Scattering by Inclusions*

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Abstract

Time-harmonic waves are scattered by obstacles, through which waves can also travel. Various examples from acoustics, elastodynamics and electromagnetics are discussed, including imperfect interfaces, chiral materials and wood. The paper gives a very subjective overview but with emphasis on the contributions and influence of Ralph Kleinman.

1 Introduction

The title refers to a basic class of problems in scattering theory: how does a time-harmonic wave interact with a bounded obstacle when the obstacle itself can support waves in its interior? This is an example of a *transmission problem.* Such examples arise in many different physical contexts, and some of these will be discussed below.

This article gives a selective review. It is subjective, not comprehensive! The following physical problems are discussed:

- acoustic scattering by a fluid inclusion or by a solid inclusion;
- scattering of elastic waves by an elastic inclusion, with the possibility of imperfect interfaces;
- electromagnetic problems, with achiral or chiral inclusions; and
- wave motion in wooden poles containing a rotten core.

The main theme, of course, is Ralph Kleinman's contributions and influence, beginning with his well-known review paper with Gary Roach.

The article is based on a lecture that I gave at the Ralph Kleinman Memorial Meeting, and, as Prof. Senior remarked, giving the lecture was a bitter-sweet experience. Ralph Kleinman was more than a fine mathematician, he was a fine man. It was a great pleasure to work with him and to have known him: we all miss his presence, contributions, laughter and company.

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2 Kleinman and Roach (1974)

This paper was published in *SIAM Review* [10]. It is concerned with various boundary-value problems for the Helmholtz equation in three dimensions. Thus, let S_{in} denote a bounded, simply-connected domain with a smooth boundary S and unbounded exterior S_{ex}. (The notation S_{in} and S_{ex} was often used by Ralph in lectures, leading to various jokes about S being the thin demarcation line! At the Memorial Meeting, it transpired that S_{in} and S_{ex} is actually *Butler's notation.*) The problem is to solve

$$(\nabla^2 + k^2)\, u = 0 \qquad \text{in } S_{\text{ex}},$$

together with a boundary condition on S, either the Dirichlet condition, $u = f$ on S, or the Neumann condition, $\partial u/\partial n = g$ on S, and the Sommerfeld radiation condition at infinity. Here, f and g are given functions on S, k^2 is real and positive, and $\partial/\partial n$ denotes normal differentiation on S.

Kleinman and Roach [10] give a systematic study of methods for solving these exterior problems, based on boundary integral equations using the free-space Green's function

$$G(P, Q; k) = -\mathrm{e}^{\mathrm{i}kR}/(2\pi R),$$

where $R = |P - Q|$ is the distance between the two points, P and Q. Interior problems are also considered, as are connections between the various integral equations, and the problem of irregular frequencies.

I have begun with this paper because it is how I began to know Ralph's (and Gary's) work. I first met Ralph, in Manchester, when he came to visit Fritz Ursell, about 20 years ago. At that time, I was Fritz's post-doc, working on the same problems as those in [10], but using the so-called *null-field equations* [12] (which are themselves related to Waterman's T-matrix method).

Why was the paper [10] influential? Two reasons come to mind. First, boundary integral equations and boundary element methods were beginning to be used by engineers to solve practical acoustic-scattering problems; see, for example, the slim proceedings of a 1975 ASME conference, edited by Cruse and Rizzo [5]. Thus, there was a need for a careful (and accessible) derivation of well-founded integral-equation methods. Second, the paper [10] gives the Big Picture: it gives a connected overview of the field. Today, the paper's influence has waned, probably because it has been subsumed by the book of Colton and Kress [4].

3 The fluid–fluid problem

Ralph encouraged me to visit Delaware; I spent a sabbatical year there, from August 1986. I arrived there with a particular interest in two topics: the use of one integral equation to solve acoustic transmission problems; and the prevalence of hypersingular operators in the treatment of certain scattering problems (especially those involving cracks).

The acoustic transmission (or 'fluid–fluid') problem models the scattering of sound in a compressible fluid by a blob of another fluid. One has to find fields u_e and u_i, where

$$(\nabla^2 + k_e^2)\, u_e = 0 \quad \text{in } S_{\text{ex}}, \quad (\nabla^2 + k_i^2)\, u_i = 0 \quad \text{in } S_{\text{in}}$$

and u_e satisfies the radiation condition. In addition, there are *transmission* (or *interface*) *conditions*,

$$u = u_i \quad \text{and} \quad \partial u/\partial n = \rho\, \partial u_i/\partial n \quad \text{on } S,$$

where $u = u_e + u_{\text{inc}}$ is the total field in S_{ex}, u_{inc} is a given incident field and ρ is a given coupling constant (usually the ratio of the two fluid densities).

We wrote a paper [9], giving a systematic study of methods for solving the fluid–fluid problem using (i) pairs of coupled boundary integral equations over S, and (ii) single integral equations. As an example of (i), we have

$$\left.\begin{aligned} (1+\rho)u + (\overline{K_e^*} - \rho\overline{K_i^*})u - (S_e - S_i)v &= 2u_{\text{inc}} \\ (1+\rho)v + \rho(N_e - N_i)u - (\rho K_e - K_i)v &= 2\rho\, \partial u_{\text{inc}}/\partial n \end{aligned}\right\} \qquad (1)$$

where $v = \partial u/\partial n$. This pair of equations is always uniquely solvable [8]. The operators S_α, $\overline{K_\alpha^*}$, K_α and N_α are the standard boundary integral operators, involving $G(P, Q; k_\alpha)$. For example, the hypersingular operator N_α is defined by

$$(N_\alpha u)(p) = \frac{\partial}{\partial n_p} \int_S u(q)\, \frac{\partial}{\partial n_q}\, G(p, q; k_\alpha)\, ds_q.$$

Note that the system (1) was contrived so that N_e and N_i occur in the combination $(N_e - N_i)$. This is an example of *regularization*: the strong singularities cancel so that $(N_e - N_i)$ is compact on suitable spaces.

Note also that the system (1) and, indeed, the transmission problem itself, behave anomalously when $\rho = -1$. This special case has been discussed by Ola [16].

Single integral equations can be derived by using an ansatz (single-layer potential, say) in one region (S_{ex}, say) and Green's theorem in the other. So, for an example of (ii), write

$$u_e(P) = \int_S \mu(q)\, G(P, q; k_e)\, ds_q, \qquad P \in S_{\text{ex}},$$

and then compute u_e and $\partial u_e/\partial n$ on S in terms of the unknown source density μ. Next, apply Green's theorem in S_{in} to u_i and $G(P, Q; k_i)$, and evaluate the normal derivative on S, giving

$$(I + K_i)(\partial u_i/\partial n) - N_i u_i = 0.$$

Finally, use the transmission conditions to obtain

$$(1 + \rho)\mu + L\mu = h,$$

a Fredholm integral equation of the second kind for μ, where h is known and

$$L = K_i(I + K_e) + K_e(I - \rho K_e) + \rho(N_e - N_i)S_e.$$

4 The solid–solid problem

As a PhD student of Gerry Wickham, I had studied the scattering of elastic waves, so it was natural for me to consider an elastic material exterior to an elastic inclusion. For this solid–solid problem, one can proceed formally as for the fluid–fluid problem, but there are difficulties. One of these is that singular integral operators are typical (even for exterior problems such as scattering by a cavity). For another, consider the elastodynamic analogue of N_α, defined by

$$(N_\alpha \mathbf{f})(p) = \frac{1}{\mu_\alpha} T_\alpha^p \int_S \mathbf{f}(q) \cdot T_\alpha^q \mathbf{G}_\alpha(q,p)\, ds_q,$$

where $\mathbf{G}_\alpha$ is the fundamental Green's tensor (Kupradze matrix) for the elastic material α, μ_α is the shear modulus and T_α^p is the traction operator at p. Then, it turns out that, unlike for acoustics, $(N_e - N_i)$ does *not* give a regularization. In two dimensions (plane strain),

$$(1-\nu_e)N_e - (1-\nu_i)N_i$$

does give a regularization, where ν_α is Poisson's ratio [13]. However, in three dimensions, the situation is much more complicated.

For the solid–solid problem, it is usual to assume that the two solids are welded together across S, so that

$$\mathbf{u} = \mathbf{u}_i \quad \text{and} \quad \mathbf{t} = \mathbf{t}_i \quad \text{on } S, \tag{2}$$

where $\mathbf{u}$ is the displacement and $\mathbf{t} = T\mathbf{u}$ is the traction. However, there is an extensive engineering literature on models of imperfect interfaces, where (2) is replaced by, for example,

$$\mathbf{u} - \mathbf{u}_i = \mathbf{F} \cdot \mathbf{t} \quad \text{and} \quad \mathbf{t} = \mathbf{t}_i \quad \text{on } S.$$

Here, the matrix $\mathbf{F}$ is chosen to model sliding and/or thin interface layers (of glue, perhaps). For a review and systematic study, see [14]. Note that Angell, Kleinman and Hettlich [1] have discussed similar models in acoustics.

5 The fluid–solid problem

In September 1989, Ralph and George Hsiao organised the Workshop on Integral and Field Equation Methods in Fluid Structure Interactions, in Newark. This stimulating meeting motivated my own work on an idealised fluid–solid problem, in which a smooth elastic body is surrounded by an inviscid, compressible fluid [11]. The transmission conditions are

$$\partial p/\partial n = \rho_{\mathrm{f}} \omega^2\, \mathbf{u} \cdot \mathbf{n} \quad \text{and} \quad -p\mathbf{n} = T\mathbf{u} \quad \text{on } S,$$

where p is the acoustic pressure, $\mathbf{u}$ is the elastic displacement, ρ_f is the fluid density, ω is the frequency and T is the traction operator. Thus, there are four scalar transmission conditions connecting four scalar unknowns, namely p in S_ex and $\mathbf{u} = (u_1, u_2, u_3)$ in S_in.

In [11], we studied the solvability of various systems of coupled boundary integral equations for the fluid–solid problem; these have four scalar unknowns. We also derived and analysed various new single integral equations over S, involving a single unknown 3-vector.

The question of uniqueness is interesting. It turns out that the fluid–solid transmission problem may exhibit *Jones frequencies* [7]. At these, there are free vibrations of the solid with $T\mathbf{u} = \mathbf{0}$ *and* $\mathbf{u} \cdot \mathbf{n} = 0$ on S; such vibrations do not couple to the fluid, and so cannot be precluded by the radiation condition. Generically (which means when S is chosen arbitrarily), Jones frequencies do not exist [6]. However, they certainly do exist for special geometries, such as all axisymmetric bodies (torsional vibrations). Note that Jones frequencies are a consequence of our simplified model; they would not occur if the exterior fluid was viscous.

6 Electromagnetic inclusion problems

Ralph was always interested in electromagnetic problems. I cut my teeth in this area by working out the electromagnetic analogue of [9] with Petri Ola [15]: electromagnetic scattering by a homogeneous dielectric obstacle, based on Maxwell's equations,

$$\operatorname{curl}\mathbf{E} - \mathrm{i}k\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl}\mathbf{H} + \mathrm{i}k\mathbf{E} = \mathbf{0},$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field and $k = \omega\sqrt{\mu\varepsilon}$ is a constant. Again, we gave a systematic study of various reformulations of the problem, involving pairs of coupled integral equations or single integral equations.

More recently, I have worked with Christos Athanasiadis and Iannis Stratis from the University of Athens on scattering by *chiral* inclusions. The chiral material is modelled by a modified form of Maxwell's equations, namely

$$\left.\begin{aligned} \operatorname{curl}\mathbf{E} - \mathrm{i}k(\mathbf{H} + \beta\operatorname{curl}\mathbf{H}) &= \mathbf{0} \\ \operatorname{curl}\mathbf{H} + \mathrm{i}k(\mathbf{E} + \beta\operatorname{curl}\mathbf{E}) &= \mathbf{0}, \end{aligned}\right\}$$

where β is the chirality parameter,

The chiral (or 'handed') nature of the material can be displayed by making use of the *Bohren decomposition*,

$$\mathbf{Q}_L = \mathbf{E} + \mathrm{i}\mathbf{H} \quad \text{and} \quad \mathbf{Q}_R = \mathbf{E} - \mathrm{i}\mathbf{H},$$

whence $\operatorname{curl}\mathbf{Q}_L = \gamma_L\mathbf{Q}_L$ and $\operatorname{curl}\mathbf{Q}_R = -\gamma_R\mathbf{Q}_R$, where $\gamma_L = k/(1 - k\beta)$ and $\gamma_R = k/(1 + k\beta)$. Thus, the left-handed component $\mathbf{Q}_L$ and the right-handed component $\mathbf{Q}_R$ propagate at different speeds if $\beta \neq 0$.

We have derived and analysed pairs of coupled integral equations for solving the chiral-inclusion problem. In practice, the chirality is small but significant, so that $0 < k\beta \ll 1$. We have shown that the fields are analytic functions of $k\beta$ for $|k\beta| < 1$, and we have shown how to compute the $O(k\beta)$ correction to the achiral ($\beta = 0$) solution [2].

We have also studied the so-called far-field operator F, which plays a role in certain methods for solving the inverse problem. (Fh is the far-field pattern corresponding to an incident Herglotz field with kernel h.) Thus, in general, we have shown that the eigenvalues of F are precisely the eigenvalues of Waterman's T-matrix [3].

My own work with Greek mathematicians parallels Ralph's own collaborations, especially his work with George Dassios on low-frequency scattering. In fact, I last saw Ralph in Greece, in July 1997, at a meeting to mark the retirement of Gary Roach.

7 Waves in wood

My final topic is the propagation of stress waves through wooden poles. This work arose from a request to understand how ultrasonics could be used to inspect telegraph poles for internal decay. Thus, the inclusion is a region of rotten wood inside a wooden cylinder. We have modelled the wood as an elastic material with cylindrical orthotropy, and then looked for solutions in the form of generalized Frobenius expansions (using Bessel functions rather than powers).

I gave a lecture on this topic at Oberwolfach in September 1998, a lecture on wooden poles in the middle of the Black Forest! The meeting itself was co-organised by Ralph and Rainer Kress. Many participants made comments in their lectures on Ralph and their interactions with him. My own, repeated at the Memorial Meeting, concerned the elusive notion of *taste* in mathematics; all I can say is that Ralph had it!

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