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THIN INTERFACE LAYERS: ADHESIVES, APPROXIMATIONS AND ANALYSIS

P.A. MARTIN

Department of Mathematics, University of Manchester
Manchester M13 9PL, England

Linear models of imperfect interfaces between elastic bodies are studied; both inclusions and laminated structures are considered. After a brief review of the literature, the problem of scattering by an inclusion with an imperfect interface is analysed. Uniqueness theorems are obtained, and boundary integral equations over the interface are derived.

1. INTRODUCTION

There are several branches of mechanics in which two solids interact across a common interface. Examples are adhesive joints [1], frictional contacts [2], laminated structures and composite materials [3, 4]. The simplest situations are those where the two solids are welded together ('perfect interface'), so that the displacement and traction vectors are continuous across the interface. The opposite extreme is when there is no interaction ('complete debonding'). Intermediate situations arise when the two solids can slip or separate, or when there is a thin layer of a different material (such as glue or lubricant) between the solids. In this paper, we are especially interested in those intermediate situations that can be modelled by simple linear modifications to the perfect-interface continuity conditions.

2. IMPERFECT PLANE INTERFACES: A REVIEW

Consider a plane interface $x_3 = z = 0$ between two elastic solids. Let $u_i^\pm$ and $\tau_{ij}^\pm$ be the components of the displacement vector and stress tensor, respectively, in $\pm z > 0$. The traction vectors on $z = 0$ are given by $t_i^\pm = \tau_{i3}^\pm$. If $z = 0$ is a perfect interface, we have

$$[\mathbf{t}] = \mathbf{0} \qquad \text{and} \qquad [\mathbf{u}] = \mathbf{0}, \tag{2.1}$$

where square brackets denote discontinuities across the interface:

$$[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-, \qquad \text{evaluated on } z = 0.$$

The perfect-interface conditions (2.1) were first modified by Newmark in 1943 [5]. He explicitly allows slipping to occur, so that (2.1) are replaced by

$$[\mathbf{t}] = \mathbf{0}, \qquad [u_3] = 0 \qquad \text{and} \qquad [u_\alpha] = M t_\alpha \tag{2.2}$$

where $\alpha = 1$ or 2, and M is a positive constant. Note that $M = 0$ corresponds to a perfect interface, whereas $M = \infty$ corresponds to a 'lubricated interface', i.e. one where there is a thin layer of inviscid fluid between the two solids.

Newmark's theory is for two beams in two dimensions, but it has been generalized to

$$[\mathbf{t}] = \mathbf{0}, \qquad [u_3] = 0 \qquad \text{and} \qquad [u_\alpha] = M_{\alpha\beta} t_\beta \tag{2.3}$$

where summation of β over $\beta = 1$ and $\beta = 2$ is implied, and $M_{\alpha\beta}$ can be a function of t_γ and $[u_\gamma]$. For a review, see [6], where $M_{\alpha\beta}$ is assumed to be a positive diagonal matrix.

Similar boundary conditions have been used by Murty [7] to model the propagation of waves through a 'loosely-bonded interface'. In [8], (2.2) are derived by assuming that there is a thin interface layer of viscous fluid. Now, the parameter M is given by

$$M = ih/(\omega\eta), \tag{2.4}$$

where h is the thickness of the layer, η is the shear viscosity coefficient of the fluid, and a harmonic time-dependence of $e^{-i\omega t}$ is implied.

It is perhaps worth noting that the problem of wave propagation through a layer of viscous fluid (or of viscoelastic solid) between two elastic solids can be analysed exactly. However, the extraction of *approximate* conditions, connecting the displacements and tractions across a *thin* layer, is not usually the aim of such analyses [9].

Jones and Whittier [10] have modelled wave propagation through a 'flexibly-bonded interface' by allowing both slip and separation. They replace (2.1) with

$$[\mathbf{t}] = \mathbf{0}, \qquad [u_3] = M_n t_3 \qquad \text{and} \qquad [u_\alpha] = M_s t_\alpha, \tag{2.5}$$

where M_n and M_s are constants. Several authors have used (2.5) [11–16]. In [15, 16], the formulae

$$M_n = h/(\lambda + \mu) \qquad \text{and} \qquad M_s = h/\mu, \tag{2.6}$$

are given, where λ and μ are the Lamé moduli of a thin elastic layer modelling the bond.

3. SCATTERING BY AN INCLUSION

Let B_i denote a bounded domain, with a smooth closed boundary S and simply-connected exterior, B_e. We seek displacements $\mathbf{u}_e(P)$ and $\mathbf{u}_i(P)$ so that

$$L_e \mathbf{u}_e(P) = \mathbf{0}, \quad P \in B_e \qquad \text{and} \qquad L_i \mathbf{u}_i(P) = \mathbf{0}, \quad P \in B_i, \tag{3.1a,b}$$

where $\mathbf{u}(P) = \mathbf{u}_e(P) + \mathbf{u}_{inc}(P)$ for $P \in B_e$, $\mathbf{u}_{inc}$ is the given incident wave and $\mathbf{u}_e$ satisfies a radiation condition at infinity. In addition, we shall impose certain continuity conditions across S; these are specified below. The operator L_a is defined by

$$L_a \mathbf{u} = k_a^{-2} \text{ grad div } \mathbf{u} - K_a^{-2} \text{ curl curl } \mathbf{u} + \mathbf{u}$$

where $\rho_a \omega^2 = (\lambda_a + 2\mu_a) k_a^2 = \mu_a K_a^2$ and $a = e$ or i. ρ_a is the density of the solid in B_a, whereas λ_a and μ_a are the Lamé moduli. The traction operator T_a is defined on S by

$$(T_a \mathbf{u})_m(p) = \lambda_a n_m \text{ div } \mathbf{u} + \mu_a n_\ell (\partial u_m / \partial x_\ell + \partial u_\ell / \partial x_m)$$

where $\mathbf{n}(p)$ is the unit normal at $p \in S$, pointing into B_e.

4. INCLUSIONS WITH IMPERFECT INTERFACES: A REVIEW

Suppose that the interface S is imperfect. For example, the inclusion might be surrounded by a thin layer of a different elastic material. For simple geometries, such problems can be treated exactly [17, 18]. The first approximate treatment, using continuity conditions across S similar to those described in §2, was given by Mal and Bose [19]. They considered spherical inclusions with the following interface conditions:

$$[\mathbf{t}] = \mathbf{0}, \qquad [u_n] = 0, \qquad \text{and} \qquad [u_\alpha] = M t_\alpha. \tag{4.1}$$

Here, $\mathbf{t} = T_e\mathbf{u}$, $\mathbf{t}_i = T_i\mathbf{u}_i$, $[\mathbf{t}] = \mathbf{t} - \mathbf{t}_i$ is the discontinuity in the tractions across S and we decompose vectors as $\mathbf{u}(p) = u_1\mathbf{s}_1 + u_2\mathbf{s}_2 + u_n\mathbf{n}$ for $p \in S$, where $\mathbf{s}_1$ and $\mathbf{s}_2$ are unit vectors in the tangent plane at p, satisfying $\mathbf{s}_1.\mathbf{s}_2 = 0$ and $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2$. The parameter M can be a complex function of ω: $M = 0$ for a perfect interface; M is given by (2.4) for a thin layer of viscous fluid; and $M = \infty$ for a lubricated interface, whence

$$[\mathbf{t}] = \mathbf{0}, \qquad t_\alpha = 0 \qquad \text{and} \qquad [u_n] = 0. \tag{4.2}$$

Similar interface conditions have been used in models of composite materials, where identical inclusions are arranged periodically prior to analysis using homogenization techniques. Let ϵ be a length scale associated with the periodicity. For elastostatics, Lene and Leguillon [20] take $M = k\epsilon$, where $k > 0$. For time-harmonic waves, Santosa and Symes [21] use $M = i\epsilon/(\omega c)$, where c is a 'viscous constant'; cf. (2.4).

Aboudi [22] has used flexibly-bonded interfaces in a different model of composites, with

$$[\mathbf{t}] = \mathbf{0}, \qquad [u_n] = M_n t_n, \qquad \text{and} \qquad [u_\alpha] = M_t t_\alpha. \tag{4.3}$$

He identifies M_n and M_t as h/E and h/μ, respectively, where the thin elastic interface layer has thickness h, Young's modulus E and shear modulus μ; cf. (2.6).

Kitahara, Nakagawa and Achenbach [23] consider an inclusion in 'spring contact' with the exterior solid. This is intended to model a thin compliant interface layer and leads to

$$[\mathbf{t}] = \mathbf{0} \qquad \text{and} \qquad [\mathbf{u}] = F.\mathbf{t}, \tag{4.4}$$

where the matrix F (called the 'flexibility matrix' [15]) is a positive diagonal matrix. Later, we shall allow F to be a full matrix, with elements that vary with position p on S.

More complicated interface conditions are obtained in [24, 25], where all terms to $O(h)$ are included for an elastic interface layer of thickness h. Non-local terms, involving various tangential derivatives, are present; see also [26]. The simplest (local) conditions discussed in [24, 25] are

$$[\mathbf{t}] = -\rho h \omega^2 \mathbf{u} \qquad \text{and} \qquad [\mathbf{u}] = \mathbf{0}, \tag{4.5}$$

where the layer has density ρ. A generalization of (4.5) is

$$[\mathbf{t}] = G.\mathbf{u} \qquad \text{and} \qquad [\mathbf{u}] = \mathbf{0}, \tag{4.6}$$

where the elements of the matrix G could vary with position p on S.

Finally, we consider a model that includes both (4.4) and (4.6), namely (cf. [27])

$$[\mathbf{t}] = G.\langle\mathbf{u}\rangle \qquad \text{and} \qquad [\mathbf{u}] = F.\langle\mathbf{t}\rangle, \tag{4.7}$$

where $\langle\mathbf{u}\rangle = \frac{1}{2}(\mathbf{u} + \mathbf{u}_i)$ is the average of $\mathbf{u}$ and $\mathbf{u}_i$ on S.

5. UNIQUENESS THEOREMS

Consider the problem of scattering by an inclusion with an imperfect interface. We can prove uniqueness theorems for interfaces characterized by (4.4), (4.6) or (4.7), by adapting standard arguments from [28]. Thus, surround S with a large sphere S_R of radius R. For $P \in B_e$, write

$$\mathbf{u}(P) = \mathbf{u}^{(p)} + \mathbf{u}^{(s)}, \qquad \text{where} \qquad \mathbf{u}^{(p)} = -k_e^{-2} \text{ grad div } \mathbf{u}, \qquad \mathbf{u}^{(s)} = \mathbf{u} - \mathbf{u}^{(p)},$$

and $\mathbf{u}_{inc} \equiv \mathbf{0}$. Then, an application of Betti's reciprocal theorem to $\mathbf{u}$ and its complex conjugate, $\overline{\mathbf{u}}$, in the region between S and S_R gives

$$k_e(\lambda_e + 2\mu_e) \lim_{R\to\infty} \int_{S_R} |\mathbf{u}^{(p)}|^2 \, ds + K_e \mu_e \lim_{R\to\infty} \int_{S_R} |\mathbf{u}^{(s)}|^2 \, ds + I = 0, \tag{5.1}$$

where

$$I = \frac{1}{2i} \int_S (\mathbf{u}.\overline{\mathbf{t}} - \overline{\mathbf{u}}.\mathbf{t}) \, ds = \mathcal{I}m \int_S \mathbf{u}.\overline{\mathbf{t}} \, ds, \tag{5.2}$$

$\mathcal{I}m$ denotes imaginary part and the radiation condition has been used (see [28], Chpt. 3, §2). If we can show that $I \geq 0$, we can deduce from (5.1) that $\mathbf{u}^{(p)} \equiv \mathbf{0}$ and $\mathbf{u}^{(s)} \equiv \mathbf{0}$, whence $\mathbf{u} \equiv \mathbf{0}$ in B_e. Then, (4.4) imply that $\mathbf{u}_i = \mathbf{0}$ and $\mathbf{t}_i = \mathbf{0}$ on S, whence $\mathbf{u}_i \equiv \mathbf{0}$ in B_i.

Now, applying Betti's theorem in B_i to $\mathbf{u}_i$ and $\overline{\mathbf{u}}_i$ gives

$$0 = \frac{1}{2i} \int_S (\mathbf{u}_i.\overline{\mathbf{t}}_i - \overline{\mathbf{u}}_i.\mathbf{t}_i) \, ds = \mathcal{I}m \int_S \mathbf{u}_i.\overline{\mathbf{t}}_i \, ds.$$

Hence, subtracting from (5.2) gives

$$I = \mathcal{I}m \int_S \overline{\mathbf{t}}.[\mathbf{u}] \, ds = \mathcal{I}m \int_S \overline{\mathbf{t}}.F\mathbf{t} \, ds,$$

after using (4.4). Thus, $I \geq 0$, provided that

$$F_{k\ell} = \overline{F}_{\ell k} \quad \text{for} \quad k \neq \ell \qquad \text{and} \qquad \mathcal{I}m(F_{kk}) \geq 0 \quad \text{(no sum)}. \tag{5.3}$$

So, if the elements of F are finite and satisfy (5.3) (for all $p \in S$ if F varies with p), we have proved that the corresponding inclusion problem has at most one solution.

The above proof fails for lubricated interfaces, given by (4.2). We obtain $I = 0$, whence $\mathbf{u}(P) \equiv \mathbf{0}$ for $P \in B_e$. It follows that $\mathbf{t}_i = \mathbf{0}$ and $\mathbf{n}.\mathbf{u}_i = 0$ on S. But, it does not follow that $\mathbf{u}_i \equiv \mathbf{0}$ in B_i, as for some geometries and frequencies the interior solid can support free oscillations which do not couple to the exterior solid. See [29] for more details.

We can obtain similar results for interfaces characterized by (4.6). For finite G, we find uniqueness, subject to

$$G_{k\ell} = \overline{G}_{\ell k} \quad \text{for} \quad k \neq \ell \qquad \text{and} \qquad \mathcal{I}m(G_{kk}) \leq 0 \quad \text{(no sum)}.$$

We can also prove uniqueness when (4.7) are used, provided that F and G are both real diagonal non-zero matrices.

6. BOUNDARY INTEGRAL EQUATIONS

We conclude by deriving (direct) boundary integral equations over S for inclusions with imperfect interfaces characterized by (4.4), in the plane case. First, we introduce two fundamental Green's tensors, $\mathbf{G}_a(P;Q)$ $(a = e, i)$:

$$(\mathbf{G}_a(P;Q))_{ij} = \frac{1}{\mu_a} \left\{ \Psi_a \delta_{ij} + \frac{1}{K_a^2} \frac{\partial^2}{\partial x_i \partial x_j} (\Psi_a - \Phi_a) \right\}$$

where $\Phi_a = -(i/2)H_0^{(1)}(k_a R)$, $\Psi_a = -(i/2)H_0^{(1)}(K_a R)$ and $R = |P - Q|$. Next, we define elastic single-layer and double-layer potentials by

$$(S_a\mathbf{u})(P) = \int_S \mathbf{u}(q).\mathbf{G}_a(q;P)\,ds_q \qquad \text{and} \qquad (D_a\mathbf{u})(P) = \int_S \mathbf{u}(q).T_a^q\mathbf{G}_a(q;P)\,ds_q,$$

respectively, where T_a^q means T_a applied at $q \in S$. Then, three applications of Betti's theorem (one in B_e to $\mathbf{u}_e$ and $\mathbf{G}_e$, one in B_i to $\mathbf{u}_{inc}$ and $\mathbf{G}_e$, and one in B_i to $\mathbf{u}_i$ and $\mathbf{G}_i$) yield the familiar representations

$$2\mathbf{u}_e(P) = (S_e\mathbf{t})(P) - (D_e\mathbf{u})(P), \qquad P \in B_e, \tag{6.1}$$

and

$$-2\mathbf{u}_i(P) = (S_i\mathbf{t}_i)(P) - (D_i\mathbf{u}_i)(P), \qquad P \in B_i. \tag{6.2}$$

Letting $P \to p \in S$, (6.1) and (6.2) give

$$(I + \overline{K_e^*})\mathbf{u} - S_e\mathbf{t} = 2\mathbf{u}_{inc} \tag{6.3}$$

and

$$(I - \overline{K_i^*})\mathbf{u}_i + S_i\mathbf{t}_i = \mathbf{0}, \tag{6.4}$$

respectively, where

$$\overline{K_a^*}\mathbf{u} = \int_S \mathbf{u}(q).T_a^q\mathbf{G}_a(q;p)\,ds_q.$$

($\overline{K_a^*}$ is a singular integral operator.) Using (4.4) in (6.4) gives

$$(I - \overline{K_i^*})\mathbf{u} + \{S_i - (I - \overline{K_i^*})F\}\mathbf{t} = \mathbf{0}. \tag{6.5}$$

The pair (6.3) and (6.5) is a system of four coupled singular integral equations for the four components of the two vectors $\mathbf{u}(p)$ and $\mathbf{t}(p)$, $p \in S$. It can be shown that this system is a *quasi-Fredholm* system [30], provided that F is a non-singular matrix (for all $p \in S$ if F varies with p). This means that all the usual Fredholm theorems hold. In particular, we can analyse the solvability of (6.3) and (6.5) by showing that the corresponding pair of homogeneous equations has only the trivial solution. These aspects will be considered elsewhere.

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