Two-dimensional Waves Around Almost Periodic Arrangements of Scatterers

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Abstract
Acoustic waves around an infinite configuration of identical circular scatterers are considered. Each scatterer is close to a node of a regular lattice: the geometrical configuration is almost periodic. Analytical estimates for the average field in such a random medium are obtained.

Introduction
Consider waves in a two-dimensional periodic structure, defined by a lattice $\Lambda$: each cell in the lattice is a parallelogram, each node in the lattice is a scatterer location. Let $d$ be the shortest distance between nodes. For scalar waves governed by the Helmholtz equation, $I 2 + k^2 = 0$, it is known how to calculate the dispersion relation, connecting the wavenumber $k$ to the Bloch vector $Q$: solutions satisfy the Bloch condition $u(r + r_j) = u(r) \exp(iQ \cdot r_j)$, for every lattice node $r_j$.

The periodic problems outlined above have been studied extensively. One important application concerns photonic crystals [1]. Fabrication of such structures inevitably introduces imperfections, leading to nearly periodic geometries or other forms of disorder. What are the effects of the disorder? There are publications on this question; the main result is that the band-gap phenomena seen with periodic structures are robust to small amounts of random disorder. Representative publications include [2], [3], [4], [5], [6]. All these papers include results from numerical simulations. Some [4], [6] use a ‘supercell’ method, which means that a periodic medium is constructed in which each period contains the same disordered arrangement of circular scatterers; evidently, such a periodic medium is not a random medium, so it is unclear how to interpret the results. The other papers [2], [3], [5] use a finite number of circular scatterers, 1152 in [2], 38 in [3] and 169 in [5].

1 The periodic problem
We consider identical circular scatterers of radius $a$. For simplicity, we suppose that $ka \ll 1$ and that each scatterer is sound soft. This permits the use of Foldy’s (deterministic) model for the scattering:

$$u(r) = \sum_{r_j \in \Lambda} B_j \psi_0^H(r - r_j) = B_i \psi_0^H(r - r_i) + u^e(r),$$

where $B_j$ are coefficients, $\psi_n^H(r) = H_n^{(1)}(kr) \exp(i \theta)$, $r, \theta$ are polar coordinates and

$$u^e(r) = u(r) - B_i \psi_0^H(r - r_i) = \sum_{j \neq i} B_j \psi_0^H(r - r_j).$$

The quantity $u^e_n(r)$ is the field incident on the $n$th scatterer in the presence of all the other scatterers. Next, suppose that $B_n = g \sigma_n^H(r_n)$, where $g$ is the scattering coefficient. Thus, the strength of the field scattered by the $n$th cylinder is proportional to the external field acting on that cylinder. All our scatterers are identical, so we use the same scattering coefficient for each. Then, evaluating (1) at $r = r_i$ gives a homogeneous linear system for the numbers $u^e_j(r_j)$. Looking for a solution of this system in the form $u^e_j(r) = \exp(iQ \cdot r)$ gives just one equation,

$$1 = g \sigma_0^H(k; Q; \Lambda),$$

where $\sigma_0^H$ is a lattice sum,

$$\sigma_n^H = \sigma_0^H(k; Q; \Lambda) = \sum_{r_j \in \Lambda} \psi_n^H(r_j) \exp(iQ \cdot r_j) \quad (3)$$

and the prime on the summation indicates that the term with $r_0 = 0$ is omitted.

Equation (2) gives a relation between $k$ and $Q$. To indicate that we are examining the periodic problem, we add a subscript $p$, giving $k_p$ and $Q_p$. We may choose to specify $Q_p$ and then (2) determines $k_p$, or we may choose to do the opposite. We are interested in how these relations are changed by the introduction of positional disorder.

2 The almost periodic problem
Let the centre of the $n$th circle be displaced from $r_n$ to $r'_n$, with $|r_n - r'_n| < \varepsilon d$, where $0 < \varepsilon < 1$: each small disc, $D_n$, of radius $R = \varepsilon d$ and centre $r_n \in \Lambda$, contains exactly one scatterer, centred at $r'_n$. 
Again, for simplicity, we use the Foldy deterministic model. Then, we compute the ensemble average, \( \langle u \rangle \), using the Lax QCA (see [7, §8.6.4], for example) and a simple choice for the pair correlation function: given \( \Lambda \), the first scatterer can be centred in any disc \( D_n \) with equal probability, the second scatterer can then be centred in any other disc with equal probability. We find that

\[
\langle u(\mathbf{r}) \rangle = \frac{g}{\pi R^2} \sum_{\mathbf{r}_j \in \Lambda} \int_{D_j} v(\mathbf{r}') \psi_0^H(\mathbf{r} - \mathbf{r}') \, d\mathbf{r}',
\]

where \( v \) solves the integral equation

\[
v(\mathbf{r}') = \frac{g}{\pi R^2} \sum_{j \neq i} \int_{D_j} v(\mathbf{r}') \psi_0^H(\mathbf{r}_i' - \mathbf{r}') \, d\mathbf{r}', \quad \mathbf{r}_i' \in D_i.
\]

This equation shows that \( (\nabla^2 + k^2)v = 0 \) inside each \( D_i \). Equation (4) shows that \( \langle u \rangle \) can be written as an acoustic ‘volume’ potential. This observation has two consequences. First, an application of \( (\nabla^2 + k^2) \) to (4) gives

\[
(\nabla^2 + k^2)\langle u(\mathbf{r}) \rangle = \begin{cases} 0, & \mathbf{r} \notin D_i, \ i \in \mathbb{Z}. \\
\frac{4g}{(\pi R^2)}v(\mathbf{r}), & \mathbf{r} \in D_i,
\end{cases}
\]

Thus, \( \langle u \rangle \) solves a certain problem for a periodic lattice of circular scatterers, each of (small) radius \( R = \varepsilon d \). Second, \( \langle u \rangle \) and its normal (radial) derivative are both continuous across the boundary of each \( D_i \).

To solve (5), we first impose the Bloch condition,

\[
v(\mathbf{r} + \mathbf{r}_j) = v(\mathbf{r}) \exp(i\mathbf{Q} \cdot \mathbf{r}_j), \quad \mathbf{r} \in D_0, \quad j \in \mathbb{Z}; \quad (6)
\]

it follows that \( \langle u \rangle \) satisfies the same condition. Use of (6) in (5) together with the two-centre expansion of \( \psi_0^H \) [7, Theorem 2.14] gives

\[
v(\mathbf{r}) = g \sum_p \sum_n (-1)^p V_n \sigma_{n-p}^H \psi_p^J(\mathbf{r}), \quad \mathbf{r} \in D_0, \quad (7)
\]

where \( \psi_p^J(\mathbf{r}) = J_n(kr)e^{in\theta} \) and

\[
V_n = \frac{1}{\pi R^2} \int_{D_0} v(s) \psi_n^J(s) \, ds.
\]

Multiply (7) by \( \psi_m^J(\mathbf{r}) \) and integrate over \( D_0 \) giving

\[
V_m = gJ_m(kR) \sum_n V_n \sigma_{n-m}^H(k; \mathbf{Q}; \Lambda), \quad m \in \mathbb{Z}, \quad (8)
\]

where \( J_m(kR) = J_m^0(kR) - J_{m-1}(kR)J_{m+1}(kR) \).

This is an infinite homogeneous system of linear algebraic equations for \( V_m \). Setting its determinant to zero yields the dispersion relation connecting \( k \) to \( \mathbf{Q} \).

For very small disorder \( (kR \ll 1) \), we can approximate the function \( J_m(kR) \) occurring in (8). As \( J_m(0) = \delta_{0m} \), (8) reduces correctly to (2) in the absence of disorder. At next order, we use \( J_0 \sim 1 - (kR)^2/4 \) and \( J_{\pm 1} \sim (kR)^2/8 \), leading to a \( 3 \times 3 \) system for \( V_{\pm 1} \) and \( V_0 \). This produces an approximate dispersion relation, valid for small disorder. This can be related to the periodic problem, giving estimates for \( \mathbf{Q} - \mathbf{Q}_p \) when \( k = k_p \) or \( k - k_p \) when \( \mathbf{Q} = \mathbf{Q}_p \), as desired.

Further work is ongoing, comparing with numerical simulations (for comparisons in one dimension, see [8]) and going beyond the Foldy representation.

References


