ON THE T-MATRIX FOR SCATTERING BY A SMALL OBSTACLE

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Abstract

Acoustic scattering by a bounded obstacle in three dimensions is considered. Relations between the T-matrix and the far-field pattern are derived, and then used to obtain new approximations for the T-matrix for a small obstacle. Various extensions and applications are suggested.

Introduction

Consider the scattering of acoustic waves by a bounded, three-dimensional obstacle, B. Choose an origin O inside B, and let C denote the smallest sphere that is centred at O and encloses B. If we know the T-matrix for B, we can calculate the scattered field outside C for any given incident field. Similarly, if we know the far-field pattern, f, we can also calculate the scattered field outside C, *but only for the incident field that generated the far-field pattern via the scattering process*: f depends on the direction of observation and on the choice of incident field.

Evidently, we can calculate the far-field pattern from the T-matrix. However, we can also calculate the T-matrix from the far-field pattern, provided we know f for all directions of observation and for all directions of incidence when the incident field is a plane wave. This simple observation means that we can use known results for low-frequency scattering of plane waves to obtain expressions for the T-matrix of small scatterers.

The main utility of these results occurs with multiple-scattering problems, where waves interact with two or (many) more obstacles. Such problems are often treated using T-matrix methods. Notice that the basic ideas are not limited to problems of acoustics, but may be generalised to electromagnetic and elastodynamic problems.

Formulation

Suppose that the scatterer B has surface S. Suppressing a time dependence of $e^{-i\omega t}$, the total field u satisfies the Helmholtz equation,

$$(\nabla^2 + k^2)u = 0,$$

in the unbounded region outside S, where $k = \omega/c$ and c is the constant sound speed. We write $u = u_{\rm in} + u_{\rm sc}$, where $u_{\rm in}$ is the known incident field and $u_{\rm sc}$ is the unknown scattered field. We require that $u_{\rm sc}$ satisfies the Sommerfeld radiation condition at infinity. Consequently,

$$u_{\rm sc}(\mathbf{r}) \sim f(\hat{\mathbf{r}})\, h_0(kr) \quad \text{as } r \to \infty,$$

where $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector in the direction of observation (from O towards P, the point with position vector $\mathbf{r}$ with respect to O), $h_n(kr) \equiv h_n^{(1)}(kr)$ is a spherical Hankel function, and $f(\hat{\mathbf{r}})$ is known as the *far-field pattern*. Note that $h_0(kr) = e^{ikr}/(ikr)$.

For direct problems, one is often interested in calculating f. For inverse problems, one often starts with f and then tries to say something about the scatterer. It is well known that if one knows $f(\hat{\mathbf{r}})$ for all $\hat{\mathbf{r}} \in \Omega$ (the unit sphere), then one can reconstruct $u_{\rm sc}(\mathbf{r})$ everywhere outside the escribed sphere C; this sphere has radius r_c. Explicitly, we have the Atkinson–Wilcox theorem,

$$u_{\rm sc}(\mathbf{r}) = h_0(kr) \sum_{n=0}^{\infty} \frac{f_n(\hat{\mathbf{r}})}{r^n} \quad \text{for } r > r_c, \tag{1}$$

where $f_0 \equiv f$. For $n = 1, 2, \ldots$, f_n is obtained by applying a second-order differential operator (essentially, the angular part of the Laplacian) to f_{n-1}. In principle, (1) can be used to continue $u_{\rm sc}$ from the far field to the near field.

The T-matrix and the far-field pattern

Outside the escribed sphere C, we have the expansion

$$u_{\rm sc}(\mathbf{r}) = \sum_{n,m} c_n^m h_n(kr)\, Y_n^m(\hat{\mathbf{r}}), \quad r > r_c, \tag{2}$$

where Y_n^m is a spherical harmonic and

$$\sum_{n,m} = \sum_{n=-\infty}^{\infty} \sum_{m=-n}^{n} .$$

We use normalised complex-valued spherical harmonics, so that $\overline{Y_n^m} = (-1)^m Y_n^{-m}$ and

$$\int_\Omega Y_n^m \overline{Y_\nu^\mu}\, d\Omega = \delta_{n\nu}\delta_{m\mu}, \tag{3}$$

where the overbar denotes complex conjugation. Using $h_n(x) \sim (-i)^n h_0(x)$ as $x \to \infty$, we have

$$f(\hat{\mathbf{r}}) = \sum_{n,m} (-i)^n c_n^m Y_n^m(\hat{\mathbf{r}}). \tag{4}$$

For the incident field, we have the expansion

$$u_{\text{in}}(\mathbf{r}) = \sum_{n,m} d_n^m j_n(kr)\, Y_n^m(\hat{\mathbf{r}}), \tag{5}$$

where j_n is a spherical Bessel function. This expansion holds in some ball centred at O. The coefficients d_n^m in (5) are known. In particular, for an incident plane wave,

$$u_{\text{in}}(\mathbf{r}) = \exp(ik\mathbf{r}\cdot\hat{\boldsymbol{\alpha}}),$$

and then we have

$$d_n^m = 4\pi i^n \overline{Y_n^m(\hat{\boldsymbol{\alpha}})}; \tag{6}$$

here, $\hat{\boldsymbol{\alpha}}$ is the direction of incidence.

The T-matrix relates the coefficients in (2) and (5):

$$c_n^m = \sum_{\nu,\mu} T_{n\nu}^{m\mu}\, d_\nu^\mu. \tag{7}$$

For properties of the T-matrix, see [1]. The T-matrix can be computed in various ways, such as by solving boundary integral equations [2].

For an incident plane wave, with the corresponding far-field pattern denoted by $f(\hat{\mathbf{r}};\, \hat{\boldsymbol{\alpha}})$, (4), (6) and (7) give

$$f(\hat{\mathbf{r}};\, \hat{\boldsymbol{\alpha}}) = 4\pi \sum_{n,m} \sum_{\nu,\mu} i^{\nu-n} T_{n\nu}^{m\mu} Y_n^m(\hat{\mathbf{r}})\, \overline{Y_\nu^\mu(\hat{\boldsymbol{\alpha}})}.$$

Then, using the orthonormality relation (3) twice, we obtain

$$T_{n\nu}^{m\mu} = \frac{i^{n-\nu}}{4\pi} \int_\Omega \int_\Omega f(\hat{\mathbf{r}};\, \hat{\boldsymbol{\alpha}})\, \overline{Y_n^m(\hat{\mathbf{r}})} Y_\nu^\mu(\hat{\boldsymbol{\alpha}})\, d\Omega(\hat{\mathbf{r}})\, d\Omega(\hat{\boldsymbol{\alpha}}). \tag{8}$$

This formula is exact. It can be found in [3]. It may be used to continue u_{sc} from the far field to the near field; cf. (1).

Small soft scatterers

As a simple example, consider Rayleigh scattering by a small sound-soft obstacle (so that $u = 0$ on S). Then, it is known that (see, for example, [4])

$$f(\hat{\mathbf{r}};\, \hat{\boldsymbol{\alpha}}) = -ik\mathcal{C} + O(k^2) \quad \text{as } k \to 0,$$

where the constant $\mathcal{C}$ is the *capacity* of S; by definition,

$$\mathcal{C} = -\frac{1}{4\pi} \int_S \frac{\partial\phi}{\partial n}\, ds,$$

where $\partial/\partial n$ denotes normal differentiation on S away from B, and the potential ϕ solves the following problem: $\nabla^2\phi = 0$ outside S, $\phi = 1$ on S and $\phi = O(r^{-1})$ as $r \to \infty$. Then, (8) gives the corresponding T-matrix as

$$T_{n\nu}^{m\mu} = -ik\mathcal{C}\overline{y_n^m} y_\nu^\mu + O(k^2) \quad \text{as } k \to 0,$$

where

$$y_n^m = \frac{(-i)^n}{\sqrt{4\pi}} \int_\Omega Y_n^m\, d\Omega = \delta_{n0}\delta_{m0},$$

using $Y_0^0 = (4\pi)^{-1/2}$. Thus, we find that every entry of the T-matrix is $O(k^2)$ except that

$$T_{00}^{00} = -ik\mathcal{C} + O(k^2) \quad \text{as } k \to 0.$$

Consequently, for *any* incident field, $u_{\text{in}}(\mathbf{r})$, we have

$$u_{\text{sc}}(\mathbf{r}) \simeq T_{00}^{00} d_0^0\, h_0(kr)\, Y_0^0$$

where, from (5), $d_0^0 Y_0^0 = u_{\text{in}}(\mathbf{0})$. Hence, we obtain the approximation

$$u_{\text{sc}}(\mathbf{r}) \simeq -ik\mathcal{C}\, u_{\text{in}}(\mathbf{0})\, h_0(kr). \tag{9}$$

Thus, as is generally known, small soft obstacles scatter isotropically (there is no dependence on $\hat{\mathbf{r}}$), with amplitude proportional to the value of the incident field at the scatterer's 'centre', $\mathbf{r} = \mathbf{0}$. This was the starting point for Foldy's famous study on multiple scattering [5]. In fact, Foldy wrote

$$u_{\text{sc}}(\mathbf{r}) \simeq g\, u_{\text{in}}(\mathbf{0})\, h_0(kr), \tag{10}$$

where g is a 'scattering coefficient'. Our asymptotic analysis gives

$$g = -ik\mathcal{C}. \tag{11}$$

However, energy considerations show that g must satisfy

$$|g|^2 + \mathrm{Re}(g) = 0, \tag{12}$$

so that a better choice for g is

$$g = -ik\mathcal{C}/(1 + ik\mathcal{C}); \tag{13}$$

this choice satisfies (12) and agrees with (11) as $k \to 0$.

Small hard scatterers

For a sound-hard obstacle, we have $\partial u/\partial n = 0$ on S. From [4], we have

$$f(\hat{\mathbf{r}};\, \hat{\boldsymbol{\alpha}}) = \frac{ik^3}{4\pi}\left\{V_B\,(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\alpha}} - 1) - \int_S (\hat{\mathbf{r}}\cdot\mathbf{n})(\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\Psi})\,ds\right\}$$

as $k \to 0$, with an error that is $O(k^4)$. In this formula, V_B is the volume of B, $\mathbf{n}(q)$ is the unit normal vector at $q \in S$ pointing away from B, and the vector field $\boldsymbol{\Psi}$ solves the following problem: $\nabla^2\boldsymbol{\Psi} = \mathbf{0}$ outside S, $\partial\boldsymbol{\Psi}/\partial n = \mathbf{n}$ on S and $\boldsymbol{\Psi} = O(r^{-2})$ as $r \to \infty$; see [4, eqn. (5.20)].

Now, following Dassios and Kleinman [4, p. 166], we define the *virtual mass tensor* $\mathbf{W}$ by

$$W_{ij} = -\int_S n_i \Psi_j\,ds = W_{ji}, \tag{14}$$

and the *magnetic polarizability tensor* $\mathbf{M}$ by

$$M_{ij} = W_{ij} + V_B\delta_{ij} = M_{ji}. \tag{15}$$

(For the special case of a sphere, $M_{ij} = \frac{3}{2}V_B\delta_{ij}$.) Then, we can express the far-field pattern concisely by

$$f(\hat{\mathbf{r}};\, \hat{\boldsymbol{\alpha}}) = \frac{ik^3}{4\pi}\{\hat{\mathbf{r}}\cdot\mathbf{M}\cdot\hat{\boldsymbol{\alpha}} - V_B\} + O(k^4) \quad \text{as } k \to 0. \tag{16}$$

Thus, the far field of a small hard scatterer depends linearly on both the observation direction and the incident direction, and it is much smaller than the far field of a small soft scatterer. Of course, this result was known to Lord Rayleigh.

We can use (16) to calculate the T-matrix for a small sound-hard scatterer. Substituting in (8), we find after some calculation that the T-matrix has ten entries that are $O(k^3)$ as $k \to 0$:

$$\begin{aligned}
&T_{00}^{00} = -ik^3 V_B/(4\pi),\\
&T_{11}^{00} = ik^3 M_{33}/(12\pi),\\
&T_{11}^{01} = -ik^3(M_{31} + iM_{32})/(12\pi\sqrt{2}) = -T_{11}^{-1,0},\\
&T_{11}^{10} = -ik^3(M_{31} - iM_{32})/(12\pi\sqrt{2}) = -T_{11}^{0,-1},\\
&T_{11}^{11} = ik^3(M_{11} + M_{22})/(24\pi) = T_{11}^{-1,-1},\\
&T_{11}^{1,-1} = ik^3(M_{22} - M_{11} + 2iM_{12})/(24\pi),\\
&T_{11}^{-1,1} = ik^3(M_{22} - M_{11} - 2iM_{12})/(24\pi).
\end{aligned}$$

Let us calculate the scattered field for any incident field, $u_{\text{in}}(\mathbf{r})$. We introduce a vector $\mathbf{U}$ with components

$$U_j = \frac{1}{k}\frac{\partial u_{\text{in}}}{\partial x_j} \quad \text{evaluated at } \mathbf{r} = \mathbf{0}. \tag{17}$$

Then, we find that $d_1^0 = \sqrt{12\pi}U_3$, $d_1^1 = -\sqrt{6\pi}(U_1 - iU_2)$ and $d_1^{-1} = \sqrt{6\pi}(U_1 + iU_2)$. Also, as before, $d_0^0 = \sqrt{4\pi}u_{\text{in}}(\mathbf{0})$. We then calculate c_n^m, using (7) and the approximations to the T-matrix given above. Eventually, we obtain

$$u_{\text{sc}}(\mathbf{r}) \simeq \frac{ik^3}{4\pi}\{\hat{\mathbf{r}}\cdot\mathbf{M}\cdot\mathbf{U}\,h_1(kr) - V_B u_{\text{in}}(\mathbf{0})\,h_0(kr)\}. \tag{18}$$

This can be used to generalise Foldy's method to collections of small hard scatterers.

Conclusions

We have described a systematic method for obtaining approximations to the T-matrix, valid for small scatterers of any shape. (The only other related results known to us are for spheroids in [6].) The method generalises to penetrable scatterers, to two dimensions (for which the low-frequency asymptotics are more complicated [7]) and to other physical situations.

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