Waves in Wood: Elastic Waves in Cylindrically Orthotropic Materials*

P. A. Martin† J. R. Berger‡

Abstract

We consider elastic waves in materials with cylindrical orthotropy, this being a plausible model for wood. For time-harmonic motions, the problem is reduced to some coupled ordinary differential equations. Previously, these have been solved using the method of Frobenius (power-series expansions). We use Neumann series (expansions in Bessel functions of various orders), motivated by the known classical solutions for homogeneous isotropic solids. This gives an effective and natural method for wave propagation in cylindrically orthotropic materials. The problem itself arose from a study of ultrasonic devices as used in the detection of rotten regions inside wooden telegraph (utility) poles.

1 Introduction

> 'Wood is a unique material. ... It is multicomponent, hygroscopic, anisotropic, inhomogeneous, discontinuous, inelastic, fibrous, porous, biodegradable, and renewable.' [2, p. vii]

Despite these complications, there is a need to have physical models for the behaviour of wood. One application (which motivated the present study) is to model the use of ultrasonic devices for the detection of rotten regions inside telegraph poles, so as to predict the strength of in-service poles. Typically, these devices use stress waves through the pole cross-section, which we can take to be circular.

Traditionally, wood is modelled as an orthotropic elastic solid [2, chapter 3]. Thus, at any point P in a wooden pole, we can identify three mutually orthogonal directions, namely longitudinal (along the grain), radial and tangential. These can be taken to specify three symmetry planes at P, and this leads to the orthotropic model. Elastic waves in orthotropic materials are discussed in detail by Musgrave [6, chapter 9].

However, the local orthotropic description ignores one obvious characteristic of trees and poles — the presence of *annual rings*. Thus [3, p. 3]: 'At the annual ring level the structure is again one of a layered composite built up with two layers corresponding to the *earlywood* and *latewood*'. Typically, the density of earlywood is about half that of latewood [3, p. 151]. Bodig & Jayne [2, §10.3.2] give more details.

The effect of this layered structure on wave-speed measurement is discussed by Bucur [3, §4.3.2.4]: 'The opinions of different authors are rather divergent'. However, it is clear that

---

*This work was partially supported by EPSRC grant GR/M76560. The paper appeared in *5th International Conference on Mathematical and Numerical Aspects of Wave Propagation* (ed. A. Bermúdez, D. Gomez, C. Hazard, P. Joly and J. E. Roberts), SIAM, Philadelphia, 2000, 182–186.

†Department of Mathematical & Computer Sciences, Colorado School of Mines, Golden, CO 80401-1887, USA

‡Division of Engineering, Colorado School of Mines, Golden, CO 80401-1887, USA

the curvature of the rings should be taken into account if wave propagation over significant distances is to be modelled; and this is precisely the case in the telegraph-pole problem.

The above considerations suggest that a pole could be modelled as a composite material with concentric layers of two different materials, giving an axisymmetric structure. For simplicity, each layer could be assumed to be homogeneous (constant material parameters) with continuity conditions across the interfaces between layers.

For the wood itself, an alternative formulation of the theory suggests itself, in which the wood is assumed to be *cylindrically anisotropic*. Thus, Bodig & Jayne [2, p. 21] wrote that one 'might model [trees or poles] as homogeneous with cylindrical anisotropy due to the layered growth ring structure', although they did not go further. By definition, cylindrical anisotropy means that the elastic stiffnesses are constants when referred to cylindrical polar coordinates. Properties of materials with cylindrical anisotropy have been studied by several authors; see, for example, [5], [1] and [7]. We shall develop the theory of cylindrical anisotropy for wave propagation through wooden poles. Specifically, we consider a material with cylindrical orthotropy, and then study the resulting system of ordinary differential equations. These equations have been solved by others using the method of Frobenius. We use a generalization of this method, in which we expand using Bessel functions rather than powers. We claim that the use of such Neumann series is more effective (and more natural) for wave propagation in cylindrical situations.

2 Formulation

Let $x_1 \equiv x$, $x_2 \equiv y$ and $x_3 \equiv z$ be Cartesian coordinates. Then, the governing equations of motion for an anisotropic elastic material are

$$\frac{\partial}{\partial x_j}\widetilde{\tau}_{ij} = \rho \frac{\partial^2}{\partial t^2}\widetilde{u}_i \quad \text{where} \quad \widetilde{\tau}_{ij} = \widetilde{C}_{ijk\ell}\frac{\partial}{\partial x_k}\widetilde{u}_\ell$$

is the stress tensor, $\widetilde{\boldsymbol{u}}$ is the displacement vector, ρ is the mass density, t is the time, $\widetilde{C}_{ijk\ell}$ are the elastic stiffnesses and the summation convention holds. As usual, we assume that $\widetilde{C}_{ijk\ell} = \widetilde{C}_{jik\ell} = \widetilde{C}_{k\ell ij} = \widetilde{C}_{ij\ell k}$.

Introduce cylindrical polar coordinates (r, θ, z), where $x = r\cos\theta$ and $y = r\sin\theta$. If $C_{ijk\ell}$ denote the elastic stiffnesses referred to (r, θ, z), we have

$$C_{ijk\ell}(\theta) = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{\ell s}\widetilde{C}_{pqrs}(\theta),$$

where

$$\Omega_{ij}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are interested in special materials for which $C_{ijk\ell}(\theta)$ are constant, so that $C_{ijk\ell}(\theta) = C_{ijk\ell}(0) = \widetilde{C}_{ijk\ell}(0)$; such materials are said to be *cylindrically anisotropic*. Cylindrical anisotropy seems to be a good model for wooden poles [2, p. 21].

Following Ting's formulation [7] for static problems, we write the equations of motion as

$$\frac{\partial}{\partial r}(r\boldsymbol{t}_r) + \frac{\partial}{\partial \theta}\boldsymbol{t}_\theta + \mathsf{K}\boldsymbol{t}_\theta + r\frac{\partial}{\partial z}\boldsymbol{t}_z = \rho r \frac{\partial^2}{\partial t^2}\widetilde{\boldsymbol{u}},$$

where

$$\boldsymbol{t}_r = \begin{pmatrix} \tau_{rr} \\ \tau_{r\theta} \\ \tau_{rz} \end{pmatrix}, \quad \boldsymbol{t}_\theta = \begin{pmatrix} \tau_{\theta r} \\ \tau_{\theta\theta} \\ \tau_{\theta z} \end{pmatrix}, \quad \boldsymbol{t}_z = \begin{pmatrix} \tau_{zr} \\ \tau_{z\theta} \\ \tau_{zz} \end{pmatrix}, \quad \widetilde{\boldsymbol{u}} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Ting [7, p. 2399] gives expressions for the traction vectors $\boldsymbol{t}_i$ in terms of $\widetilde{\boldsymbol{u}}$. If we assume that $\widetilde{\boldsymbol{u}}$ does not depend on z, we find that two-dimensional motions are governed by

$$\begin{aligned}(1)\qquad & r\mathsf{Q}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\widetilde{\boldsymbol{u}}\right)+r(\mathsf{R}+\mathsf{R}^T)\frac{\partial^2}{\partial r\,\partial\theta}\widetilde{\boldsymbol{u}}+\mathsf{T}\frac{\partial^2}{\partial\theta^2}\widetilde{\boldsymbol{u}} \\ & +r(\mathsf{RK}+\mathsf{KR}^T)\frac{\partial}{\partial r}\widetilde{\boldsymbol{u}}+(\mathsf{TK}+\mathsf{KT})\frac{\partial}{\partial\theta}\widetilde{\boldsymbol{u}}+\mathsf{KTK}\widetilde{\boldsymbol{u}} = \rho r^2\frac{\partial^2}{\partial t^2}\widetilde{\boldsymbol{u}},\end{aligned}$$

generalising [7, eqn. (3.1)] to dynamic problems. The 3×3 matrices occurring here are given by Ting [7] as

$$\mathsf{Q}=\begin{pmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55}\end{pmatrix},\quad \mathsf{R}=\begin{pmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45}\end{pmatrix},\quad \mathsf{T}=\begin{pmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44}\end{pmatrix};$$

R^T is the transpose of R, and we have used the contracted notation $C_{\alpha\beta}$ for the elastic stiffnesses [8, §2.3].

We look for time-harmonic solutions in the form

$$\widetilde{\boldsymbol{u}}(r,\theta,t)=\mathrm{Re}_{\mathrm{i}}\left\{\boldsymbol{u}_m(r)\,\mathrm{e}^{\mathrm{j}m\theta}\,\mathrm{e}^{-\mathrm{i}\omega t}\right\},$$

where i and j are two non-interacting complex units, m is an integer, ω is the radian frequency, and Re_{i} denotes the real part with respect to i. Use of $\mathrm{e}^{\mathrm{j}m\theta}$ rather than $\cos m\theta$ and $\sin m\theta$ allows us to retain the nice matrix notation in what follows. Thus, from (1), we find that $\boldsymbol{u}_m(r)$ solves

$$(2)\qquad r^2\mathsf{Q}\boldsymbol{u}_m''+r(\mathsf{Q}+\mathsf{RK}_m+\mathsf{K}_m\mathsf{R}^T)\boldsymbol{u}_m'+(\rho\omega^2r^2+\mathsf{K}_m\mathsf{TK}_m)\boldsymbol{u}_m=\boldsymbol{0},$$

where $\mathsf{K}_m=\mathsf{K}+\mathrm{j}m\mathsf{I}$. If $m=0$ (axisymmetry) and $\omega=0$ (static), (2) reduces to [7, eqn. (3.2)].

Setting $\boldsymbol{u}_m=(u_m,v_m,w_m)$, (2) gives three coupled ordinary differential equations for the three components of $\boldsymbol{u}_m$. For the special case of cylindrically orthotropic materials (i.e. wood), these equations simplify slightly; for such materials, the non-trivial stiffnesses are [8, pp. 36 & 45]

$$\begin{aligned} C_{11}&=C_{1111}, & C_{12}&=C_{1122}, & C_{13}&=C_{1133},\\ C_{22}&=C_{2222}, & C_{23}&=C_{2233}, & C_{33}&=C_{3333},\\ C_{44}&=C_{2323}, & C_{55}&=C_{1313}, & C_{66}&=C_{1212}.\end{aligned}$$

We find that the 3×3 system (2) uncouples into a 2×2 system for u_m and v_m, and a single equation for w_m. We can solve for w_m explicitly in terms of certain Bessel functions; cf. [9].

The ordinary differential equations for $u_m(r)$ and $v_m(r)$ are

$$\begin{aligned}(3)\qquad & r^2C_{11}u_m''+r\left[C_{11}u_m'+\mathrm{j}m(C_{66}+C_{12})v_m'\right]\\ & +(\rho\omega^2r^2-m^2C_{66}-C_{22})u_m-\mathrm{j}m(C_{66}+C_{22})v_m = 0,\\ (4)\qquad & r^2C_{66}v_m''+r\left[C_{66}v_m'+\mathrm{j}m(C_{66}+C_{12})u_m'\right]\\ & +(\rho\omega^2r^2-C_{66}-m^2C_{22})v_m+\mathrm{j}m(C_{66}+C_{22})u_m = 0.\end{aligned}$$

They can be solved exactly when $m=0$ (axisymmetric motions) but, so far, we have not found any explicit solutions when $m\neq 0$.

3 Expansion Methods

For non-axisymmetric motions ($m \neq 0$), we begin with a slight simplification of notation. Thus, we define dimensionless stiffnesses by

$$c_1 = C_{11}/C_{66}, \quad c_{12} = C_{12}/C_{66} \quad \text{and} \quad c_2 = C_{22}/C_{66}.$$

Then, (3) and (4) become

$$\begin{aligned}
&(5) \quad c_1(r^2 u_m'' + r u_m') + \mathrm{j}m(1+c_{12}) r v_m' + (\kappa^2 r^2 - m^2 - c_2) u_m - \mathrm{j}m(1+c_2) v_m &= 0,\\
&(6) \quad r^2 v_m'' + r v_m' + \mathrm{j}m(1+c_{12}) r u_m' + (\kappa^2 r^2 - 1 - m^2 c_2) v_m + \mathrm{j}m(1+c_2) u_m &= 0,
\end{aligned}$$

where $\kappa^2 = \rho\omega^2/C_{66}$.

3.1 The method of Frobenius

An obvious way of treating (5) and (6) is to look for solutions in the form of power series,

$$u_m(r) = \sum_{n=0}^{\infty} \hat{a}_n \, (\kappa r)^{2n+\alpha} \quad \text{and} \quad v_m(r) = \mathrm{j} \sum_{n=0}^{\infty} \hat{b}_n \, (\kappa r)^{2n+\alpha}, \tag{7}$$

where the coefficients $\hat{a}_n$, $\hat{b}_n$ and α are to be determined. (It turns out that there is no loss of generality in using $(\kappa r)^{2n}$ rather than $(\kappa r)^n$.) We find that α is given as the solution of the indicial equation,

$$\alpha^4 c_1 + \alpha^2 \{ m^2 (2c_{12} - c_1 c_2 + c_{12}^2) - (c_1 + c_2) \} + (m^2 - 1)^2 c_2 = 0. \tag{8}$$

Once a value of α has been selected, we can then generate the coefficients $\hat{a}_n$ and $\hat{b}_n$ recursively.

This procedure for computing the coefficients is efficient. It has been used previously by, for example, Chou and Achenbach [4] and by Yuan and Hsieh [10]. Its main drawback is that it is essentially a *static* method: power-series expansions in r are only expected to be good for small values of r.

3.2 Neumann series

As an alternative procedure, we can use a generalization of the method of Frobenius, in which u_m and v_m are expanded as *Neumann series*,

$$u_m(r) = \sum_{n=0}^{\infty} a_n \, J_{2n+\alpha}(kr) \quad \text{and} \quad v_m(r) = \mathrm{j} \sum_{n=0}^{\infty} b_n \, J_{2n+\alpha}(kr). \tag{9}$$

Here J_ν is a Bessel function and the coefficients a_n, b_n and α are to be determined. Note that the parameter k is at our disposal. We are motivated to use Neumann series rather than power series because we know that, in the isotropic case, both u_m and v_m can be written as linear combinations of just two Bessel functions.

We have investigated two methods for finding a_n and b_n, which we call *direct* and *indirect*. In the direct method, we substitute the expansions (9) in (5) and (6) and then group terms. This requires manipulating series of Bessel functions, and so is more complicated than at the analogous stage of the method of Frobenius. It turns out that α solves the same indicial equation as before, namely (8). Eventually, we obtain some recurrence relations for a_n and b_n, which we do not record here. It suffices to say that they are fairly complicated but that they are well behaved numerically.

For the indirect method, we begin with the standard method of Frobenius, leading to the computation of the coefficients $\hat{a}_n$ and $\hat{b}_n$. From these, we then compute the coefficients a_n and b_n, using the known expansion of an arbitrary power in terms of Bessel functions:

$$(\tfrac{1}{2}kr)^\nu = \sum_{n=0}^{\infty} \frac{(2n+\nu)\,\Gamma(n+\nu)}{n!}\, J_{2n+\nu}(kr).$$

(Compare this with the definition of a Bessel function,

$$J_\nu(kr) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}(\tfrac{1}{2}kr)^{2n+\nu},$$

which can itself be obtained by the method of Frobenius.)

Comparisons between these methods are currently being made, with applications. Preliminary results suggest that the use of Neumann series is more efficient for the problems of wave propagation in wood that we have described above.

References

[1] S. A. Ambartsumyan, *Theory of Anisotropic Plates*, Technomic Publ. Co., Stamford, CT, 1970.
[2] J. Bodig and B. A. Jayne, *Mechanics of Wood and Wood Composites*, Van Nostrand Reinhold, New York, 1982.
[3] V. Bucur, *Acoustics of Wood*, CRC Press, Boca Raton, Florida, 1995.
[4] F. -H. Chou and J. D. Achenbach, *Three-dimensional vibrations of orthotropic cylinders*, Proc. ASCE, J. Engng. Mech. Div., 98 (1972), pp. 813–822.
[5] S. G. Lekhnitskii, *Anisotropic Plates*, Gordon & Breach, New York, 1968.
[6] M. J. P. Musgrave, *Crystal Acoustics*, Holden-Day, San Francisco, 1970.
[7] T. C. T. Ting, *Pressuring, shearing, torsion and extension of a circular tube or bar of cylindrically anisotropic material*, Proc. Roy. Soc. A, 452 (1996), pp. 2397–2421.
[8] T. C. T. Ting, *Anisotropic Elasticity*, Oxford University Press, 1996.
[9] K. Watanabe and R. G. Payton, *SH wave in a cylindrically anisotropic elastic solid. A general solution for a point source*, Wave Motion, 25 (1997), pp. 197–212.
[10] F. G. Yuan and C. C. Hsieh, *Three-dimensional wave propagation in composite cylindrical shells*, Composite Structures, 42 (1998), pp. 153–167.