Torsional Waves in Composite Co-axial Cylinders with Imperfect Interfaces*

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Abstract

We consider the propagation of time-harmonic torsional waves in composite bi-material elastic cylinders. The interface between the core and the surrounding cladding is assumed to be imperfect: the tractions are continuous across the interface but the displacement jump is proportional to the stress acting on the interface. For torsional waves, this requires the introduction of a single constant of proportionality, F. We derive a frequency equation for the cylinder. The analysis recovers the known dispersion curves for a bimaterial rod with a perfect (welded) interface ($F = 0$), and has the correct limiting behavior for large F. We show that the modes, at any given frequency, are orthogonal, and outline how the problem of reflection of a torsional mode by a planar defect (such as a circumferential crack) can be treated.

1 Introduction

Electromagnetic–acoustic transducers (EMATs) have found a variety of applications in the non-destructive evaluation of materials and structures. For reviews, see [6] and [8]. We are especially interested in their use to launch and receive time-harmonic torsional waves in reinforced cables, so as to detect breaks and other defects. We model the cable as an infinitely long bimaterial cylinder, with a core of circular cross-section surrounded by a coaxial cladding; the core and the cladding are assumed to be made from different homogeneous isotropic elastic solids. EMATs have been used to study standing torsional modes in a single-material circular cylinder [9]. This is a classical problem, originally studied by Pochhammer [1, §6.10].

Propagation of torsional waves in a rod composed of two or more elastic layers has also been studied; see [13] for a review. Early work was done by Armenàkas [2]. He studied the dispersion of harmonic waves in a bimaterial cylinder and obtained the frequency equation. There has also been work on the free vibrations of a bimaterial rod of finite length [4] and on infinite rods with many co-axial layers [11].

We consider a bimaterial elastic cylinder with an *imperfect* interface between the core and the cladding. We do this because it is unrealistic to assume a perfectly bonded (welded) interface for our intended application to reinforced cables. We model the imperfect interface

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using a (linear) modification to the standard perfect-interface conditions, allowing some slippage. The interface conditions involve a single dimensionless parameter F. We have studied the effect of varying F on the dispersion relations; results for a perfectly bonded interface can be recovered by setting $F = 0$.

EMATs can be used to excite propagating modes with a specified axial wavelength λ, where λ is determined by the physical spacing between the magnets of alternating polarity. One then adjusts the frequency ω until one of the propagating torsional modes is excited. When such a mode interacts with a defect in the composite cylinder, other allowable modes at the frequency ω, but with various wavelengths, will be stimulated; evanescent modes (decaying exponentially with distance from the defect) will also be present, in general. We show that the torsional modes at a given frequency are orthogonal, extending a proof due to Gregory [7]. Finally, we outline how our knowledge of the modal structure for the composite cylinder can be used to model the problem of reflection of a torsional mode by a thin defect in a cross-sectional plane. The EMAT system can only receive waves with the same wavelength as the incident mode, so that some information at the excitation frequency ω is lost; but the experiment can be repeated at other modal frequencies.

2 Formulation

We consider an infinite isotropic elastic bimaterial cylinder, consisting of a solid core, $r < a$, surrounded by an annular cladding, $a < r < b$, where (r, θ, z) are cylindrical polar coordinates. The core and cladding are made of materials 1 and 2, respectively; material m has Lamé moduli λ_m and μ_m, $m = 1, 2$. Our analysis generally follows [2]. For torsional waves, the only non-trivial displacement component is the tangential displacement v, and v itself is required to be independent of θ. We can write

$$v = -\partial\psi/\partial r, \tag{1}$$

where the potential ψ satisfies the wave equation $\nabla^2\psi = c^{-2}\partial^2\psi/\partial t^2$ and c is the shear wave-speed. The only non-trivial stress components are

$$\sigma_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \quad \text{and} \quad \sigma_{\theta z} = \mu\,\frac{\partial v}{\partial z}. \tag{2}$$

For waves propagating in the positive z-direction, we can write

$$\psi(r, z, t) = \mathrm{Re}\left\{\Psi(r)\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}\right\}, \tag{3}$$

where k and ω are real, and Ψ solves Bessel's equation of order zero,

$$r^{-1}(d/dr)\left(r\Psi'\right) + \left((\omega/c)^2 - k^2\right)\Psi = 0, \tag{4}$$

with solutions that depend on the sign of $\omega^2 - k^2c^2$. Thus, we define

$$Z_n = J_n, \quad W_n = Y_n \quad \text{and} \quad q = \sqrt{(\omega/c)^2 - k^2} \quad \text{if } \omega^2 > k^2c^2, \text{ and} \tag{5}$$

$$Z_n = (-1)^n I_n, \quad W_n = K_n \quad \text{and} \quad q = \sqrt{k^2 - (\omega/c)^2} \quad \text{if } \omega^2 < k^2c^2, \tag{6}$$

where J_n, Y_n are Bessel functions and I_n, K_n are modified Bessel functions. The factor $(-1)^n$ will allow a unified treatment for all frequencies. The appropriate solution of (4) is

$$\Psi(r) = q^{-2}A\,Z_0(qr) + b^2B\,W_0(qr), \tag{7}$$

where A and B are arbitrary dimensionless constants.

Omitting the time-dependence, substituting (3) and (7) in (1) gives

$$v = \left\{ q^{-1} A\, Z_1(qr) + qb^2 B\, W_1(qr) \right\} \mathrm{e}^{\mathrm{i}kz} \tag{8}$$

as $Z_0'(x) = -Z_1(x)$ and $W_0'(x) = -W_1(x)$. From (2), we obtain for the stress

$$\sigma_{r\theta} = -\mu \left\{ A\, Z_2(qr) + (qb)^2 B\, W_2(qr) \right\} \mathrm{e}^{\mathrm{i}kz}, \tag{9}$$

as $Z_1'(x) - x^{-1} Z_1(x) = -Z_2(x)$ and $W_1'(x) - x^{-1} W_1(x) = -W_2(x)$.

Let us now use the expressions above, using subscripts 1 and 2 to indicate quantities in the core and cladding, respectively. Thus, from (8), the displacement in the cladding is

$$v_2 = \left\{ q_2^{-1} A_2\, Z_1(q_2 r) + q_2 b^2 B_2\, W_1(q_2 r) \right\} \mathrm{e}^{\mathrm{i}kz}. \tag{10}$$

For the core, the solution for v_1 must be bounded at the origin so we have

$$v_1 = q_1^{-1} A_1\, Z_1(q_1 r)\, \mathrm{e}^{\mathrm{i}kz}. \tag{11}$$

In these expressions, q_j is defined by

$$q_j = \sqrt{k_j^2 - k^2} \text{ if } k_j^2 > k^2 \text{ and } q_j = \sqrt{k^2 - k_j^2} \text{ if } k_j^2 < k^2, \text{ for } j = 1, 2, \tag{12}$$

where $k_j = \omega / c_j$. Note that the wavenumber, k, is the same in the expressions for q_1 and q_2; this observation gives a relation between q_1 and q_2.

It remains to specify boundary and interface conditions on the displacement field given by (10) and (11). At the outer surface, we have the traction-free boundary condition

$$\sigma_{r\theta} = 0 \quad \text{at } r = b. \tag{13}$$

For the imperfect interface at $r = a$, we suppose that

$$\sigma_{r\theta}(a^-) = \sigma_{r\theta}(a^+) \quad \text{and} \quad [v] = (a/\mu_1)\, F\, \sigma_{r\theta}(a), \tag{14}$$

where $[v] = v_2(a^+) - v_1(a^-)$ and F is a dimensionless scalar. Note that if $F = 0$, the perfect interface conditions of continuity of traction and displacement are recovered. For a review of interface conditions such as (14), and their derivation, see [10].

3 Frequency Equation for the Rod

Substituting the displacement field of (10) in the boundary condition, (13), yields

$$A_2\, Z_2(q_2 b) + (q_2 b)^2 B_2\, W_2(q_2 b) = 0. \tag{15}$$

Similarly, (10), (11), and (14) give

$$\begin{aligned}
(16)\quad & (\mu_1/\mu_2) A_1\, Z_2(q_1 a) - A_2\, Z_2(q_2 a) - (q_2 b)^2 B_2\, W_2(q_2 a) &= 0, \\
(17)\quad & (q_1 b)^{-1} A_1 \{ Z_1(q_1 a) - F q_1 a Z_2(q_1 a) \} - (q_2 b)^{-1} A_2 Z_1(q_2 a) - q_2 b B_2 W_1(q_2 a) &= 0.
\end{aligned}$$

Equations (15)–(17) provide three equations in the three unknown constants A_1, A_2 and B_2. In matrix form, the system of equations is $\mathbf{Db} = \mathbf{0}$, where the elements of the non-symmetric matrix $\mathbf{D}$ are obtained directly from (15)–(17) and $\mathbf{b} = (A_1, A_2, B_2)^T$. For a non-trivial solution we then require

$$\det \mathbf{D} = 0. \tag{18}$$

This is the frequency equation for the rod.

The quantity $\det \mathbf{D}$ seems to depend on only five dimensionless parameters, namely

$$q_1 b, \quad q_2 b, \quad a/b, \quad \mu_1/\mu_2 \quad \text{and} \quad F; \tag{19}$$

in particular, the density ratio (or, equivalently, c_1/c_2) does not appear explicitly. However, this is illusory: we have to know how to choose Z_n (J_n or $(-1)^n I_n$?) and W_n (Y_n or K_n?) in each material, and these choices depend on the relative sizes of k^2, k_1^2 and k_2^2, information that we cannot extract from a knowledge of (19) alone. Thus, we proceed as follows. Assume that we are given values for a/b, μ_1/μ_2, F and $c_2/c_1 = k_1/k_2 = \alpha$, say. Choose a value for the axial wavenumber kb. We then seek values of $k_2 b$, say, so that (18) is satisfied. Note that $k_1 b = \alpha k_2 b$, and then $q_1 b$ and $q_2 b$ are defined by (12), with the associated selections of Z_n and W_n dictated by (5) and (6).

Numerical results are presented and discussed in [3], with emphasis on the role of F. That paper also considers what happens when $q \to 0$ (the 'first' torsional mode) and the construction of evanescent modes (which decay exponentially with z).

4 Discussion and Mode Orthogonality

We have found a variety of torsional modes for the bimaterial cylinder in the general form $\mathbf{u}(r,\theta,z,t) = \mathrm{Re}\left\{\mathbf{U}(r,\theta)\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}\right\}$. In our computations, we fixed the axial wavenumber k and then calculated the frequencies ω of the allowable modes. This is appropriate for the application to EMATs, as these can be used to excite propagating modes of a specified axial wavelength. However, once such a mode has been excited, we want to study its reflection by defects in the cylinder. This is most conveniently done by specifying the frequency and then determining all the allowable modes at that frequency. Thus, we write a typical mode as

$$\mathbf{u}^{(n)}(r,\theta,z,t) = \mathrm{Re}\left\{\mathbf{U}^{(n)}(r,\theta)\,\mathrm{e}^{\mathrm{i}(k^{(n)}z-\omega t)}\right\},$$

where the wavenumber $k^{(n)}$ need not be real. These modes are *bi-orthogonal.* To be more explicit, denote the stresses corresponding to $\mathbf{u}^{(n)}$ by

$$\boldsymbol{\sigma}^{(n)}(r,\theta,z,t) = \mathrm{Re}\left\{\mathbf{S}^{(n)}(r,\theta)\,\mathrm{e}^{\mathrm{i}(k^{(n)}z-\omega t)}\right\}.$$

Then, if $\mathcal{A}$ is the cross-section of the composite cylinder, and if $k^{(n)} \neq \pm k^{(m)}$, we have

$$\int_{\mathcal{A}} \left\{U_z^{(m)} S_{zz}^{(n)} - S_{rz}^{(m)} U_r^{(n)} - S_{\theta z}^{(m)} U_\theta^{(n)}\right\} r\,dr\,d\theta = 0. \tag{20}$$

This relation can be proved by a simple extension of the proof given by Gregory [7]. (Apply the elastic reciprocal theorem twice, once in the core and once in the cladding, and then add the results; the interface conditions imply that the contributions from integrating over the two sides of the interface cancel.) In fact, (20) holds for *all* modes in composite cylinders of *any* cross-section, and with *any* number of imperfect (cylindrical) interfaces. For our problem, with torsional modes given by

$$v^{(n)}(r,z,t) = \mathrm{Re}\left\{V^{(n)}(r)\,\mathrm{e}^{\mathrm{i}(k^{(n)}z-\omega t)}\right\},$$

equation (20) reduces to

$$\int_0^b V^{(m)}(r)\,V^{(n)}(r)\,r\,dr = 0, \qquad m \neq n, \tag{21}$$

so that torsional modes are actually orthogonal. This orthogonality relation is useful when the reflection of a torsional mode by certain defects is examined. For example, we may consider a bimaterial cylinder with a planar break (crack) perpendicular to the cylinder's axis, giving an idealised model of a damaged cable. Specifically, we partition the cross-section $\mathcal{A}$ into a broken part $\mathcal{A}_b$ and an unbroken part $\mathcal{A}_u$, so that $\mathcal{A} = \mathcal{A}_b \cup \mathcal{A}_u$. The boundaries of $\mathcal{A}_b$ and $\mathcal{A}_u$ are concentric circles. Then, if a torsional mode is incident on the defect, the reflected and transmitted fields can be written as modal sums. This is a standard approach for planar obstacles in waveguides. In the context of torsional waves, see [5] for an analysis of the effect of a step-change in radius of homogeneous circular cylinders. For the present problem, application of the boundary conditions at the defect plane leads to a system of equations for the reflection and transmission coefficients; of particular interest are the reflected and transmitted modes with the same wavelength as the incident mode, because these are the only modes that can be detected by the EMAT. Again, in a standard way, one can derive integral equations and/or variational expressions for the reflection and transmission coefficients; see, for example, [12] for a discussion on related scattering problems. It remains to make detailed computations for a cracked composite cylinder.

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