Two-dimensional Waves Around Almost Periodic Arrangements of Scatterers*

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Abstract

Acoustic waves around an infinite configuration of identical circular scatterers are considered. Each scatterer is close to a node of a regular lattice: the geometrical configuration is almost periodic. Analytical estimates for the average field in such a random medium are obtained.

Introduction

Consider waves in a two-dimensional periodic structure, defined by a lattice Λ: each cell in the lattice is a parallelogram, each node in the lattice is a scatterer location. Let d be the shortest distance between nodes. For scalar waves governed by the Helmholtz equation, $(\nabla^2+k^2)u=0$, it is known how to calculate the dispersion relation, connecting the wavenumber k to the Bloch vector $\mathbf{Q}$: solutions satisfy the Bloch condition $u(\mathbf{r}+\mathbf{r}_j)=u(\mathbf{r})\exp(\mathrm{i}\mathbf{Q}\cdot\mathbf{r}_j)$, for every lattice node $\mathbf{r}_j$.

The periodic problems outlined above have been studied extensively. One important application concerns photonic crystals [1]. Fabrication of such structures inevitably introduces imperfections, leading to nearly periodic geometries or other forms of disorder. What are the effects of the disorder? There are publications on this question; the main result is that the band-gap phenomena seen with periodic structures are robust to small amounts of random disorder. Representative publications include [2], [3], [4], [5], [6]. All these papers include results from numerical simulations. Some [4], [6] use a 'supercell' method, which means that a periodic medium is constructed in which each period contains the same disordered arrangement of circular scatterers; evidently, such a periodic medium is not a random medium, so it is unclear how to interpret the results. The other papers [2], [3], [5] use a finite number of circular scatterers, 1152 in [2], 38 in [3] and 169 in [5].

1 The periodic problem

We consider identical circular scatterers of radius a. For simplicity, we suppose that $ka \ll 1$ and that each scatterer is sound soft. This permits the use of Foldy's (deterministic) model for the scattering:

$$u(\mathbf{r})=\sum_{\mathbf{r}_j\in\Lambda}\mathcal{B}_j\psi_0^H(\mathbf{r}-\mathbf{r}_j)=\mathcal{B}_i\psi_0^H(\mathbf{r}-\mathbf{r}_i)+u_i^{\mathrm{e}}(\mathbf{r}),$$

where $\mathcal{B}_j$ are coefficients, $\psi_n^H(\mathbf{r})=H_n^{(1)}(kr)\mathrm{e}^{\mathrm{i}n\theta}$, r,θ are polar coordinates and

$$u_i^{\mathrm{e}}(\mathbf{r})=u(\mathbf{r})-\mathcal{B}_i\psi_0^H(\mathbf{r}-\mathbf{r}_i)=\sum_{j\neq i}\mathcal{B}_j\psi_0^H(\mathbf{r}-\mathbf{r}_j). \quad (1)$$

The quantity $u_n^{\mathrm{e}}(\mathbf{r})$ is the field incident on the nth scatterer in the presence of all the other scatterers. Next, suppose that $\mathcal{B}_n=g u_n^{\mathrm{e}}(\mathbf{r}_n)$, where g is the scattering coefficient. Thus, the strength of the field scattered by the nth cylinder is proportional to the external field acting on that cylinder. All our scatterers are identical, so we use the same scattering coefficient for each. Then, evaluating (1) at $\mathbf{r}=\mathbf{r}_i$ gives a homogeneous linear system for the numbers $u_j^{\mathrm{e}}(\mathbf{r}_j)$. Looking for a solution of this system in the form $u_j^{\mathrm{e}}(\mathbf{r})=\exp(\mathrm{i}\mathbf{Q}\cdot\mathbf{r})$ gives just one equation,

$$1=g\sigma_0^H(k;\mathbf{Q};\Lambda), \quad (2)$$

where σ_0^H is a lattice sum,

$$\sigma_n^H=\sigma_n^H(k;\mathbf{Q};\Lambda)=\sum_{\mathbf{r}_j\in\Lambda}{}'\,\psi_n^H(\mathbf{r}_j)\exp(\mathrm{i}\mathbf{Q}\cdot\mathbf{r}_j) \quad (3)$$

and the prime on the summation indicates that the term with $\mathbf{r}_0=\mathbf{0}$ is omitted.

Equation (2) gives a relation between k and $\mathbf{Q}$. To indicate that we are examining the periodic problem, we add a subscript p, giving k_{p} and $\mathbf{Q}_{\mathrm{p}}$. We may choose to specify $\mathbf{Q}_{\mathrm{p}}$ and then (2) determines k_{p}, or we may choose to do the opposite. We are interested in how these relations are changed by the introduction of positional disorder.

2 The almost periodic problem

Let the centre of the nth circle be displaced from $\mathbf{r}_n$ to $\mathbf{r}_n'$ with $|\mathbf{r}_n-\mathbf{r}_n'|<\varepsilon d$, where $0<\varepsilon\ll 1$: each small disc, $\mathcal{D}_n$, of radius $R=\varepsilon d$ and centre $\mathbf{r}_n\in\Lambda$, contains exactly one scatterer, centred at $\mathbf{r}_n'$.

Again, for simplicity, we use the Foldy deterministic model. Then, we compute the ensemble average, $\langle u\rangle$, using the Lax QCA (see [7, §8.6.4], for example) and a simple choice for the pair correlation function: given Λ, the first scatterer can be centred in any disc $\mathcal{D}_n$ with equal probability, the second scatterer can then be centred in any other disc with equal probability. We find that

$$\langle u(\mathbf{r})\rangle = \frac{g}{\pi R^2}\sum_{\mathbf{r}_j\in\Lambda}\int_{\mathcal{D}_j} v(\mathbf{r}')\,\psi_0^H(\mathbf{r}-\mathbf{r}')\,\mathrm{d}\mathbf{r}', \qquad (4)$$

where v solves the integral equation

$$v(\mathbf{r}_i') = \frac{g}{\pi R^2}\sum_{j\neq i}\int_{\mathcal{D}_j} v(\mathbf{r}')\,\psi_0^H(\mathbf{r}_i'-\mathbf{r}')\,\mathrm{d}\mathbf{r}', \quad \mathbf{r}_i'\in\mathcal{D}_i. \qquad (5)$$

This equation shows that $(\nabla^2+k^2)v = 0$ inside each $\mathcal{D}_i$. Equation (4) shows that $\langle u\rangle$ can be written as an acoustic 'volume' potential. This observation has two consequences. First, an application of (∇^2+k^2) to (4) gives

$$(\nabla^2+k^2)\langle u(\mathbf{r})\rangle = \begin{cases} 0, & \mathbf{r}\notin\mathcal{D}_i, \\ [4\mathrm{i}g/(\pi R^2)]v(\mathbf{r}), & \mathbf{r}\in\mathcal{D}_i, \end{cases} \quad i\in\mathbb{Z}.$$

Thus, $\langle u\rangle$ solves a certain problem for a *periodic* lattice of circular scatterers, each of (small) radius $R = \varepsilon d$. Second, $\langle u\rangle$ and its normal (radial) derivative are both continuous across the boundary of each $\mathcal{D}_i$.

To solve (5), we first impose the Bloch condition,

$$v(\mathbf{r}+\mathbf{r}_j) = v(\mathbf{r})\exp(\mathrm{i}\mathbf{Q}\cdot\mathbf{r}_j), \quad \mathbf{r}\in\mathcal{D}_0, \quad j\in\mathbb{Z}; \quad (6)$$

it follows that $\langle u\rangle$ satisfies the same condition. Use of (6) in (5) together with the two-centre expansion of ψ_0^H [7, Theorem 2.14] gives

$$v(\mathbf{r}) = g\sum_n\sum_p(-1)^p V_n\sigma_{n-p}^H\psi_p^J(\mathbf{r}), \quad \mathbf{r}\in\mathcal{D}_0, \quad (7)$$

where $\psi_n^J(\mathbf{r}) = J_n(kr)\mathrm{e}^{\mathrm{i}n\theta}$ and

$$V_n = \frac{1}{\pi R^2}\int_{\mathcal{D}_0} v(\mathbf{s})\,\psi_{-n}^J(\mathbf{s})\,\mathrm{d}\mathbf{s}.$$

Multiply (7) by $\psi_{-m}^J(\mathbf{r})$ and integrate over $\mathcal{D}_0$ giving

$$V_m = g\mathcal{J}_m(kR)\sum_n V_n\sigma_{n-m}^H(k;\mathbf{Q};\Lambda), \quad m\in\mathbb{Z}, \quad (8)$$

where $\mathcal{J}_m(kR) = J_m^2(kR) - J_{m-1}(kR)J_{m+1}(kR)$. This is an infinite homogeneous system of linear algebraic equations for V_n. Setting its determinant to zero yields the dispersion relation connecting k to $\mathbf{Q}$.

For very small disorder ($kR \ll 1$), we can approximate the function $\mathcal{J}_m(kR)$ occurring in (8). As $\mathcal{J}_m(0) = \delta_{0m}$, (8) reduces correctly to (2) in the absence of disorder. At next order, we use $\mathcal{J}_0 \sim 1-(kR)^2/4$ and $\mathcal{J}_{\pm1}\sim(kR)^2/8$, leading to a 3×3 system for $V_{\pm1}$ and V_0. This produces an approximate dispersion relation, valid for small disorder. This can be related to the periodic problem, giving estimates for $\mathbf{Q}-\mathbf{Q}_\mathrm{p}$ when $k = k_\mathrm{p}$ or $k-k_\mathrm{p}$ when $\mathbf{Q} = \mathbf{Q}_\mathrm{p}$, as desired.

Further work is ongoing, comparing with numerical simulations (for comparisons in one dimension, see [8]) and going beyond the Foldy representation.

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