppose, now, that P lies outside C_+ (see Figure 2). Then, we can again use the bilinear expansion of G_0, (3.1), in (3.5) (together with (N.F) and (3.3)) to obtain

$$\phi(P) = \sum_{m=0}^{\infty}\sum_{\sigma=1}^{2} c_m^\sigma\phi_m^\sigma(P), \tag{3.6}$$

where the coefficients c_m^σ are given by

$$c_m^\sigma = \frac{1}{2\pi}\int_{\partial D}\{\alpha_m^\sigma(q)V(q) - \phi(q)\frac{\partial}{\partial n_q}\alpha_m^\sigma(q)\}ds_q,$$

$$\sigma = 1, 2;\ m = 0, 1, \ldots .$$

(3.6) implies that, exterior to C_+, $\phi(P)$ has an expansion in terms of a set of functions $\{\phi_m^\sigma\}$, each of which satisfies Laplace's equation in D, the free-surface condition and the radiation condition. Expansions of this type were first used by Ursell (1949) to solve the problem of the heaving, half-immersed circular cylinder (see section 4). Later (1950), he proved that the set $\{\phi_m^\sigma\}$ is complete. Therefore, we can <u>assume</u> that (3.6) holds exterior to C_+ and then <u>proceed</u> to give an alternative derivation of the null-field equations: Apply Green's theorem to $\phi(P)$ and $\phi_m^\sigma(P)$ in the region bounded by ∂D, F and S, where S is a large semi-circle, of radius r, enclosing ∂D and centred on O. There is no contribution from integrating over F, since ϕ and ϕ_m^σ both satisfy (1.1). We can show that the contribution from integrating over S vanishes as $r \to \infty$, by using (3.6), and then using asymptotic properties of ϕ_m^σ to prove that

$$\lim_{r\to\infty}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\{\phi_m^\sigma(r,\theta)\frac{\partial}{\partial r}\phi_n^\nu(r,\theta) - \phi_n^\nu(r,\theta)\frac{\partial}{\partial r}\phi_m^\sigma(r,\theta)\}r\,d\theta = 0. \tag{3.7}$$

(Here, we have assigned plane polar coordinates (r, θ), $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$, to points on S.) If we now choose appropriate values for m and σ, we obtain the complete set of null-field equations, (N.F).

Thus, we see that the null-field equations do not depend, essentially, on the bilinear expansion of the wave-source, (3.1), or on the interior integral relation (2.4), but on the expansion of potentials which satisfy the free-surface and radiation conditions, as (3.6).

4. The Half-Immersed Circular Cylinder

Consider the special case of a half-immersed circular cylinder, with wetted surface C, floating in the free surface of deep water. We define circular polar coordinates (r,θ) such that points on C have coordinates (a,θ), with $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$. The symmetric boundary-value problem $\mathcal{P}$ corresponding to vertical oscillations of the cylinder was first solved by Ursell (1949a). We shall now briefly describe his method (the 'method of multipoles').

For the particular geometry considered here, C_* coincides with C. Thus, we can represent ϕ as an infinite series of multipole potentials, throughout the entire fluid domain D, i.e. we can write

$$\phi(r,\theta) = \sum_{n=0}^{\infty} c_n \phi_n^1(r,\theta) \text{ for } r \ge a, \ -\tfrac{1}{2}\pi \le \theta \le \tfrac{1}{2}\pi, \quad (4.1)$$

where ϕ_0^1 is the potential due to a simple wave-source at O and ϕ_n^1, $n > 0$, are symmetric wave-free potentials; see the appendix.

(4.1) satisfies Laplace's equation in D, the free-surface condition (1.1) and the radiation condition. (4.1) also satisfies the boundary condition (1.2) on C if the coefficients c_n can be chosen such that

$$V(a,\theta) = U_0\cos\theta = \sum_{n=0}^{\infty} c_n \left\langle \frac{\partial}{\partial r}\phi_n^1(r,\theta)\right\rangle, \ 0 \le \theta \le \tfrac{1}{2}\pi, \quad (4.2)$$

where U_0 is a constant and angular brackets indicate that r is to be put equal to a.

To find the unknown coefficients c_n, Ursell suggested two methods. In one of these, he multiplied (4.2) by the complete set $\{\cos 2m\theta\}$, $m = 0,1,\ldots$, and integrated over C to give

$$\int_0^{\frac{1}{2}\pi} V(a,\theta)\cos 2m\theta \, a\,d\theta = \sum_{n=0}^{\infty} c_n \int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial}{\partial r}\phi_n^1(r,\theta)\right\rangle \cos 2m\theta\, d\theta, \quad m = 0,1,\ldots .$$

This is an infinite system of linear algebraic equations for c_n; approximate values for c_n may be obtained by numerically solving a truncated system of equations.

Instead of multiplying (4.2) by $\{\cos 2m\theta\}$, let us multiply by the complete set $\{\phi_m^1(a,\theta)\}$, $m = 0,1,\ldots$, and integrate over C. We obtain

$$\int_0^{\frac{1}{2}\pi} V(a,\theta)\phi_m^1(a,\theta) a\,d\theta = \sum_{n=0}^{\infty} c_n \int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial}{\partial r}\phi_n^1(r,\theta)\right\rangle \phi_m^1(a,\theta)\,d\theta. \quad (4.3)$$

We now apply Green's theorem to ϕ_n^1 and ϕ_m^1 in the region bounded by C, F and S_∞, where S_∞ is a large semi-circle of radius r_∞. There is no contribution from integrating over F and the contribution from integrating over S_∞ also vanishes as $r_\infty \to \infty$, by (3.7). Hence, (4.3) becomes

$$\int_0^{\frac{1}{2}\pi} V(a,\theta)\phi_m^1(a,\theta)a\,d\theta = \sum_{n=0}^{\infty} c_n \int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial}{\partial r}\phi_m^1(r,\theta)\right\rangle \phi_n^1(a,\theta)\,d\theta \quad (4.4)$$

$$= \int_0^{\frac{1}{2}\pi} \phi(a,\theta)\left\langle a\frac{\partial}{\partial r}\phi_m^1(r,\theta)\right\rangle d\theta,$$

$m = 0,1,\ldots$, by (4.1). We see that (4.4) are precisely the null-field equations for the symmetric oscillations of a half-immersed circular cylinder. This may be compared with the corresponding exterior problem of acoustics: for an oscillating circular cylinder, the null-field equations simply yield the Fourier components of the well-known exact solution; for all other geometries, the null-field equations of acoustics must be solved numerically. For water-wave problems, the null-field equations must always be solved numerically; this will be discussed in the next section.

5. Numerical Solution of the null-field equations

The null-field equations may be written as

$$\int_{\partial D} \phi(q)\frac{\partial}{\partial n_q}\phi_m^\sigma(q)\,ds_q = V_m^\sigma, \ \sigma = 1,2; \ m = 0,1,\ldots \quad (5.1)$$

where $\phi(q)$ is to be determined and the known constants V_m^σ are given by

$$V_m^\sigma = \int_{\partial D} V(q)\phi_m^\sigma(q)\,ds_q .$$

Before considering how to solve (5.1), we remark that the null-field equations are <u>not</u> integral equations; they form an infinite set of moment-like equations.

Let $\{\psi_n(q)\}$ be a set of functions which is complete over ∂D. Write

$$\phi(q) = \sum_{n=0}^{\infty} a_n\psi_n(q) \quad (5.2)$$

where the coefficients a_n are to be determined. Substituting (5.2) into (5.1) gives

$$\sum_{n=0}^{\infty} K_{mn}a_n = V_m, \quad m = 0,1,\ldots, \quad (5.3)$$

where

$$K_{mn} = \int_{\partial D} \psi_n(q)\frac{\partial}{\partial n_q}\phi_m(q)\,ds_q \quad (5.4)$$

and we have suppressed the dependence on σ. Clearly, the best choice for $\{\psi_n\}$ would be $\{\Psi_n\}$, where

$$\int_{\partial D} \Psi_n(q)\frac{\partial}{\partial n_q}\phi_m(q)\,ds_q = \delta_{mn}, \ m,n = 0,1,\ldots \quad (5.5)$$

(δ_{mn} is the Kronecker delta), for then (5.3) yields $a_m = V_m$ and (5.2) becomes

$$\phi(q) = \sum_{n=0}^{\infty} V_n \Psi_n(q).$$

However, we do not know a priori which functions $\{\Psi_n\}$ satisfy the orthogonality relation, (5.5), for a given boundary ∂D; for each value of n, the determination of Ψ_n is equivalent to solving the null-field equations (with V_m replaced by δ_{mn}). Computationally, it is not worth while to determine $\{\Psi_n\}$ (see Bates and Wall (1977, p.57) for some considerations of this point, in acoustics). However, it may be possible to choose $\{\psi_n\}$ so that (5.5) is nearly satisfied, i.e. so that the infinite system of equations (5.3) is diagonally-dominant. Assuming that such a choice can be made, we truncate the infinite system and obtain

$$\sum_{n=0}^{N} K_{mn} a_n = V_m, \quad m = 0,1,\ldots,N. \qquad (5.6)$$

This is a system of $N+1$ simultaneous, linear, algebraic equations for the $N+1$ coefficients a_n, $n = 0,1,\ldots,N$. Note that we have now made two approximations: we have only used the first $N+1$ of the null-field equations and we have only used $N+1$ of the complete set $\{\psi_n\}$ to represent ϕ on ∂D.

To obtain a practical numerical method, we must now choose the set of functions $\{\psi_n\}$. Simple choices, such as trigonometric functions or orthogonal polynomials, will often be satisfactory, but sometimes, other choices (perhaps functions which depend on K) will be more appropriate. Also, from computational experience gained by solving the analogous exterior problem of acoustics, we expect that the system (5.6) will be ill-conditioned for elongated bodies. Several methods for alleviating this difficulty have been devised (see, e.g., Wall, 1980), but, even in acoustics, no satisfactory algorithm exists for choosing $\{\psi_n\}$. Nevertheless, if we have made a reasonable choice for $\{\psi_n\}$, we can solve (5.6) and obtain a good approximation to $\phi(q)$; in the next section, we shall describe the use of this procedure to solve a particular problem $\mathcal{P}$.

6. The Half-Immersed Elliptic Cylinder

We consider the vertical oscillations of a half-immersed elliptic cylinder. The corresponding boundary-value problem $\mathcal{P}$ has been treated by several authors. For example, Porter (1960) (cf. Ursell, 1949b) has used conformal mapping and the method of multipoles, whilst Kim (1965) has solved the source integral equation (2.8).

Let an arbitrary point $q \equiv (x, y)$ on the wetted surface of the cylinder have coordinates

$$x = a\sin\eta, \quad y = b\cos\eta, \quad -\tfrac{1}{2}\pi \le \eta \le \tfrac{1}{2}\pi,$$

where $2a$ and b are the beam and draught, respectively, of the cylinder. In circular polar coordinates (r,θ), we have

$$x = r\sin\theta, \quad y = r\cos\theta, \quad -\tfrac{1}{2}\pi \le \theta \le \tfrac{1}{2}\pi,$$

where

$$r = b(\cos^2\eta + H^2\sin^2\eta)^{\frac{1}{2}}, \quad \tan\theta = H\tan\eta$$

and $H = a/b$. Since the motion is symmetric about $\theta = 0$ $(\eta = 0)$, we only require the even $(\sigma = 1)$ null-field equations, and only need to integrate over half of ∂D. Moreover, we have $V(q)ds_q = U_0 a\cos\eta\, d\eta$, where the cylinder is oscillating with vertical velocity $\mathrm{Re}\{U_0 e^{-i\omega t}\}$. Thus, the null-field equations become

$$\int_0^{\frac{1}{2}\pi} \phi(\eta)\{b\frac{\partial}{\partial n}\phi_m^1(q)\}(\sin^2\eta + H^2\cos^2\eta)^{\frac{1}{2}} d\eta = U_0 a V_m,$$
$$m = 0,1,\ldots \qquad (6.1)$$

where the constants V_m are given by

$$V_m = \int_0^{\frac{1}{2}\pi} \phi_m^1(r,\theta)\cos\eta\, d\eta. \qquad (6.2)$$

The source potential, ϕ_0^1, may be evaluated using the expansion given by Yu and Ursell (1961), namely

$$\phi_0^1(r,\theta) = -\{(\log Kr - i\pi + \gamma)\cos(Kr\sin\theta)$$
$$+ \theta\sin(Kr\sin\theta)\}\exp(-Kr\cos\theta)$$
$$+ \sum_{m=1}^{\infty} \frac{(-Kr)^m}{m!}\{\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{m}\}\cos m\theta,$$

where $\gamma = 0.5772\ldots$ is Euler's constant. The wave-free potentials, ϕ_m^1, are real and are given by

$$\phi_m^1(r,\theta) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{(2m-1)}\frac{\cos(2m-1)\theta}{r^{2m-1}},$$
$$m = 1,2,\ldots .$$

The normal derivative of ϕ_m^1, $m = 0,1,\ldots$, may be evaluated using

$$\frac{\partial}{\partial n}\phi(r,\theta) = \cos(\alpha-\theta)\frac{\partial\phi}{\partial r} + \sin(\alpha-\theta)\frac{1}{r}\frac{\partial\phi}{\partial\theta},$$

where $H\tan\alpha = \tan\eta$.

To solve the null-field equations, we use the method outlined in section 5: write

$$\phi(\eta) = U_0 a \sum_{n=0}^{N} a_n \psi_n(\eta)$$

and substitute into the first $N+1$ of (6.1). We obtain the system (5.11), with V_m given by (6.2) and K_{mn} given by

$$K_{mn} = \int_0^{\frac{1}{2}\pi} \psi_n(\eta)\{b\frac{\partial}{\partial n}\phi_m^1(r,\theta)\}(\sin^2\eta + H^2\cos^2\eta)^{\frac{1}{2}} d\eta.$$

V_m and K_{mn} may be evaluated numerically using any suitable quadrature formula (the integrands are non-singular). In our numerical work, we tried $\psi_n(\eta) = \cos 2n\eta$ and $\psi_n(\eta) = T_{2n}(2\eta/\pi)$ ($T_n(x)$ is a Chebyshev polynomial: $T_n(\cos\theta) = \cos n\theta$); although other choices could have been made, we found the Chebyshev polynomials to be quite satisfactory for our problem.

In Table 1, we give values of the virtual-mass coefficient for various values of Ka and H, where

$$\text{virtual-mass coefficient} = -\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \mathrm{Re}\left\{\frac{\phi(\eta)}{U_0 a}\right\}\cos\eta \, d\eta.$$

(Here, the virtual mass has been normalised by the mass of the fluid displaced by a half-immersed circular cylinder of radius a.) All the results shown were obtained using Chebyshev polynomials and $N \leq 7$. Comparing our numerical values with the graphical results of Porter (1960, Fig.13) and Kim (1965, Fig.15), we see that the agreement is good (Kim's results must be multiplied by $2/\pi$). We believe that these are correct to 4 significant figures.

As expected our simple numerical scheme for solving the null-field equations does not converge for very thin ellipses ($H < 0.4$ and $H > 3$, approximately), even though the complete set of equations is guaranteed to have a unique solution. It is hoped that this difficulty can be overcome by using a more sophisticated numerical technique; see, e.g., Varadan and Varadan (1980), where various techniques for solving the null-field equations of acoustics are described. However, when the equations are not too ill-conditioned, our simple scheme is very efficient; the machine-time required is rather less than that required by integral-equation methods. In addition, the null-field method does not exhibit irregular frequencies; this is important, computationally, because the location of these frequencies in the spectrum is unknown a priori for an arbitrary curve ∂D. Thus, the null-field method may be computationally attractive, especially for problems which are fully three-dimensional (see section 7).

H	Wavenumber, Ka: 0.5	1.0	1.5	2.0
0.6	0.54584	0.64426	0.74904	0.81580
0.7	0.56600	0.62149	0.71830	0.78765
0.8	0.59115	0.60915	0.69457	0.76333
0.9	0.61795	0.60440	0.67711	0.74303
1.0	0.64463	0.60498	0.66493	0.72658
1.1	0.67029	0.60917	0.65702	0.71362
1.2	0.69453	0.61575	0.65248	0.70371
1.3	0.71718	0.62384	0.65056	0.69637
1.4	0.73824	0.63282	0.65063	0.69119
1.5	0.75777	0.64227	0.65219	0.68778
1.6	0.77586	0.65190	0.65486	0.68580
1.7	0.79262	0.66152	0.65834	0.68499
1.8	0.80816	0.67098	0.66239	0.68509
1.9	0.82257	0.68020	0.66683	0.68593
2.0	0.[illegible]598	0.68912	0.67153	0.68734

Table 1. Virtual-mass coefficient for heaving elliptic cylinder, for various Ka and H, where $H = a/b$ = half-beam/draught.

7. Conclusion

The best known method for treating water-wave radiation problems is to solve an integral equation of the second kind over the (mean) wetted surface. However, it is also well-known that the usual boundary integral equations are not uniquely solvable at the irregular values of K. In this paper, we have described an alternative method, which is to solve the infinite system of null-field equations. These equations (which appear to be new in the context of water-wave problems) always have a unique solution - the unphysical irregular values do not occur. Moreover, this solution may be used to solve the original boundary-value problem $\mathcal{P}$.

For simplicity, we have only presented the null-field equations for two-dimensional motions in deep water. However, the method may be extended to three-dimensional motions and to water of constant finite depth. For the two-dimensional, finite-depth case, we simply replace ϕ_m^σ by the corresponding multipole potentials; see Thorne (1953) or Ursell (1981). The three-dimensional case is similar and is treated by Martin (1981); the corresponding multipole potentials were also obtained by Thorne (1953), and the bilinear expansion of the point wave-source, corresponding to (3.1), has been given by Martin (1981). In each case, it can be shown that the infinite set of null-field equations has precisely one solution, for all values of K.

In section 5, we examined several aspects of the numerical solution of the null-field equations. We described a simple exact method for reducing the null-field equations to an infinite system of linear algebraic equations. Truncating this system leads to a practical method for solving the null-field equations. In section 6, we described a successful application of this method to the two-dimensional problem $\mathcal{P}$ corresponding to the vertical oscillations of a half-immersed, elliptic cylinder.

It is clear that much work remains to be done on the development of more sophisticated techniques for solving the null-field equations. Nevertheless, the null-field method does provide an alternative approach which, it is hoped, will be as computationally useful as it is already known to be in other branches of mathematical physics.

Appendix

The following theorem has been proved by Ursell (1981):

Theorem. When $r_P < r_Q$, the two-dimensional source potential, defined by (2.1) can be expanded as

$$G_0(P,Q) = \sum_{m=0}^{\infty} \sum_{\sigma=1}^{2} \alpha_m^\sigma(P)\phi_m^\sigma(Q),$$

where

$$\phi_0^1(P) = \oint_0^\infty e^{-ky}\cos kx \frac{dk}{k-K}, \quad \phi_0^2(P) = -\frac{\partial}{\partial x}\phi_0^1(P),$$

$$\phi_m^1(P) = \frac{\cos 2m\theta}{r^{2m}} + \frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}},$$

$$\phi_m^2(P) = \frac{\sin(2m+1)\theta}{r^{2m+1}} + \frac{K}{2m}\frac{\sin 2m\theta}{r^{2m}},$$

$$\alpha_0^1(P) = -2e^{-Ky}\cos Kx, \quad \alpha_0^2(P) = \frac{-2}{K}e^{-Ky}\sin Kx,$$

$$\alpha_m^1(P) = \frac{-2(2m-1)!}{K^{2m}}\sum_{q=2m}^{\infty}\frac{(-Kr)^q}{q!}\cos q\theta,$$

$$\alpha_m^2(P) = \frac{2(2m)!}{K^{2m+1}}\sum_{q=2m+1}^{\infty}\frac{(-Kr)^q}{q!}\sin q\theta,$$

$m = 1,2,\ldots,$ and the point $P \equiv (x, y)$ has circular polar coordinates given by $x = r\sin\theta$, $y = r\cos\theta$ (with $r = r_p$).

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