INTERNAL GRAVITY WAVES AND HYPERBOLIC BOUNDARY-VALUE PROBLEMS[†]

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Talk Abstract

Three-dimensional time-harmonic internal gravity waves are generated by oscillating a bounded object in an unbounded stratified fluid. Energy is found in conical wave beams. The problem is to calculate the wave fields for an object of arbitrary shape. It can be formulated as a hyperbolic boundary-value problem. The following aspects are discussed: reduction to boundary integral equations; single-layer and double-layer potentials; estimation of far fields and radiation conditions. The problem is complicated because the group and phase velocities are orthogonal. In addition, singular boundary integrals arise: their integrands are infinite along a certain curve (not just at a point) on the boundary, and this happens even when the field point is off the boundary (but within one of the conical wave beams).

Introduction

Boundary-value problems (BVPs) for hyperbolic partial differential equations (PDEs) are unfamiliar to most mathematicians. However, they do arise in applications, as we shall see later. To see that they do differ from elliptic BVPs, start with interior Dirichlet problems for a function u in a disc, $r < a$, with $u = 0$ at $r = a$. For Laplace's equation, $u_{xx} + u_{yy} = 0$, the only solution is $u \equiv 0$, whereas for the "wave" equation, $u_{xx} - u_{yy} = 0$, there is a simple non-trivial solution, namely $u = a^2 - r^2$; this example was noted by Bateman [1, p. 611] in 1929. Studies of $u_{xx} = u_{yy}$ in rectangles were made by Bourgin and Duffin [2] in 1939 and John [3] in 1941. Since then, the pure mathematical literature is sparse.

In applications, interior hyperbolic BVPs arise with certain models of granular flow [4] and with internal waves [5]. We shall also consider internal waves, but our interest is with exterior problems.

Governing equations

Consider an inviscid unbounded fluid with a uniform density stratification, under gravity. There is a bounded 3D object (with boundary S) in the fluid. Time-harmonic
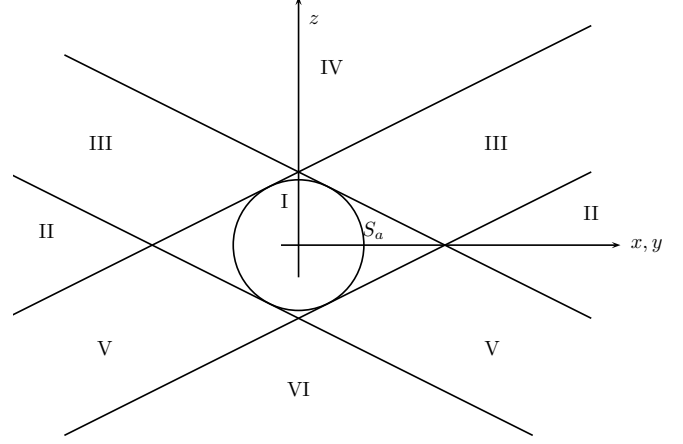
Figure 1: The oscillating body S is located inside the sphere, S_a. Regions III and V are the conical wave beams bounded by characteristic cones. The angle between the conical surfaces and the z-axis is θ_{c}.

internal waves are generated by oscillating S. It is found that the significant wave motion is confined to beams forming a "Saint Andrew's cross" (in 2D), as shown in famous images obtained by Mowbray and Rarity [6]; in 3D, the beams are conical (see Figure 1).

Under the Boussinesq approximation, the wave motion can be found by calculating the pressure $\mathrm{Re}\{p\,e^{-i\omega t}\}$; $p(x, y, z)$ solves

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \Upsilon\frac{\partial^2 p}{\partial z^2} = 0, \text{ where } \Upsilon = \frac{\omega^2}{\omega^2 - N^2} \quad (1)$$

is a constant, z is the vertical coordinate and N is the constant Brunt–Väisälä frequency. The two frequencies, ω and N, satisfy $0 < \omega < N$, so that $\Upsilon < 0$ and the PDE (1) is hyperbolic. It is to be solved subject to boundary and far-field conditions. The boundary condition is natural: $\mathbf{v} \cdot \mathbf{n}$ is prescribed on S, where $\mathbf{n}$ is a normal to S and the velocity $\mathbf{v} = (u, v, w)$ is given in terms of p by

$$u = -\frac{i}{\omega}\frac{\partial p}{\partial x}, \quad v = -\frac{i}{\omega}\frac{\partial p}{\partial y}, \quad w = -\frac{i\Upsilon}{\omega}\frac{\partial p}{\partial z}. \quad (2)$$

The far-field conditions are discussed next.

Far-field conditions

For comparison purposes, start with linear acoustics, governed by the Helmholtz equation, $(\nabla^2 + k^2)u = 0$. In this context, the Sommerfeld radiation condition (SRC) is imposed: $r(u_r - iku) \to 0$ as $r \to \infty$. Physically, the SRC ensures that radiated waves go away from S. Mathematically, the SRC ensures that we have a well-posed BVP. The SRC is linear. Thus, we can use superposition: as $G = e^{ik|P-q|}/|P-q|$ satisfies the SRC, so does

$$u(P) = (\mathrm{S}\mu)(P) \equiv \int_S \mu(q)\, G(P,q)\, dS_q, \qquad (3)$$

for any reasonable μ; $\mathrm{S}\mu$ is a single-layer potential.

What are appropriate far-field conditions for internal waves? Of course, we want to capture the expected physical behaviour, so we begin by summarising what is known. Let r be the distance from the point P to an origin inside S. Referring to Figure 1, we expect $p = O(r^{-1})$ as $r \to \infty$ when P is in Regions II, IV or VI; in these regions, p behaves like a solution of Laplace's equation, and there is no energy transport. All energy transport occurs in the conical wave beams, Regions III and V. In these beams, p is larger, $O(r^{-1/2})$ as $r \to \infty$ [7, p. 467]. Specifically,

$$p \sim r^{-1/2} F(\sigma) \quad \text{as } r \to \infty \text{ in Regions III and V}, \quad (4)$$

where F is a complex-valued function of the lateral coordinate σ, across the beam. F may be thought of as the far-field pattern; it does not depend on r. Notice that we do not see wavelike behaviour in (4), as a function of r. That is because the waves travel *across* the beam whereas the energy propagates *along* the beam: the group velocity is parallel to the beam and it is perpendicular to the phase velocity [7, p. 445]. This is very different to linear acoustics, where the phase and group velocities are in the same direction: we can ensure that energy travels away by ensuring that the waves travel away, which explains why the SRC is physically correct.

For internal waves, we can examine energy directly, and then require that energy travels away. Physically, this condition is attractive but, mathematically, it is awkward because energy is a quadratic quantity. In detail, for P in Region III, let $\mathbf{e}$ be a unit vector that is parallel to the beam, pointing from S towards P. Apart from inessential positive factors, the time-averaged energy transport is given by the vector $\mathrm{Re}\,(p\overline{\mathbf{v}}) \sim r^{-1} I(\sigma)\mathbf{e}$, where the overbar denotes complex conjugation and

$$I(\sigma) = \mathrm{Im}\,(\overline{F}F'). \qquad (5)$$

A plausible far-field condition is to require $I(\sigma) > 0$ for $|\sigma| < a$, where σ varies from $-a$ to a across the beam. This pointwise condition turns out to be too restrictive. We should require that $\int I > 0$, where the integration is over the beam cross-section, but it is unclear if this is sufficient to define the physically-meaningful solution.

A fundamental solution and layer potentials

One way to construct outgoing solutions proceeds as follows. Solve (1) when $\omega > N$ ($\Upsilon > 1$, so (1) is elliptic). Then use analytic continuation in the complex ω-plane to determine the solution for $0 < \omega < N$ ($\Upsilon < 0$). This approach was pioneered by Pierce [8] and Hurley [9]. It is more fundamental than looking at energy flow because it is based on causality in the time domain, which means there should be no motion before a disturbance is excited. As we have used a time-dependence of $e^{-i\omega t}$, causality implies that there should be no singularities or branch cuts in the upper half of the ω-plane.

For $\omega > N$ ($\Upsilon > 1$), it is easy to see that

$$G(P,Q) \equiv G(x_0, y_0, z_0;\ x, y, z)$$
$$= \{(x - x_0)^2 + (y - y_0)^2 + \Upsilon^{-1}(z - z_0)^2\}^{-1/2} \quad (6)$$

solves (1), where P is a fixed point. Evidently, G is a fundamental solution for Laplace's equation with a rescaling of the z coordinate, and $G(P,Q)$ is singular at $P = Q$. We can then use G to construct single-layer and double-layer potentials (such as (3)), just as we would for the Helmholtz equation [10]. Integral representations for p in terms of the boundary values of p and $\mathbf{v} \cdot \mathbf{n}$ can also be developed, using an appropriate reciprocal theorem [11].

Next, we effect the analytic continuation to obtain formulas valid for $0 < \omega < N$. Define an angle θ_c by

$$\omega = N \cos\theta_\mathrm{c} \quad \text{with} \quad 0 < \theta_\mathrm{c} < \pi/2.$$

Define spherical polar coordinates (R, Θ, Φ) by $x_0 - x = R\sin\Theta\cos\Phi$, $y_0 - y = R\sin\Theta\sin\Phi$, $z_0 - z = R\cos\Theta$. Then, for P in the fluid, we obtain the representation [11]

$$p(P) = \mathrm{D}p - \mathrm{S}(\mathbf{v} \cdot \mathbf{n}), \qquad (7)$$

where $\mathrm{D}\mu$ is a double-layer potential. We define $\mathrm{S}\mu$ by

$$(\mathrm{S}\mu)(P) = i(\omega^2 - N^2)^{1/2} \int_S \mu(q)\, G(P,q)\, \frac{dS_q}{4\pi}$$
$$= \int_S \mu(x, y, z)\mathcal{M}(\Theta)\, \frac{dS(x, y, z)}{4\pi R}, \qquad (8)$$

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where the first (second) line defines $S\mu$ before (after) analytic continuation and

$$\mathcal{M}(\Theta) = \begin{cases} \dfrac{-N\cos\theta_c\sin\theta_c}{\sqrt{\cos^2\theta_c - \cos^2\Theta}}, & |\cos\Theta| < \cos\theta_c, \\[2em] \dfrac{iN\cos\theta_c\sin\theta_c}{\sqrt{\cos^2\Theta - \cos^2\theta_c}}, & \cos\theta_c < |\cos\Theta|, \end{cases} \tag{9}$$

so that $\mathcal{M}(\Theta)$ is weakly singular at $\Theta = \theta_c$ and at $\Theta = \pi - \theta_c$. $D\mu$ is defined similarly [11], except that the singularities are stronger. Then, given $\mathbf{v}\cdot\mathbf{n}$ on S, we can obtain a boundary integral equation (BIE) for p on S from (7). Alternatively, instead of using (7), we could try using $p(P) = (S\mu)(P)$; the boundary condition would then give a BIE for μ.

Discussion, questions and anxieties

The strategy outlined above (which is familiar in classical potential theory and in linear acoustics) would seem to offer a way to calculate the pressure field in a stratified fluid due to the oscillations of a body S of arbitrary shape. Indeed, Sturova [12], working in 2D, starts with $p = S\mu$ and an appropriate choice for G. Similar representations were first used for internal waves by Robinson [13]. He considered a thin vertical barrier in a finite-depth ocean, and he constructed G so as to satisfy boundary conditions at $z = 0$ and $z = H$. Similar methods have been used for barriers [14], [15] and for other 2D bottom topographies [16], [17], [18]. All these papers start with $\psi = S\mu$, where ψ is a stream function, leading to a first-kind BIE. Implicitly, it is assumed that, since G has been constructed so as to satisfy the far-field conditions [7], then so will $S\mu$, once μ has been found from a BIE.

Singularities along curves

In order to check, we need a method for estimating the far-field behaviour of $S\mu$, i.e., for calculating $F(\sigma)$ in (4). Such a method has been developed [11] but it is quite complicated. To see the difficulties, suppose that $P = (x_0, y_0, z_0)$ is in Region III and $q = (x, y, z)$ is on S. The singularity in (9) occurs at $\Theta = \theta_c$: for fixed P, we have to integrate over q, so we must identify where the singularities are on S. We have

$$\{(x - x_0)^2 + (y - y_0)^2\}\cos^2\theta_c - (z - z_0)^2\sin^2\theta_c$$
$$= R^2(\cos^2\theta_c - \cos^2\Theta).$$

Thus, the singularities in (9) occur when

$$(z - z_0)^2 = \{(x - x_0)^2 + (y - y_0)^2\}\cot^2\theta_c. \tag{10}$$

This defines a double cone in xyz-space with apex at P.

If P were in Region II, IV or VI, the double cone would not intersect S, and there would be no singularities in the integrations over S in (8). Conventional methods then give the far-field behaviour.

The situation is different when P is in Region III. Then, the lower half of the double cone (10),

$$z - z_0 = -\sqrt{(x - x_0)^2 + (y - y_0)^2}\,\cot\theta_c,$$

intersects S in a curve $\mathcal{C}$. Thus, as q is at (x, y, z), we see that the singularities in the integration over S in (8) occur at all points q on the curve $\mathcal{C}$. This curve is characterised as being where $\Theta = \theta_c$. (A similar construction can be made when P is in Region V.)

We emphasise that there are singularities in the boundary integrals over S defining $S\mu$ and $D\mu$ even when P is not on S (but is in Region III or V). This is very different from classical potential theory, for example, where typical boundary integrals only contain singularities when the field point P is on the boundary, and then the singularity is at P, not along a curve on S. All this is a consequence of the hyperbolic nature of the governing PDE.

Far-field behaviour within the wave beams

Having identified where the singularities are on S, the next step is to estimate the integrals when P is far from S but within Region III. As the singularities are characterised by being where $\Theta = \theta_c$, it is natural to make a change of variables in the boundary integrals, expressing them as double integrals over a region $\mathcal{E}$ in the $\Theta\Phi$-plane. This change puts the singularities along the straight line $\Theta = \theta_c$ (which passes through $\mathcal{E}$) and so they can be handled by one-dimensional calculations; the Φ integrations are benign. Strongly singular integrals can also be handled by moving into the complex Θ-plane. In addition, as the observation point recedes to infinity within the wave beams, the domain $\mathcal{E}$ shrinks so that approximations can be made.

The result of the computations are formulas for $F(\sigma)$ in (4) corresponding to $S\mu$ and $D\mu$. These have been verified by comparing with known solutions for spherical objects.

The main idea of changing variables so that singularities on a curve are moved to singularities on a coordinate line goes back to work by I.M. Lifshitz in the 1940s on the Green's function for waves in a regular lattice; see [19] for details and references.

Some consequences

As we now have a way of calculating F, we can calculate I, defined by (5). It turns out that simple choices

for μ (such as linear functions of z when S is a sphere) in $S\mu$ lead to functions $I(\sigma)$ that are negative in part of the wave beams. This is consistent with known solutions for spheres and with our recent work for oscillating horizontal discs.

Instead of writing $p = S\mu$, we could use (7), involving boundary integrals of p and $\mathbf{v} \cdot \mathbf{n}$. This has two advantages: the representation is known to be valid (in the sense that if the BVP has a solution, then the solution can be represented as claimed) and it involves physical quantities (as opposed to μ). Again, we can estimate far-field quantities.

Computationally, the treatment of the BIEs is not standard, mainly due to the singularity structure. Mathematically, we do not have a rigorous theory for the solvability of the hyperbolic BVP. Evidently, there is much that remains to be done.

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