Scattering by Inhomogeneities*

P. A. Martin

Department of Mathematical and Computer Sciences
Colorado School of Mines, Golden, CO 80401, USA

Abstract. Acoustic scattering problems are considered when the density and speed of sound are functions of position within a bounded region. An integro-differential equation for the pressure in this region is obtained. Solving this equation is equivalent to solving the scattering problem. Problems of this kind are often solved by regarding the effects of the inhomogeneity as an unknown source term driving a Helmholtz equation, leading to an equation of Lippmann–Schwinger type. It is shown that this approach is erroneous when the density is discontinuous.

1 Introduction

Time-harmonic acoustic waves in an inhomogeneous compressible fluid can be modelled using *Bergmann's equation* (see [1], [5] and [7, p. 408])

$$\mathcal{B}p \equiv \rho \operatorname{div} \left(\rho^{-1} \operatorname{grad} p \right) + k^2 p = 0, \tag{1}$$

where ρ is the density, p is the total acoustic pressure, $k = \omega/c$, ω is the frequency and c is the speed of sound; in general, ρ and c can vary with position.

Equation (1) can be written as $(\nabla^2 + k_{\mathrm{e}}^2)p = Vp$, where k_{e}^2 is constant and the operator V is defined by

$$Vu = (k_{\mathrm{e}}^2 - k^2)u + \rho^{-1}(\operatorname{grad} \rho) \cdot \operatorname{grad} u. \tag{2}$$

We are interested in scattering problems, so that $p = p_{\mathrm{inc}} + p_{\mathrm{sc}}$, where the incident field p_{inc} satisfies $(\nabla^2 + k_{\mathrm{e}}^2)p_{\mathrm{inc}} = 0$ and the scattered field p_{sc} satisfies the Sommerfeld radiation condition. Thus

$$(\nabla^2 + k_{\mathrm{e}}^2)p_{\mathrm{sc}} = Vp. \tag{3}$$

Formally, we may regard the right-hand side of (3) as known, so that

$$p(P) = p_{\mathrm{inc}}(P) + \int G_{\mathrm{e}}(P,Q) \, (Vp)(Q) \, dV_Q. \tag{4}$$

where P and Q are typical points in three-dimensional space. Here, $G_{\mathrm{e}}(P,Q)$ is the free-space Green's function, defined by

$$G_{\mathrm{e}}(P,Q) = -\exp\left(ik_{\mathrm{e}}R\right)/(4\pi R), \tag{5}$$

---

where R is the distance between P and Q.

In this calculation, we have used the fact that

$$u(P) = \int G_{\mathrm{e}}(P,Q)\, f(Q)\, dV_Q \quad \text{solves} \quad (\nabla^2 + k_{\mathrm{e}}^2)u = f \tag{6}$$

and satisfies the radiation condition. Formal derivations of this kind are often found in textbooks; see, for example, [2, §8.9.1]. The result (6) can be justified readily if one assumes that f is (Hölder) continuous. However, for discrete scatterers, there will be interfaces across which $k(Q)$ and the normal derivative of p, $\partial p/\partial n$, will be discontinuous (although p and $\rho^{-1}\partial p/\partial n$ are both continuous across such interfaces).

We shall derive a new equation that respects the proper transmission conditions across interfaces. Solving this equation is equivalent to solving the transmission problem for the acoustic pressure. The new equation reduces to the well-known Lippmann–Schwinger equation when the density in the inhomogeneity is constant and equal to the density of the surrounding homogeneous fluid. It also reduces to the equation derived formally above, namely (4), *but only when there is no discontinuity in the density across the boundary of the inhomogeneity.* If there is such a discontinuity (as is typical in applications), an extra term is needed; see (18) below.

2   Formulation

Consider the scattering of sound waves in a homogeneous compressible fluid by a bounded inhomogeneous obstacle. In the exterior fluid, B_{e}, we can write $p_{\mathrm{e}} = p_{\mathrm{inc}} + p_{\mathrm{sc}}$, where p_{e} is the total pressure. The governing equation for p_{sc} is

$$(\nabla^2 + k_{\mathrm{e}}^2)p_{\mathrm{sc}} = 0 \quad \text{in } B_{\mathrm{e}}, \tag{7}$$

where $k_{\mathrm{e}} = \omega/c_{\mathrm{e}}$ and c_{e} is the constant speed of sound.

Within the obstacle, B, the governing equation is Bergmann's equation, $\mathcal{B}p_{\mathrm{i}} = 0$, where p_{i} is the pressure; the interior density ρ and speed of sound c can vary with position in B. At the interface S between B and B_{e}, we have a pair of transmission conditions, expressing continuity of pressure and normal velocity. These are

$$p_{\mathrm{e}} = p_{\mathrm{i}} \quad \text{and} \quad \frac{1}{\rho_{\mathrm{e}}}\frac{\partial p_{\mathrm{e}}}{\partial n} = \frac{1}{\rho}\frac{\partial p_{\mathrm{i}}}{\partial n} \quad \text{on } S, \tag{8}$$

where ρ_{e} is the (constant) density of the fluid in B_{e}.

Summarising, we have the following problem to solve.

SCATTERING PROBLEM.   Let p_{inc} be a given incident field. Find a pair of functions, $\{p_{\mathrm{e}}, p_{\mathrm{i}}\}$, where $p_{\mathrm{sc}} = p_{\mathrm{e}} - p_{\mathrm{inc}}$ satisfies (7) and the Sommerfeld radiation condition, p_{i} satisfies (1), and p_{e} and p_{i} satisfy the transmission conditions (8) across the interface S.

Werner wrote an important paper on the Scattering Problem in 1963 [8]. He reduced the problem to a system of coupled integral equations, using single-layer, double-layer and volume potentials; this is an example of an *indirect method*. He proved that the Scattering Problem has exactly one solution. We shall use a *direct method*, and we shall only use volume potentials.

Much has been written on the case where ρ is constant, so that the second term on the right-hand side of (2) can be deleted [3, Chapter 8]. Here, we do not make this assumption: we allow both ρ and k to vary with position. Note that if the material in B is actually homogeneous, so that ρ and k are both constants, boundary integral equations over S can be used; see [4] for a review.

3 An integro-differential equation

We shall consider integral representations obtained using the free-space Green's function for the exterior fluid, G_e, defined by (5). Two applications of Green's second theorem, one in B_e to p_{sc} and G_e and one in B to p_{inc} and G_e, give

$$\int_S \left\{ G_e(P,q) \frac{\rho_e}{\rho} \frac{\partial p_i}{\partial n_q} - p_i(q) \frac{\partial}{\partial n_q} G_e(P,q) \right\} ds_q = \begin{cases} p_{sc}(P), & P \in B_e, \\ -p_{inc}(P), & P \in B, \end{cases} \quad (9)$$

where we have used $p_e = p_{sc} + p_{inc}$ and the transmission conditions (8). The first of these gives an integral representation for $p_{sc}(P)$. Such representations are common in scattering theory. However, it is not very convenient here because we do not know p_i or $\partial p_i/\partial n$ on S.

To make progress, recall Green's first theorem,

$$\int_B \left\{ \phi \, \nabla^2 \psi + (\text{grad}\,\phi) \cdot (\text{grad}\,\psi) \right\} dV = \int_S \phi \frac{\partial \psi}{\partial n} \, ds,$$

where ϕ and ψ are sufficiently smooth in B. Choose $\phi(Q) = p_i(Q)$ and $\psi(Q) = G_e(P,Q)$ with $P \in B_e$, whence

$$\int_S p_i \frac{\partial}{\partial n_q} G_e(P,q) \, ds_q = \int_B \left\{ (\text{grad}\,p_i) \cdot (\text{grad}_Q G_e) - k_e^2 p_i \, G_e(P,Q) \right\} dV_Q, \quad (10)$$

where we have used $(\nabla^2 + k_e^2) G_e(P,Q) = 0$ for $P \neq Q$. Similarly, if we choose $\psi(Q) = p_i(Q)$ and $\phi(Q) = (\rho_e/\rho) G_e(P,Q)$ with $P \in B_e$, we obtain

$$\int_S \frac{\rho_e}{\rho} \frac{\partial p_i}{\partial n_q} G_e(P,q) \, ds_q = \int_B \frac{\rho_e}{\rho} \left\{ (\text{grad}\,p_i) \cdot (\text{grad}_Q G_e) - k_e^2 N p_i \, G_e \right\} dV_Q, \quad (11)$$

where $N = (k/k_e)^2 = (c_e/c)^2$ is the *refractive index* and we have used $\mathcal{B}p_i = 0$. Subtracting (10) from (11) gives the left-hand side of (9) for $P \in B_e$, whence $p_{sc}(P) = (\mathcal{L}p_i)(P)$ for $P \in B_e$, where

$$(\mathcal{L}v)(P) = \int_B \left\{ (\alpha - 1)(\text{grad}\,v) \cdot (\text{grad}_Q G_e(P,Q)) + (1 - N\alpha)k_e^2 v G_e \right\} dV_Q \quad (12)$$

and $\alpha(P) = \rho_e/\rho(P)$.

We repeat the calculations for $P \in B$, having excised a small sphere centred at P. The singularity at $P = Q$ has no effect on (11) but it causes $-p_i(P)$ to be added to the left-hand side of (10). Then, (9) for $P \in B$ becomes

$$-p_{inc}(P) = -p_i(P) + (\mathcal{L}p_i)(P), \quad P \in B.$$

At this stage, we have proved one half of the following theorem.

Theorem 1. *Let the pair $\{p_e, p_i\}$ solve the Scattering Problem. Then $v(P) \equiv p_i(P)$ solves*

$$v(P) - (\mathcal{L}v)(P) = p_{inc}(P), \quad P \in B, \tag{13}$$

where $\mathcal{L}v$ is defined by (12). Conversely, let v solve (13). Then the pair $\{p_e, p_i\}$, defined by

$$p_e(P) = p_{inc}(P) + (\mathcal{L}v)(P) \quad \text{for } P \in B_e \tag{14}$$

and $p_i(P) = v(P)$ for $P \in B$, solves the Scattering Problem.

The second half of the theorem is proved in [6].

4 Discussion

4.1 Solvability

Solving the Scattering Problem is equivalent to solving equation (13), which is an integro-differential equation for $v(P)$, $P \in B$. This equation is uniquely solvable. To see this, we appeal to Werner's existence result [8]: the solution $\{p_e, p_i\}$ of the Scattering Problem exists and, by the first half of Theorem 1, p_i solves (13). For uniqueness, suppose that $v_0(P)$ solves (13) with $p_{inc} \equiv 0$. Construct $p_e = (\mathcal{L}v_0)(P)$ for $P \in B_e$ and $p_i = v_0(P)$ for $P \in B$. By the second half of Theorem 1, these fields solve the homogeneous Scattering Problem; they must vanish identically by the uniqueness theorem for the Scattering Problem. In particular, $v_0(P) \equiv 0$ for $P \in B$, as required.

4.2 The Lippmann–Schwinger equation

As a special case, suppose that $\rho(Q) = \rho_e$ for all $Q \in B$, so that the density of the scatterer is the same as that of the surrounding homogeneous fluid. Then, the integro-differential equation (13) reduces to the integral equation

$$v(P) - k_e^2 \int_B \{1 - N(Q)\}v(Q)\, G_e(P, Q)\, dV_Q = p_{inc}(P), \quad P \in B, \tag{15}$$

where $N(Q) = (k/k_e)^2 = \{c_e/c(Q)\}^2$. This integral equation and its numerical treatment have been discussed in [2, §8.9.1].

Let us define $N(P) = 1$ for $P \in B_{\mathrm{e}}$, $w(P) = p_{\mathrm{e}}(P)$ for $P \in B_{\mathrm{e}}$, and $w(P) = p_{\mathrm{i}}(P)$ for $P \in B$. Then, we can combine (15) with (14) to obtain

$$w(P) - k_{\mathrm{e}}^2 \int (1 - N)w G_{\mathrm{e}}(P, Q)\, dV_Q = p_{\mathrm{inc}}(P), \quad \text{for all } P \in B \cup B_{\mathrm{e}}, \quad (16)$$

where the integration is over all Q. We recognise this equation as the *Lippmann–Schwinger equation*; see, for example, [3, §8.2]. Notice that our derivation shows that (16) is valid even when $N(Q)$ is discontinuous as Q crosses S.

4.3   An alternative equation

As we know that $v \equiv p_{\mathrm{i}}$ solves $\mathcal{B}v = 0$ in B, we can use this fact to rewrite the expression for $\mathcal{L}v$. Thus $(\alpha - 1)(\operatorname{grad} v) \cdot (\operatorname{grad}_Q G_{\mathrm{e}}) = \operatorname{div}\{(\alpha - 1)G_{\mathrm{e}}\operatorname{grad} v\} - G_{\mathrm{e}}\operatorname{div}\{(\alpha - 1)\operatorname{grad} v\}$ and

$$\begin{aligned}
\operatorname{div}\{(\alpha - 1)\operatorname{grad} v\} &= (\alpha - 1)\nabla^2 v + \rho_{\mathrm{e}}\left(\operatorname{grad} \rho^{-1}\right) \cdot \operatorname{grad} v \\
&= (\alpha - 1)\left\{\nabla^2 v + \rho\left(\operatorname{grad} \rho^{-1}\right) \cdot \operatorname{grad} v\right\} + \rho\left(\operatorname{grad} \rho^{-1}\right) \cdot \operatorname{grad} v \\
&= (1 - \alpha)\, k_{\mathrm{e}}^2 N v - \rho^{-1}(\operatorname{grad} \rho) \cdot \operatorname{grad} v.
\end{aligned}$$

Hence, substituting in (12), we obtain

$$(\mathcal{L}v)(P) = \int_B G_{\mathrm{e}}(P, Q)\,(Vv)(Q)\, dV_Q + (\mathcal{L}_E v)(P),$$

where Vv is defined by (2) and

$$(\mathcal{L}_E v)(P) = \int_B \operatorname{div}\{(\alpha - 1)G_{\mathrm{e}}\operatorname{grad} v\}\, dV_Q = \int_S (\alpha - 1)G_{\mathrm{e}}(P, q)\,\frac{\partial v}{\partial n}\, ds_q, \quad (17)$$

by the divergence theorem. Thus, the Scattering Problem reduces to solving

$$p_{\mathrm{i}}(P) = p_{\mathrm{inc}}(P) + \int_B G_{\mathrm{e}}(P, Q)\,(Vp_{\mathrm{i}})(Q)\, dV_Q + p_E(P), \quad P \in B, \quad (18)$$

where (using $(8)_2$ in (17))

$$p_E(P) = (\mathcal{L}_E p_{\mathrm{i}})(P) = \int_S \left(\frac{\partial p_{\mathrm{e}}}{\partial n} - \frac{\partial p_{\mathrm{i}}}{\partial n}\right) G_{\mathrm{e}}(P, q)\, ds_q.$$

If we had tried to solve the Scattering Problem using the formal method described in Section 1, we would have obtained precisely (18) but with $p_E(P) \equiv 0$. In general, this extra term is not zero. Observe that, from (17), p_E does vanish if $\rho(q) = \rho_{\mathrm{e}}$ for all $q \in S$, which means that the density is continuous across S. Otherwise, the single-layer potential $p_E(P)$ should be retained.

References

1. P.G. Bergmann: J. Acoust. Soc. Amer. **17**, 329 (1946)
2. W.C. Chew: *Waves and Fields in Inhomogeneous Media* (Van Nostrand, New York 1990)
3. D. Colton, R. Kress: *Inverse Acoustic and Electromagnetic Scattering Theory* (Springer, Berlin, 1992)
4. R.E. Kleinman, P.A. Martin: SIAM J. Appl. Math. **48**, 307 (1988)
5. P.A. Martin: J. Acoust. Soc. Amer. **111**, 2013 (2002)
6. P.A. Martin: SIAM J. Appl. Math. **64**, 297 (2003)
7. P.M. Morse, K.U. Ingard: *Theoretical Acoustics* (Princeton University Press, 1986)
8. P. Werner: Arch. Rational Mech. & Analysis **12**, 155 (1963)